Minimum SER Zero-Forcing Transmitter Design for MIMO Channels with Interference Pre-subtraction

Michael Botros Shenouda, *Student Member, IEEE,* and Timothy N. Davidson, *Member, IEEE*

***Abstract*— We consider point-to-point multiple antenna communication systems in which multiple data streams are transmitted simultaneously. We consider systems which use Tomlinson-Harashima (TH) precoding to pre-subtract the interference among these data streams at the transmitter. In a conventional Tomlinson-Harashima precoding system, transmitter feedback and receiver feedforward processing matrices are used for interference pre-subtraction and channel spatial equalization. In addition to these matrices, we consider a transmitter precoding matrix that generalizes the permutation matrix used for ordering the precoded symbols in existing designs. This extra degree of freedom offers the potential for improved performance. In particular, under a mild signal to noise ratio (SNR) constraint, we find an optimum zero-forcing precoding matrix that minimizes the average symbol error rate (SER) of the data streams subject to a transmitter power constraint. We also show that the proposed design is optimal from an average bit error rate (BER) perspective. Simulation studies show significant improvement over conventional zero-forcing Tomlinson-Harashima precoders.**

I. Introduction

Tomlinson-Harashima (TH) precoding was originally developed for temporal equalization of channels with inter-symbol interference [1], [2]. Based on the transmitter's knowledge of the channel, it subtracts the effect of the interference that would be created by previously transmitted symbols. The same concept can be applied to spatial equalization of Multiple-Input Multiple-Output (MIMO) channels in which multiple data streams are transmitted simultaneously [3], [4]. In this scenario, the data symbols are successively precoded and the interference created by previously precoded symbols is pre-subtracted. This pre-subtraction at the transmitter is a dual, in some sense, to Decision Feedback Equalization (DFE) in which the effect of previously detected symbols is subtracted [5]. By operating at the transmitter, TH precoding avoids the effects of error propagation, but it requires channel knowledge at the transmitter in return.

In conventional point-to-point Tomlinson-Harashima precoding, interference pre-subtraction and channel spatial equalization are implemented using a feedback matrix at the transmitter and a feedforward matrix at the receiver. For point-to-point MIMO systems, designs of TH precoding systems under zero-forcing criteria were considered in [3], [4], and mean square error (MMSE) designs were derived in [6], [7].

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada. The work of the second author is also supported by the Canada Research Chairs Program.

The authors are with the Department of Electrical and Computer Engineering, McMaster University, Hamilton, Ontario, Canada (e-mail: botrosmw@mcmaster.ca; davidson@mcmaster.ca)

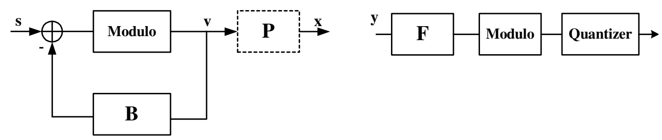

Fig. 1. Multiple antenna transmitter and receiver using Tomlinson-Harashima precoding.

Analogous designs were obtained for the multi-user MIMO scenarios in [8], [9] In addition to these processing matrices, further improvements were obtained by optimizing the order of the precoded symbols in a way similar to BLAST ordering [10], [11]. This is equivalent to ordering the precoded symbols and can be represented by a permutation matrix at the transmitter. In our work, we consider the use of a transmitter precoding matrix that generalizes the permutation matrix used for channel ordering. Under a mild SNR constraint, we find an optimum precoding matrix that minimizes the average symbol error rate of the data streams subject to a total transmitter power constraint. We also show that the proposed design is optimal from an average bit error rate (BER) perspective. An extension of our work to the case of MMSE based TH Precoding can be found in [12].

II. System Model

We consider a point-to-point multiple antenna communication system with n_t transmit antennas and n_r receive antennas. The use of multiple antennas will enable the transmission of K data streams simultaneously, where K is less than or equal the rank of the channel matrix $\mathbf{H}$. We consider communication systems in which Tomlinson-Harashima precoding is used to pre-subtract the interference between these data streams. As shown in Fig. 1, the joint task of interference pre-subtraction and channel spatial equalization is performed using a transmit feedback matrix $\mathbf{B} \in \mathbb{C}^{K\times K}$ and a receive feedforward matrix $\mathbf{F} \in \mathbb{C}^{K\times n_r}$. In addition to the feedback and feedforward processing of conventional TH precoding, we propose the use of precoding matrix $\mathbf{P} \in \mathbb{C}^{n_t\times K}$ that generalizes the permutation matrix used in conventional TH design. The vector $\mathbf{s} \in \mathbb{C}^K$ contains the data symbols of each stream, and we assume that $\mathbf{s}_k$ is chosen from a square QAM constellation $\mathcal{S}$ with cardinality M. The Voronoi region of the constellation $\mathcal{V}$ is a square whose side length is D; i.e., $D = \sqrt{M}\, d$, where d is the distance between two successive constellation points along any of the basis directions.

