LECTURE 2: Introduction into the Theory of Radiation

(*Maxwell's equations – revision. Power density and Poynting vector – revision. Radiated power – definition. Basic principle of radiation. Vector and scalar potentials – revision. Far fields and vector potentials.*)

1. Maxwell's equations – revision

(a) the law of induction (Faraday's law):

$$-\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} + \mathbf{M}^{\clubsuit} \tag{2.1}$$

$$\oint_c \mathbf{E} \cdot d\mathbf{c} = -\frac{\partial}{\partial t} \iint_{S_{[c]}} \mathbf{B} \cdot d\mathbf{s} \quad \Leftrightarrow \quad e = -\frac{\partial \Psi}{\partial t} \tag{2.1-i}$$

E (V/m)	electric field (electric field intensity)
B (T=Wb/m^2)	magnetic flux density
M (V/m^2)	magnetic current density[♣]
Ψ (Wb=V$\cdot$s)	magnetic flux
e (V)	electromotive force

(b) Ampere's law, generalized by Maxwell to include the displacement current $\partial\mathbf{D}/\partial t$:

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \tag{2.2}$$

$$\oint_c \mathbf{H} \cdot d\mathbf{c} = \iint_{S_{[c]}} \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \right) \cdot d\mathbf{s} \quad \Leftrightarrow \quad I = \oint_c \mathbf{H} \cdot d\mathbf{c} \tag{2.2-i}$$

H (A/m)	magnetic field (magnetic field intensity)
D (C/m^2)	electric flux density (electric displacement)
J (A/m^2)	electric current density
I (A)	electric current

[♣] **M** is a fictitious quantity, which renders Maxwell's equations symmetrical and which proves a useful mathematical tool when solving EM boundary value problems applying equivalence theorem.

(c) Gauss' electric law:

$$\nabla \cdot \mathbf{D} = \rho \tag{2.3}$$

$$\oiint_{S} \mathbf{D} \cdot d\mathbf{s} = \iiint_{V_{[S]}} \rho dv = Q \tag{2.3-i}$$

ρ (C/m^3)	electric charge density
Q (C)	electric charge

Equation (2.3) follows from equation (2.2) and the continuity relation:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}. \tag{2.4}$$

Hint: Take the divergence of both sides of (2.2).

(d) Gauss' magnetic law:

$$\nabla \cdot \mathbf{B} = \rho_m \,^{♣♣} \tag{2.5}$$

The equation $\nabla \cdot \mathbf{B} = 0$ follows from equation (2.1), provided that $\mathbf{M} = 0$.

Maxwell's equations alone are insufficient to solve for the four vector quantities: **E**, **D**, **H**, and **B** (twelve scalar quantities). Two additional vector equations are needed.

(e) Constitutive relationships

The constitutive relationships describe the properties of matter with respect to electric and magnetic forces.

$$\mathbf{D} = \overset{\leftrightarrow}{\boldsymbol{\varepsilon}} \cdot \mathbf{E} \tag{2.6}$$

$$\mathbf{B} = \overset{\leftrightarrow}{\boldsymbol{\mu}} \cdot \mathbf{H}. \tag{2.7}$$

In an anisotropic medium, the dielectric permittivity and the magnetic permeability are *tensors*. In vacuum, which is isotropic, the permittivity and the permeability are constants (or tensors whose *diagonal elements only* are non-zero and are the same): $\varepsilon_0 = 8.854187817 \times 10^{-12}$ F/m, $\mu_0 = 4\pi \times 10^{-7}$ H/m. In an isotropic medium, the vectors **D** and **E** are collinear, and so are the vectors **B** and **H**.

The dielectric properties relate to the electric field (electric force).

[♣♣] ρ_m is a fictitious quantity introduced via the continuity relation $\nabla \cdot \mathbf{M} = -\partial \rho_m / \partial t$. As per experimental evidence, $\nabla \cdot \mathbf{B} = 0$.

Dielectric materials with relative permittivity (dielectric constant) ε_r > 1 are built of atomic/molecular sub-domains, which have the properties of dipoles. In external electric field, the dipoles tend to orient in such a way that their own fields have a cancellation effect on the external field. The electric force $\mathbf{F}_e = Q\mathbf{E}$ exerted on a point charge Q from a source Q_s in such medium is ε_r times weaker than the electric force of the same source in vacuum.