In absence of the modulo operation, the output symbols of the feedback loop in Fig. 1, $\mathbf{v}_k$, would be generated

1550-2252/$25.00 ©2007 IEEE

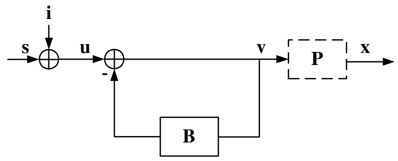

Fig. 2. Equivalent linear model for the transmitter.

successively according to the following relation:

$$\mathbf{v}_k = \mathbf{s}_k - \sum_{j=1}^{k-1} \mathbf{B}_{k,j}\mathbf{v}_j, \quad (1)$$

where at the k^{th} step, only the previously precoded symbols $v_1, .., v_{k-1}$ are subtracted. Hence, $\mathbf{B}$ is a strictly lower triangular matrix. The summation in (1) suggests that the magnitude of $\mathbf{v}_k$ may grow beyond the boundaries of $\mathcal{V}$. The role of the modulo operation is to bring the magnitude back inside the boundaries of $\mathcal{V}$. The effect of the modulo operation is equivalent to the addition of the complex quantity $\mathbf{i}_k = \mathbf{i}_k^{re}D + \mathbf{i}_k^{imag}D$ to $\mathbf{s}_k$, where $\mathbf{i}_k^{re}$, $\mathbf{i}_k^{imag} \in \mathbb{Z}$. Using this observation, we obtain the standard linear model of the transmitter that does not involve a modulo operation, as shown in Fig. 2; e.g., [3]. In this model, the constellation of the modified data symbols in the vector $\mathbf{u} = \mathbf{s} + \mathbf{i}$ is simply the periodic extension of the original constellation $\mathcal{S}$ in both the real and imaginary directions. From this equivalent model, it is clear that $\mathbf{v}$ is linearly related to the modified data vector $\mathbf{u}$,

$$\mathbf{v} = (\mathbf{I} + \mathbf{B})^{-1}\mathbf{u} = \mathbf{C}^{-1}\mathbf{u}, \quad (2)$$

where $\mathbf{C} = \mathbf{I} + \mathbf{B}$.

Since the precoded symbols $\mathbf{v}$ will depend on the order used for the interference pre-subtraction, the ordering can be optimized to improve the performance. The use of a given ordering is equivalent to using $\mathbf{P}$ which is a permutation matrix when $K = n_t$ or consisting of columns of a permutation matrix when $K < n_t$, [10]. We consider a model in which $\mathbf{P}$ is not necessarily a permutation matrix and this additional degree of freedom offers the potential for performance improvement. For this model, the transmitted and received signals are given by:

$$\mathbf{x} = \mathbf{P}\mathbf{v}, \quad (3)$$

$$\mathbf{y} = \mathbf{H}\mathbf{P}\mathbf{C}^{-1}\mathbf{u} + \mathbf{n}, \quad (4)$$

respectively, where $\mathbf{n}$ is the vector of additive noise which is assumed to have zero-mean and a covariance matrix $\mathrm{E}\{\mathbf{n}\mathbf{n}^H\} = \sigma_n^2\mathbf{I}$. At the receiver, the feedforward processing matrix $\mathbf{F}$ is used to obtain an estimate $\hat{\mathbf{u}}$ of the modified data symbols $\mathbf{u}$:

$$\hat{\mathbf{u}} = \mathbf{F}\mathbf{H}\mathbf{P}\mathbf{C}^{-1}\mathbf{u} + \tilde{\mathbf{n}}, \quad (5)$$

where $\tilde{\mathbf{n}} = \mathbf{F}\mathbf{n}$. Following the linear receive processing, the modulo operation is used to remove the effect of the periodic extension of the constellation.

A. Zero-Forcing Design

The goal of zero-forcing designs is the elimination of the interference among data streams. A design that is based on ZF criteria was proposed in [3], [4]. In that design the matrices $\mathbf{F}, \mathbf{C}$ are obtained using a QL decomposition of the ordered channel matrix $\mathbf{H}_{\text{ordered}} = \mathbf{H}\mathbf{P} = \mathbf{Q}\mathbf{L}$, where $\mathbf{Q} \in \mathbb{C}^{n_r \times K}$ is a matrix with orthonormal columns and $\mathbf{L} \in \mathbb{C}^{K \times K}$ is a lower triangular matrix. The ordering $\mathbf{P}$ is similar to BLAST ordering [11]. Assuming a full column rank channel $\mathbf{H}$ and the decomposition $\mathbf{H}\mathbf{P} = \mathbf{Q}\mathbf{L}$, the matrices $\mathbf{F}$ and $\mathbf{C}$ are given by [3], [4]:

$$\mathbf{F} = \mathbf{Q}^H, \quad (6)$$

$$\mathbf{C} = \mathrm{Diag}\left(\frac{1}{\mathbf{L}_{11}}, \ldots, \frac{1}{\mathbf{L}_{KK}}\right) \mathbf{L}. \quad (7)$$

This QL decomposition is guaranteed to exist [3], and can be obtained using a QR decomposition algorithm starting with the last column. Without loss of generality, the $\mathbf{L}$ factor can be assumed to have positive diagonal elements. Once TH precoding at the transmitter and spatial equalization at the receiver have been performed, the resulting composite channel will be interference free, as one would expect from a zero-forcing design. The output can be written as:

$$\hat{\mathbf{u}} = \mathrm{Diag}\left(\mathbf{L}_{11}, \ldots, \mathbf{L}_{KK}\right) \mathbf{u} + \tilde{\mathbf{n}}, \quad (8)$$

where $\mathrm{E}\{\tilde{\mathbf{n}}\tilde{\mathbf{n}}^H\} = \sigma_n^2\mathbf{I}$. The modulo operation will then be used to eliminate $\mathbf{i}$ from $\hat{\mathbf{u}}$ (either by scaling down by factors $\mathbf{L}_{ii}$ and taking modulo D for each stream or by directly taking modulo $D\mathbf{L}_{ii}$). Assuming a reasonable SNR, the modulo operation can be assumed to remove the constellation expansion induced by the transmitter, and the output of the modulo operation can be modeled as:

$$\hat{\mathbf{s}} = \mathrm{Diag}\left(\mathbf{L}_{11}, \ldots, \mathbf{L}_{KK}\right) \mathbf{s} + \tilde{\mathbf{n}}. \quad (9)$$

B. Performance Metric

Our performance metric is the average symbol error rate of the K data streams P^{avg}. Assuming that each data stream employs the same constellation, and using (9), P^{avg} is given by:

$$P^{\text{avg}}(\mathbf{L}_{11}, \ldots, \mathbf{L}_{KK}) = \frac{1}{K}\sum_{i=1}^{K} P_{\text{e}}(\mathbf{L}_{ii}), \quad (10)$$

where $P_{\text{e}}(\mathbf{L}_{ii})$ is the average probability of symbol error of the i^{th} data stream. This depends on the constellation type and the SNR of the i^{th} stream, namely $\frac{\mathbf{L}_{ii}^2}{\sigma_n^2}$. For example, for M-ary QAM:

$$P_{\text{e}}(\mathbf{L}_{ii}) = 4aQ\left(\sqrt{\frac{3\ \mathbf{L}_{ii}^2}{(M-1)\ \sigma_n^2}}\right) - 4a^2Q^2\left(\sqrt{\frac{3\ \mathbf{L}_{ii}^2}{(M-1)\ \sigma_n^2}}\right)$$

where $a = 1 - 1/\sqrt{M}$ and $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-z^2/2}\, dz$.

C. Transmitter Power Constraints

We want to find an optimum precoding matrix $\mathbf{P}$ to minimize the average SER in (10) subject to a constraint on the transmitted power $\mathrm{E}\{\mathbf{x}^H\mathbf{x}\} = \mathrm{tr}(\mathbf{P}\mathbf{R}_v\mathbf{P}^H) = P_{\text{total}}$, where $\mathbf{R}_v = \mathrm{E}\{\mathbf{v}\mathbf{v}^H\}$. To obtain $\mathbf{R}_v$, will make the standard observation that the elements of $\mathbf{v}$ are almost uncorrelated and uniformly distributed over the Voronoi region of the constellation $\mathcal{V}$ [3, Th. 3.1], [13]. In this model, the symbols $\mathbf{v}$ will have slightly higher energy average power than the input symbols $\mathbf{s}$. For example, for a square M-QAM constellation, we have $\sigma_v^2 = \mathrm{E}\{|\mathbf{v}_k|^2\} = \frac{M}{M-1}\mathrm{E}\{|\mathbf{s}_k|^2\}$ for all k except the first one [4]. Using these properties and the assumption that $\mathrm{E}\{\mathbf{s}\mathbf{s}^H\} = \mathbf{I}$, it follows that $\mathbf{R}_v$ can be approximated by $\sigma_v^2\mathbf{I}$ [3], [4]. Consequently, the power constraint can be rewritten as $\mathrm{tr}(\mathbf{P}\mathbf{P}^H) = \frac{P_{\text{total}}}{\sigma_v^2}$.

III. Minimum SER Precoder Design

Our approach to optimum precoder design will consist of two steps. We will start by finding an optimum choice of values of $\mathbf{L}_{ii},\ i = 1, \ \ldots, \ K$ that minimizes the average SER expression in (10). Then, we will find the optimal precoding matrix such that the equivalent channel $\mathbf{H}_{\text{eq}} = \mathbf{H}\mathbf{P}$ will yield QL decomposition with the desired $\mathbf{L}$ factor.

A. Choice of $\mathbf{L}_{ii}$

In this section we show that under a mild constraint on the SNR of each stream, $P_{\text{e}}(\mathbf{L}_{ii})$ exhibits desirable convexity properties that enable us to characterize optimal values of $\mathbf{L}_{ii}$ using results from majorization theory, a concept that has been useful in the design of of linear transceiver [14]. We will start our presentation by brief introduction of the relevant concepts from majorization theory [15].