On the contrary, magnetic materials with relative permeability (magnetic constant) μ_r > 1 are made of sub-domains, which tend to orient in external magnetic field in such a way, that their own magnetic fields align with the external field. The magnetic force $\mathbf{F}_m = Q\mathbf{v}\times\mathbf{B}$ exerted on a moving point charge Q in such a medium is μ_r times stronger than the force that this same source (e.g. electric currents) would create in vacuum.

We are mostly concerned with isotropic media, i.e., media where the equations $\mathbf{B} = \mu_0\mu_r\mathbf{H}$ and $\mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E}$ hold.

(f) Time-harmonic field analysis

In harmonic analysis of EM fields, the field phasors are introduced:

$$\begin{aligned}\mathbf{e}(x,y,z,t) &= \operatorname{Re}\left\{\mathbf{E}(x,y,z)e^{j\omega t}\right\}\\ \mathbf{h}(x,y,z,t) &= \operatorname{Re}\left\{\mathbf{H}(x,y,z)e^{j\omega t}\right\}.\end{aligned} \tag{2.8}$$

For clarity, from this point on, we will denote time-dependent field vectors with lower-case letters, while their phasors will be denoted with upper-case letters. Complex-conjugate quantities will be denoted with an asterisk *.

The phasor equations are obtained from the time-dependent equations by a simple substitution using the following correspondences:

$$f(x,y,z,t) \doteq F(x,y,z)$$

$$\frac{\partial f_{(x,y,z,t)}}{\partial t} \doteq j\omega F(x,y,z)$$

$$\frac{\partial f}{\partial \xi} \doteq \frac{\partial F}{\partial \xi} \quad , \quad \xi = x,y,z \ .$$

For example, Maxwell's equations in phasor form are:

$$\nabla\times\mathbf{H} = j\omega\bar{\varepsilon}\mathbf{E} + \mathbf{J}, \quad \bar{\varepsilon} = \varepsilon' - j\left(\varepsilon'' + \sigma/\omega\right) \tag{2.9}$$

$$-\nabla\times\mathbf{E} = j\omega\bar{\mu}\mathbf{H} + \mathbf{M}, \ \bar{\mu} = \mu' - j\mu''. \tag{2.10}$$

These Maxwell's equations include the equivalent (fictitious) magnetic currents

M. The dielectric polarization loss $\omega\varepsilon''$ and the conductivity loss σ constitute the imaginary part of the *complex dielectric permittivity* $\bar{\varepsilon}$. Often, the dielectric loss is represented by the dielectric loss angle δ_d:

$$\bar{\varepsilon} = \varepsilon'\left[1 - j\left(\frac{\varepsilon''}{\varepsilon'} + \frac{\sigma}{\omega\varepsilon'}\right)\right] = \varepsilon'\left[1 - j\left(\tan\delta_d + \frac{\sigma}{\omega\varepsilon'}\right)\right]. \qquad (2.11)$$

Similarly, the magnetic loss is described by the imaginary part of the *complex magnetic permeability* $\bar{\mu}$ or by the magnetic loss angle δ_m:

$$\bar{\mu} = \mu' - j\mu'' = \mu'\left(1 - j\frac{\mu''}{\mu'}\right) = \mu'(1 - j\tan\delta_m). \qquad (2.12)$$

In antenna theory, we are mostly concerned with *isotropic*, *homogeneous* and *loss-free* propagation media.

2. Power density, vector of Poynting, radiated power

2.1. Poynting vector – revision

In the time-domain analysis, the Poynting vector is defined as

$$\mathbf{p}(t) = \mathbf{e}(t) \times \mathbf{h}(t), \text{ W/m}^2. \qquad (2.13)$$

As follows from Poynting's theorem, **p** is a vector representing the density and the direction of the EM power flow. Thus, the total power leaving certain volume V is obtained as

$$\Pi(t) = \oint\!\!\oint_{S_{[V]}} \mathbf{p}(t) \cdot d\mathbf{s}, \text{ W}. \qquad (2.14)$$