Majorization: Let $\mathbf{a},\ \mathbf{b} \in \mathbb{R}^K$ and let $a_{[1]} \geq \ldots \geq a_{[K]}$ denote the components of $\mathbf{a}$ in descending order. The vector $\mathbf{b}$ is said to majorize the vector $\mathbf{a}$, $\mathbf{a} \prec \mathbf{b}$, if:

$$\sum_{i=1}^{j} \mathbf{a}_{[i]} \leq \sum_{i=1}^{j} \mathbf{b}_{[i]} \qquad j = 1, \ldots, K-1, \tag{11}$$

$$\sum_{i=1}^{K} \mathbf{a}_{[i]} = \sum_{i=1}^{K} \mathbf{b}_{[i]}. \tag{12}$$

A useful result that follows from the above definition is that any vector $\mathbf{a} \in \mathbb{R}^K$ majorizes its mean vector $\bar{\mathbf{a}}$ whose elements are all equal to the mean; i.e., $\bar{\mathbf{a}}_i = \frac{1}{K}\sum_{i=1}^{K}\mathbf{a}_i$. That is:

$$\bar{\mathbf{a}} \prec \mathbf{a}. \tag{13}$$

Schur-convex and Schur-concave functions: A real-valued function $f(\mathbf{x})$ defined on a subset $\mathcal{A}$ of $\mathbb{R}^K$ is said to be Schur-convex if $\mathbf{a} \prec \mathbf{b}$ on $\mathcal{A} \Rightarrow f(\mathbf{a}) \leq f(\mathbf{b})$, and is said to be Schur-concave if $\mathbf{a} \prec \mathbf{b}$ on $\mathcal{A} \Rightarrow f(\mathbf{a}) \geq f(\mathbf{b})$.

Now, let $\mathbf{L}$ be the lower triangular matrix resulting from the QL factorization of $\mathbf{H}_{\text{eq}}$. We know that:

$$\prod_{i=1}^{K} \mathbf{L}_{ii}^2 = \det(\mathbf{L}^H\mathbf{L}) = \det({\mathbf{H}_{\text{eq}}}^H\mathbf{H}_{\text{eq}}). \tag{14}$$

Define $\boldsymbol{\lambda} = (\ln(\mathbf{L}_{11}), \ \ldots, \ \ln(\mathbf{L}_{KK}))$. Then, $\sum_{i=1}^{K}\boldsymbol{\lambda}_i = \frac{1}{2}\ln\det({\mathbf{H}_{\text{eq}}}^H\mathbf{H}_{\text{eq}})$. It follows from (13) that:

$$\bar{\boldsymbol{\lambda}} \prec \boldsymbol{\lambda}, \tag{15}$$

where $\bar{\boldsymbol{\lambda}}_i = \frac{1}{2K}\ln\det({\mathbf{H}_{\text{eq}}}^H\mathbf{H}_{\text{eq}})$. Using this majorization relation with the following lemma we can characterize the optimal $\mathbf{L}_{ii}$.

Lemma 1: If each $P_{\text{e}}(\mathbf{L}_{ii}) = P_{\text{e}}(e^{\boldsymbol{\lambda}_i})$ is a convex function of $\boldsymbol{\lambda}_i$, then $P^{\text{avg}}(\mathbf{L}_{11}, \ldots, \mathbf{L}_{KK})$ is minimized when $\mathbf{L}_{11} = \ldots = \mathbf{L}_{KK} = \sqrt[2K]{\det({\mathbf{H}_{\text{eq}}}^H\mathbf{H}_{\text{eq}})}$

Proof: The proof relies on the fact that if $f(x)$ is convex function, then $\phi(\mathbf{x}) = \sum_{i=1}^{K} f(\mathbf{x}_i)$ is a Schur-convex function of $\mathbf{x}$ [15]. In particular, if $P_{\text{e}}(e^{\boldsymbol{\lambda}_i})$ is convex function of $\boldsymbol{\lambda}_i$ then $P^{\text{avg}} = \frac{1}{K}\sum_{i=1}^{K} f(\mathbf{x}_i)$ is a Schur-convex function of $\boldsymbol{\lambda}$. Using (13), P^{avg} minimized by $\boldsymbol{\lambda}_1 = \cdots = \boldsymbol{\lambda}_K = \frac{1}{2K}\ln\det({\mathbf{H}_{\text{eq}}}^H\mathbf{H}_{\text{eq}})$. ■

The condition that $P_{\text{e}}(e^{\boldsymbol{\lambda}_i})$ is convex function of $\boldsymbol{\lambda}_i$ is satisfied under a mild assumption on the SNR. For example, for BPSK constellations, $P_{\text{e}}(e^{\boldsymbol{\lambda}_i})$ is convex for $\mathbf{L}_{ii}^2 \geq \sigma_n^2$. Appendix I provides similar thresholds for M-ary QAM constellations.

The optimal $\mathbf{L}_{ii}$ in Lemma 1 can also be shown to be optimal from the average BER perspective. Indeed if we assume that each data stream employs the same constellation, then using (9), the average bit error rate, BER^{avg}, is:

$$BER^{\text{avg}}(\mathbf{L}_{11}, \ldots, \mathbf{L}_{KK}) = \frac{1}{K}\sum_{i=1}^{K} BER(\mathbf{L}_{ii}), \tag{16}$$

where for M-ary QAM, BER is closely approximated by [16], [17]:

$$BER(\mathbf{L}_{ii}) \simeq b\, Q\left(\sqrt{\frac{3\log_2 M\, \mathbf{L}_{ii}^2}{(M-1)\sigma_n^2}}\right) + c\, Q\left(3\sqrt{\frac{3\log_2 M\, \mathbf{L}_{ii}^2}{(M-1)\sigma_n^2}}\right) \tag{17}$$

where $b = 2(\sqrt{M}-1)/(\sqrt{M}\log_2\sqrt{M})$ and $c = 2(\sqrt{M}-2)/(\sqrt{M}\log_2\sqrt{M})$. Using the approach of the proof of Lemma 1, it can be shown that (17) is a convex function of $\boldsymbol{\lambda}_i = \ln\mathbf{L}_{ii}$ under a mild constraint on SNR of each stream. Hence, the design of equal $\mathbf{L}_{ii}$ is optimal from the average BER perspective; See Appendix I.

B. Optimum Design of $\mathbf{P}$

Using the fact that the optimal solution results in all $\mathbf{L}_{ii}$'s taking the same value, the optimized average SER will be given by:

$$P^{\text{avg}} = P_{\text{e}}(\mathbf{L}_{ii}) = P_{\text{e}}\left(\sqrt[2K]{\det({\mathbf{H}_{\text{eq}}}^H\mathbf{H}_{\text{eq}})}\right) \tag{18}$$

A similar expression can be obtained for the average BER. What remains is to determine the optimal value for the $\mathbf{L}_{ii}$'s and a $\mathbf{P}$ that achieves it. Since P_{e} and BER are both decreasing functions of their argument, our objective reduces to designing a $\mathbf{P}$ to maximize $\det({\mathbf{H}_{\text{eq}}}^H\mathbf{H}_{\text{eq}})$ subject to the power constraint $\mathrm{tr}(\mathbf{P}\mathbf{P}^H) = \frac{P_{\text{total}}}{\sigma_v^2}$ and subject to the constraint that diagonal elements of the $\mathbf{L}$ factor of $\mathbf{H}_{\text{eq}}$ are all equal. We will start by characterizing the family of solutions that maximize $\det({\mathbf{H}_{\text{eq}}}^H\mathbf{H}_{\text{eq}})$ subject to the power constraint, then we will choose from this family the one that yields the desired $\mathbf{L}$ factor. The family of precoders that maximize $\det({\mathbf{H}_{\text{eq}}}^H\mathbf{H}_{\text{eq}})$ is derived in Appendix II and is given by:

$$\mathbf{P} = \sqrt{\frac{P_{\text{total}}}{K\sigma_v^2}}\, \mathbf{U}_1\mathbf{V}, \tag{19}$$

where $\mathbf{U}_1$ is the matrix consisting of the eigen vectors corresponding to the K largest eigen values of $\mathbf{H}^H\mathbf{H}$ and $\mathbf{V}$ is a unitary matrix degree of freedom.

To complete the design of $\mathbf{P}$, we will select $\mathbf{V}$ such that the QL decomposition of $\mathbf{H}_{\text{eq}} = \sqrt{\frac{P_{\text{total}}}{K\sigma_v^2}}(\mathbf{H}\mathbf{U}_1)\mathbf{V} = \mathbf{QL}$ yields an $\mathbf{L}$ with equal diagonal elements. In [18], it was shown that there exists a unitary matrix such that the QR decomposition $\mathbf{A}\mathbf{V}_r = \mathbf{Q}_r\mathbf{R}$ will have R factor with equal diagonal elements. Using this result to obtain $\mathbf{Q}_r, \mathbf{V}_r, \mathbf{R}$ of $\mathbf{A} = \mathbf{H}\mathbf{U}_1$, we can obtain $\mathbf{Q}, \mathbf{V}, \mathbf{L}$ by reversing the order of columns of $\mathbf{Q}_r, \mathbf{V}_r$ to obtain $\mathbf{Q}, \mathbf{L}$ and reversing the columns and rows of $\mathbf{R}$ to obtain $\mathbf{L}$.

IV. Simulation Studies

In our simulations, we use QPSK and 16-QAM signaling over an independent Rayleigh fading channel. The coefficients of the $\mathbf{H}$ matrix are modelled as being independent proper complex Gaussian random variables with zero mean and unit variance. We plot the average bit error rate (BER) of the K data streams against the signal-to-noise-ratio, which is defined as the ratio of the total average transmitted power P_{total} to the total receiver noise power $\mathrm{E}\{\mathbf{n}^H\mathbf{n}\}$.

We compare the performance of the proposed zero-forcing TH precoding with minimum SER and BER criteria (ZFTHP-MinBER) to that of the zero-forcing TH precoder designs with BLAST ordering (ZFTHP-BLAST ordering), and no ordering, $\mathbf{P} = \mathbf{I}$, (ZFTHP-No ordering) [3], [4]. We also compare with zero-forcing linear precoding design with minimum BER criteria (ZFLinear-MinBER) in [19]. In Fig. 3 we plot the average BER for a system with 4 transmit and receive antennas transmitting 4 data steams using 16-QAM signaling. We observe the significant BER improvement resulting from using a precoding matrix $\mathbf{P}$ that is not merely an ordering (permutation) matrix. In Fig. 4 we plot the average BER for a system with 3 transmit and receive antennas transmitting 3 data steams using QPSK signaling. Similar performance improvement is observed specially at medium and high SNR when the condition of optimality of the proposed design is valid. We observe also that the minimum BER linear transceiver is performing better than Tomlinson-Harashima system at low SNRs. This is a consequence the periodic extension of the constellation in TH precoding, which results in additional neighbors for the constellation points. This is more apparent in the case of QPSK signalling than the case of 16-QAM signaling shown in Fig. 3.