Since

$$\mathbf{e}(t) = \operatorname{Re}\left\{\mathbf{E}e^{j\omega t}\right\} = \frac{1}{2}\left(\mathbf{E}e^{j\omega t} + \mathbf{E}^* e^{-j\omega t}\right), \qquad (2.15)$$

and

$$\mathbf{h}(t) = \operatorname{Re}\left\{\mathbf{H}e^{j\omega t}\right\} = \frac{1}{2}\left(\mathbf{H}e^{j\omega t} + \mathbf{H}^* e^{-j\omega t}\right), \qquad (2.16)$$

the instantaneous power density appears as

$$\mathbf{p}(t) = \underbrace{\frac{1}{2}\operatorname{Re}\left\{\mathbf{E}\times\mathbf{H}^*\right\}}_{\mathbf{p}_{av}} + \frac{1}{2}\operatorname{Re}\left\{\mathbf{E}\times\mathbf{H}\cdot e^{2\cdot j\omega t}\right\}. \qquad (2.17)$$

The first term in (2.17) $\mathbf{p}_{av}$ has no time dependence. It is the average value, about which the power flux density fluctuates. It is a vector of unchanging direction showing a constant outflow (positive value) or inflow (negative value) of EM power. It describes the active power flow, which is the time-average power flux

$$\Pi_{av} = \oiint_{S[V]} \mathbf{p}_{av} \cdot d\mathbf{s} . \qquad (2.18)$$

The second term in (2.17) is a vector changing its direction with a double frequency (2ω). It describes the reactive power flow, i.e., the power, which fluctuates in space (propagates to and fro) without contribution to the overall transport of energy in any direction.

Definition: The complex Poynting vector is the vector

$$\mathbf{P} = \frac{1}{2}\mathbf{E}\times\mathbf{H}^*, \qquad (2.19)$$

whose real part is equal to the average power flux density: $\mathbf{p}_{av} = \operatorname{Re}\mathbf{P}$.

2.2. Radiated power

Definition: Radiated power is the average power radiated by the antenna:

$$\Pi_{rad} = \oiint_{S[V]} \mathbf{p}_{av} \cdot d\mathbf{s} = \oiint_{S[V]} \operatorname{Re}\mathbf{P} \cdot d\mathbf{s} = \frac{1}{2}\oiint_{S[V]} \operatorname{Re}\left\{\mathbf{E}\times\mathbf{H}^*\right\} \cdot d\mathbf{s} . \qquad (2.20)$$

3. Basic principle of radiation

Radiation is produced by accelerated or decelerated charge (time-varying current element).

3.1. Current element

Definition: A current element ($I\Delta l$), $\text{A}\times\text{m}$, is a filament of length Δl and current I.

The concept of current element is essential since the time-varying current element is the elementary source of EM radiation. It has the same significance as the concept of a point charge in electrostatics. The field radiated by a complex antenna in a linear medium can be analyzed by making use of the superposition principle after decomposing the antenna into elementary sources, i.e., into current elements.

Assume the existence of a piece of a very thin wire where electric current can be excited. The current i flowing through the wire cross-section ΔS is defined as the amount of charge passing through ΔS in 1 second:

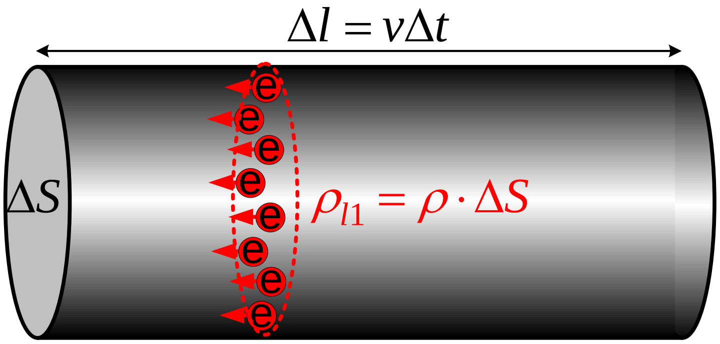
$$i = \rho \cdot \Delta S \cdot \Delta l_1 = \rho \cdot \Delta S \cdot v, \quad \text{A}, \tag{2.21}$$

where

ρ (C/m^3) is the electric charge volume density,

v (m/s) is the charge velocity normal to the cross-section,

Δl_1 (m/s) is the distance traveled by a charge in 1 second.