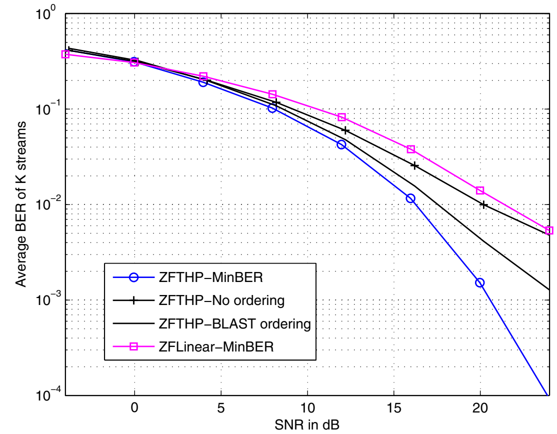

Fig. 3. Comparison between the performance of the proposed minimum BER zero-forcing TH precoding design with conventional design of zero-forcing TH precoding [3], [4] and minimum BER zero-forcing linear precoding [19] for a system with $n_t = 4$, $n_r = 4$ and $K = 4$ simultaneous data streams using 16-QAM signaling.

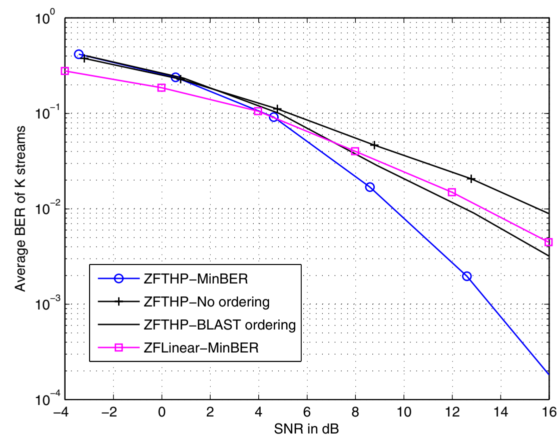

Fig. 4. Comparison between the performance of the proposed minimum BER zero-forcing TH precoding design with conventional design of zero-forcing TH precoding [3], [4] and minimum BER zero-forcing linear precoding [19] for a system with $n_t = 3$, $n_r = 3$ and $K = 3$ simultaneous data streams using QPSK signaling.

V. Conclusion

We have considered the design of Tomlinson-Harashima precoders for single user multiple antenna communication systems. We have considered the use of a transmitter precoding matrix in addition to the feedforward and feedback matrices of conventional Tomlinson-Harashima precoding. This precoding matrix can be viewed as a generalization of permutation matrices used for ordering the precoded signals. We designed an optimum precoding matrix for zero-forcing systems that minimizes the average probability of symbol error of the data streams. We also showed that the proposed design is optimal from an average bit error rate (BER) perspective. Simulation studies showed significant BER improvements over existing designs of zero-forcing Tomlinson-Harashima precoding for single user multiple antenna systems.

Appendix I
Convexity of $P_e(e^{\boldsymbol{\lambda}_i})$ and $BER(e^{\boldsymbol{\lambda}_i})$

In this appendix will find the regions for which $P_e(e^{\boldsymbol{\lambda}_i})$ and $BER(e^{\boldsymbol{\lambda}_i})$ are convex for different constellations.

Convexity of $P_e(e^{\boldsymbol{\lambda}_i})$ for BPSK: The average probability of symbol error in each data stream is equal to $P_{\text{e}}(\mathbf{L}_{ii}) = Q(\mathbf{L}_{ii}/\sigma_n) = Q(e^{\boldsymbol{\lambda}_i}/\sigma_n)$, where $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-z^2/2}\,dz$. The second derivative of P_{e} with respect to $\boldsymbol{\lambda}_i$ is given by:

$$\frac{d^2 P_{\text{e}}(e^{\boldsymbol{\lambda}_i})}{d{\boldsymbol{\lambda}_i}^2} = \frac{e^{\left(\boldsymbol{\lambda}_i - (e^{2\boldsymbol{\lambda}_i})/2\sigma_n^2\right)}}{\sqrt{2\pi\sigma_n^2}}\left(\frac{e^{2\boldsymbol{\lambda}_i}}{\sigma_n^2} - 1\right), \qquad (20)$$

which is non-negative for $\frac{e^{2\boldsymbol{\lambda}_i}}{\sigma_n^2} = \frac{\mathbf{L}_{ii}^2}{\sigma_n^2} \geq 1$. Hence, $P_e(e^{\boldsymbol{\lambda}_i})$ is convex in this region.

Convexity of $P_e(e^{\boldsymbol{\lambda}_i})$ for M-ary QAM: The average probability of error of each stream is:

$$\begin{aligned} P_e(\mathbf{L}_{ii}) &= 4aQ\left(\sqrt{\frac{3}{M-1}}\frac{\mathbf{L}_{ii}}{\sigma_n}\right) - 4a^2Q^2\left(\sqrt{\frac{3}{M-1}}\frac{\mathbf{L}_{ii}}{\sigma_n}\right), \\ &= 4aQ\left(\frac{e^{\boldsymbol{\lambda}_i}}{\sigma_M}\right) - 4a^2Q^2\left(\frac{e^{\boldsymbol{\lambda}_i}}{\sigma_M}\right), \end{aligned}$$

where $a = 1 - 1/\sqrt{M}$ and $\sigma_M = \sqrt{\frac{M-1}{3}}\sigma_n$. The second derivative of P_e with respect to $\boldsymbol{\lambda}_i$ is given by:

$$\begin{aligned} \frac{d^2P_e}{d\boldsymbol{\lambda}_i{}^2} &= 4a\frac{e^{\left(\boldsymbol{\lambda}_i - \left(e^{2\boldsymbol{\lambda}_i}\right)/2\sigma_M^2\right)}}{\sqrt{2\pi\sigma_M^2}} \\ &\times\left(\left[1 - 2aQ\left(\frac{e^{\boldsymbol{\lambda}_i}}{\sigma_M}\right)\right]\left[\frac{e^{2\boldsymbol{\lambda}_i}}{\sigma_M^2} - 1\right] - 2a\frac{e^{\left(\boldsymbol{\lambda}_i - \left(e^{2\boldsymbol{\lambda}_i}\right)/2\sigma_M^2\right)}}{\sqrt{2\pi\sigma_M^2}}\right). \end{aligned} \tag{21}$$

If we denote $x = \frac{e^{\boldsymbol{\lambda}_i}}{\sigma_M} = \frac{\mathbf{L}_{ii}}{\sigma_M}$, it can be shown that the second derivative is non-negative for values of $x \geq \gamma$, where γ is the root of the equation:

$$f(x) = (1 - 2aQ(x))\left(x^2 - 1\right) - \frac{2a}{\sqrt{2\pi}}xe^{-x^2/2}. \tag{22}$$

As a result, $P_e(e^{\boldsymbol{\lambda}_i})$ is convex for values $\frac{\mathbf{L}_{ii}}{\sigma_M} \geq \gamma$. For example, for 16-QAM $\gamma = 1.1935$.

Convexity of $BER(e^{\boldsymbol{\lambda}_i})$: For BPSK, the expression for $BER(e^{\boldsymbol{\lambda}_i})$ is the same as $P_e(e^{\boldsymbol{\lambda}_i})$. For M-QAM, we will use the approximation given by (17), [16], [17]. Using this expression and defining $\sigma_B = \sqrt{\frac{M-1}{3\log_2 M}}\sigma_n$, it can be verified that the second derivative of $BER(\mathbf{e}^{\boldsymbol{\lambda}_i})$ is:

$$\frac{d^2BER}{d\boldsymbol{\lambda}_i{}^2} = \frac{b}{\sqrt{2\pi}}[ye^{\frac{-y^2}{2}}(y^2-1)] + \frac{3c}{\sqrt{2\pi}}[ye^{\frac{-9y^2}{2}}(9y^2-1)], \tag{23}$$

where b and c are defined in Section III and $y = \frac{e^{\boldsymbol{\lambda}_i}}{\sigma_B} = \frac{\mathbf{L}_{ii}}{\sigma_B}$. The second derivative is non-negative for values of $y \geq 1$. Therefore, $BER(e^{\boldsymbol{\lambda}_i})$ is convex for values $\mathbf{L}_{ii} \geq \sigma_B$.

APPENDIX II
DERIVATION OF OPTIMAL $\mathbf{P}$

Let $\mathbf{H}^H\mathbf{H} = \mathbf{G}\boldsymbol{\Lambda}\mathbf{G}^H$ be the eigen vector decomposition of the $\mathbf{H}^H\mathbf{H}$ such that eigen values are in descending order. Let the singular value decomposition of the precoding matrix be given by:

$$\mathbf{P} = \mathbf{U}\begin{bmatrix}\boldsymbol{\Phi} \\ \mathbf{0}\end{bmatrix}\mathbf{V} = \mathbf{U}_1\boldsymbol{\Phi}\mathbf{V}. \tag{24}$$

Now, the objective of the maximization can written as:

$$\begin{aligned} \det(\mathbf{H}_{eq}{}^H\mathbf{H}_{eq}) &= \det(\mathbf{V}^H\boldsymbol{\Phi}\mathbf{U}_1^H\mathbf{G}\boldsymbol{\Lambda}\mathbf{G}^H\mathbf{U}_1\boldsymbol{\Phi}\mathbf{V}) \\ &= \det(\mathbf{W}_1{}^H\boldsymbol{\Lambda}\mathbf{W}_1)\det(\boldsymbol{\Phi}^2) \end{aligned} \tag{25}$$

where $\mathbf{W}_1 = \mathbf{G}^H\mathbf{U}_1$ is a matrix with orthonormal columns ($\mathbf{W}_1^H\mathbf{W}_1 = \mathbf{I}$). Using Hadamard inequality, the first term in Eq. (25) is maximized when $\mathbf{W}_1$ consists of first K columns of $\mathbf{I}$. Accordingly, $\mathbf{U}_1$ is the first K columns of $\mathbf{G}$.