Equation (2.21) can be also written as

$$\mathbf{j} = \rho \cdot \mathbf{v}, \quad \text{A}/\text{m}^2, \tag{2.22}$$

where **j** is the electric current density. The product $\rho_l = \rho \cdot \Delta S$ is the charge per unit length (charge line density) along the wire. Thus, from (2.21) it follows that

$$i = v \cdot \rho_l, \ \text{A}. \tag{2.23}$$

It is then obvious that

$$\frac{di}{dt} = \rho_l \frac{dv}{dt} = \rho_l \cdot a, \ \text{A/s}, \tag{2.24}$$

where a (m/s^2) is the acceleration of the charge. The time-derivative of a current element $i\Delta l$ is then proportional to the amount of charge q enclosed in the volume of the current element and to its acceleration:

$$\Delta l \frac{di}{dt} = \Delta l \cdot \rho_l \cdot a = q \cdot a, \ \text{A} \times \text{m/s}. \tag{2.25}$$

3.2. Mathematical description of the accelerated charge as a radiation source

It is not immediately obvious from Maxwell's equations that the time-varying current is the source of radiation. A simple transformation of the Maxwell's equations

$$\left|\begin{aligned} -\nabla\times\mathbf{e} &= \mu\frac{\partial\mathbf{h}}{\partial t} \\ \nabla\times\mathbf{h} &= \varepsilon\frac{\partial\mathbf{e}}{\partial t}+\mathbf{j} \end{aligned}\right. \tag{2.26}$$

into a single second-order equation either for **E** or for **H** proves this statement. By taking the curl of both sides of the first equation in (2.26) and by making use of the second equation in (2.26), we obtain

$$\nabla\times\nabla\times\mathbf{e}+\mu\varepsilon\frac{\partial^2\mathbf{e}}{\partial t^2}=-\mu\frac{\partial\mathbf{j}}{\partial t}. \tag{2.27}$$

From (2.27), it is obvious that the time derivative of the electric currents is the source for the wave-like vector **e**.

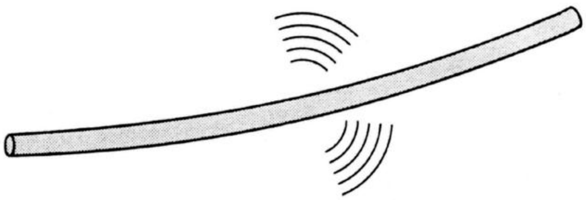
In an analogous way, one can obtain the wave equation for the magnetic field **H** and its sources:

$$\nabla\times\nabla\times\mathbf{h}+\mu\varepsilon\frac{\partial^2\mathbf{h}}{\partial t^2}=\nabla\times\mathbf{j}. \tag{2.28}$$

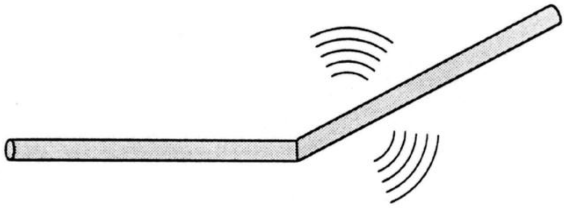
Notice that, as follows from (2.28) and (2.26), curl-free currents (e.g., $j=\nabla\psi$) do not radiate either.

To accelerate/decelerate charges, one needs sources of electromotive force and/or discontinuities of the medium in which the charges move. Such discontinuities can be bends or open ends of wires, change in the electrical properties of the region, etc. In summary:

- If charge is not moving, current is zero ⇒ no radiation.
- If charge is moving with a uniform velocity ⇒ no radiation.
- If charge is accelerated due to electromotive force in a non-dissipative medium or due to discontinuities, such as terminations, bends, curvatures ⇒ radiation occurs.



(a)



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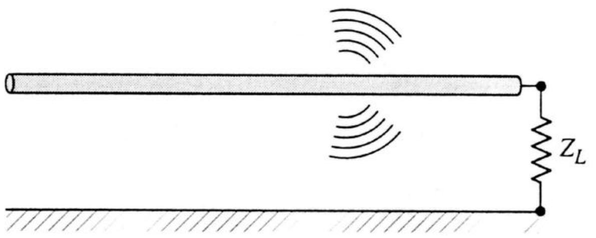