Next, we choose $\boldsymbol{\Phi} = \mathrm{Diag}(\boldsymbol{\Phi}_{11}, \ldots, \boldsymbol{\Phi}_{KK})$ to maximize $\det(\boldsymbol{\Phi}^2)$ subject to $\sum_{i=1}^K \boldsymbol{\Phi}_{ii}^2 = P_{total}/\sigma_v^2$. We observe that $\ln\det(\boldsymbol{\Phi}^2) = \sum_{i=1}^K \ln(\boldsymbol{\Phi}_{ii}^2)$ is a Schur-concave function of $\boldsymbol{\Phi}_{ii}^2$ and is maximized when all $\boldsymbol{\Phi}_{ii}^2$ are equal to $\frac{P_{total}}{K\sigma_v^2}$ [15].

REFERENCES

[1] M. Tomlinson, "New automatic equalizer employing modulo arithmetic," *Electron. Lett.*, vol. 7, pp. 138 –139, Mar. 1971.

[2] H. Harashima and H. Miyakawa, "Matched-transmission technique for channels with intersymbol interference," *IEEE Trans. Commun.*, vol. 20, no. 4, pp. 774–780, Aug. 1972.

[3] R. F. H. Fischer, *Precoding and Signal Shaping for Digital Transmission*. New York: Wiley, 2002.

[4] C. Windpassinger, R. F. H. Fischer, T. Vencel, and J. B. Huber, "Precoding in multiantenna and multiuser communications," *IEEE Trans. Wireless Commun.*, vol. 3, no. 4, pp. 1305–1316, Jul. 2004.

[5] N. Al-Dhahir and A. Sayed, "The finite-length multi-input multi-output MMSE-DFE," *IEEE Trans. Signal Processing*, vol. 48, no. 10, pp. 2921–2936, Oct. 2000.

[6] O. Simeone, Y. Bar-Ness, and U. Spagnolini, "Linear and nonlinear preequalization/equalization for MIMO systems with long-term channel state information at the transmitter," *IEEE Trans. Wireless Commun.*, vol. 3, no. 2, pp. 373–378, Mar. 2004.

[7] R. F. H. Fischer, C. Windpassinger, A. Lampe, and J. B. Huber, "Space-time transmission using Tomlinson-Harashima precoding," in *Proc. 4th Int. ITG Conf. Source and Channel Coding*, vol. 5, Berlin, Jan. 2002, pp. 139–147.

[8] ——, "MIMO precoding for decentralized receivers," in *Proc. Int. Symp. Inform. Theory*, Lausanne, Switzerland, Jul. 2002, p. 496.

[9] M. Joham and W. Utschick, "Ordered spatial Tomlinson Harashima precoding," in *Smart Antennas-State of the Art*, T. Kaiser, A. Bourdoux, H. Boche, J. R. Fonollosa, J. B. Andersen, and W. Utschick, Eds. New York: Hindawi Publishing, 2006, vol. 3, ch. 3, pp. 401–422.

[10] C. Windpassinger, T. Vencel, and R. F. H. Fischer, "Precoding and loading for BLAST-like systems," in *Proc. IEEE Int. Conf. Commun.*, vol. 5, Anchorage, May 2003, pp. 3061–3065.

[11] G. D. Golden, C. J. Foschini, R. A. Valenzuela, and P. W. Wolniansky, "Detection algorithm and initial laboratory results using V-BLAST space-time communication architecture," *Electron. Lett.*, vol. 35, no. 1, pp. 14–16, 7 Jan. 1999.

[12] M. Botros Shenouda and T. N. Davidson, "A framework for designing MIMO systems with decision feedback equalization or Tomlinson-Harashima precoding," To appear in *Proc. IEEE Int. Conf. on Acoustics, Speech, Signal Processing*, Hawaii, April 2007.

[13] G. D. Forney Jr. and M. V. Eyuboğlu, "Combined equilization and coding using precoding," *IEEE Commun. Mag.*, vol. 29, pp. 25–34, Dec. 1991.

[14] D. P. Palomar, J. M. Cioffi, and M. A. Lagunas, "Joint Tx-Rx beamforming design for multicarrier MIMO channels: a unified framework for convex optimization," *IEEE Trans. Signal Processing*, vol. 51, no. 9, pp. 2381–2401, Sept. 2003.

[15] A. W. Marshal and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*. New York: Academic Press, 1979.

[16] K. Cho and D. Yoon, "On the general BER expression of one- and two-dimensional amplitude modulations," *IEEE Trans. Commun.*, vol. 50, no. 7, pp. 1074–1080, July 2002.

[17] L. Yang and L. Hanzo, "A recursive algorithm for the error probability evaluation of M-QAM," *IEEE Commun. Lett.*, vol. 4, no. 10, pp. 304–306, Oct. 2000.

[18] J. Zhang, A. Kavcic, and K. M. Wong, "Equal-diagonal QR decomposition and its application to precoder design for successive-cancellation detection," *IEEE Trans. Inform. Theory*, vol. 51, no. 1, pp. 154–172, Jan. 2005.

[19] Y. Ding, T. N. Davidson, Z.-Q. Luo, and K. M. Wong, "Minimum BER block precoders for zero-forcing equalization," *IEEE Trans. Signal Processing*, vol. 51, no. 9, pp. 2410–2423, Sept. 2003.