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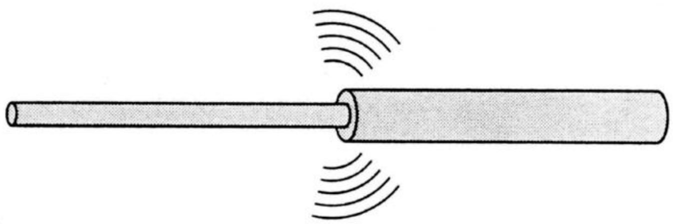

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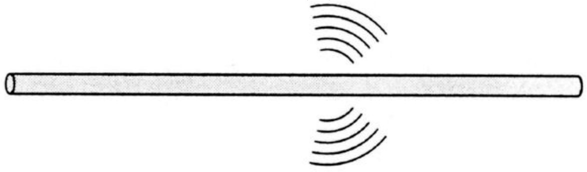
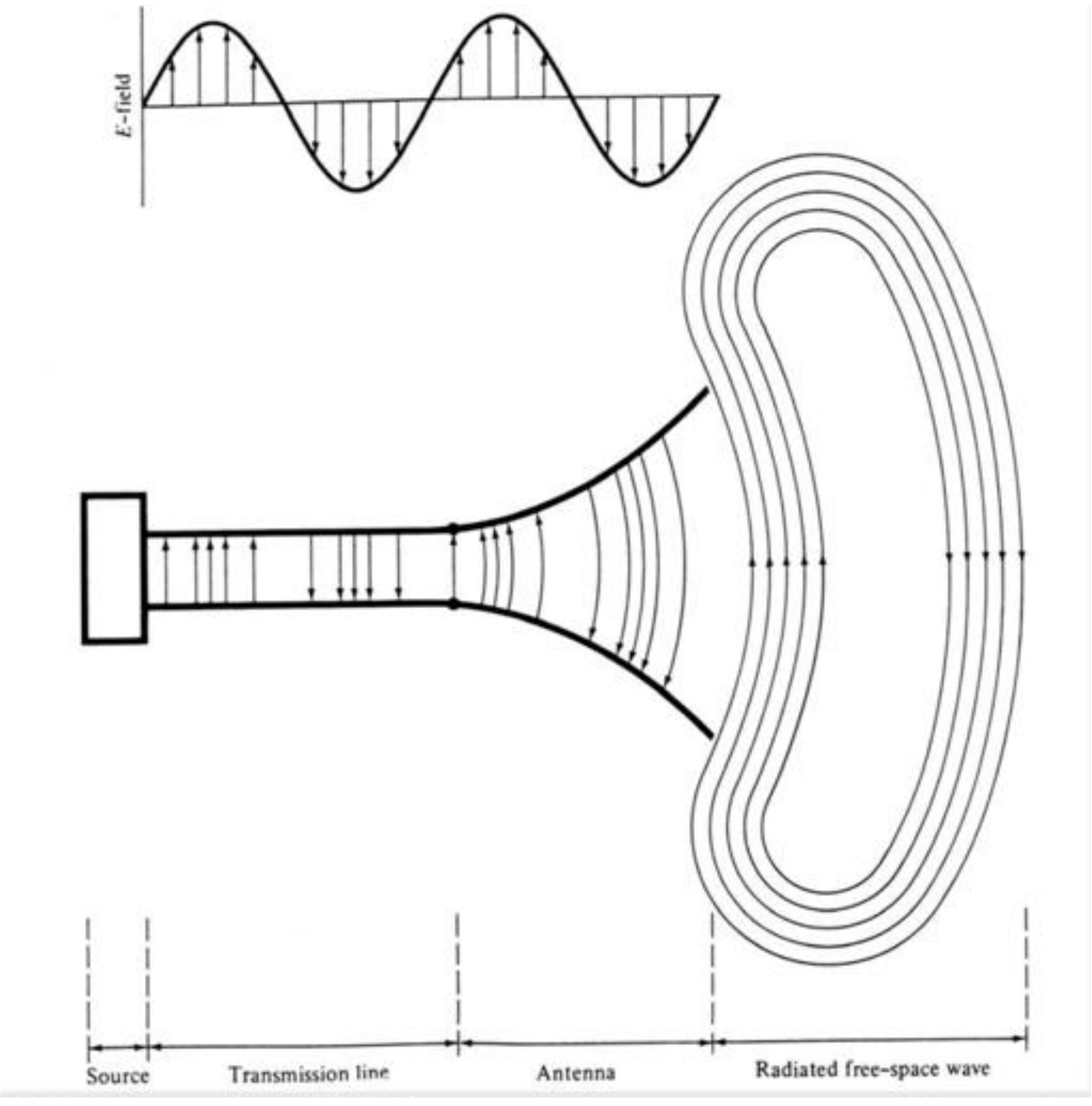
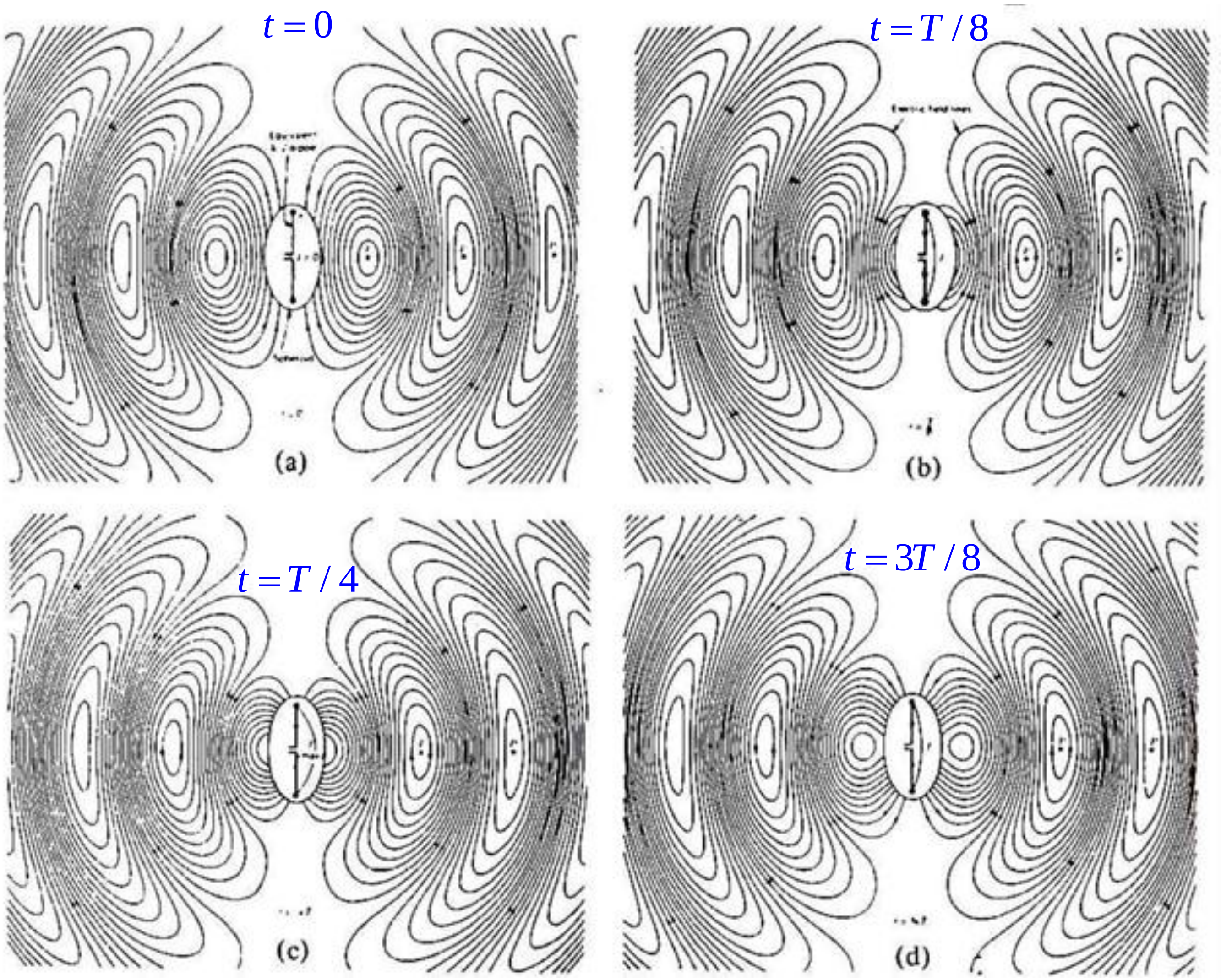


(e)

3.1. Intuitive representation of radiation from simple sources



(a) Illustration of the **E**-field lines in a transmission (feed) line and at the antenna aperture



(b) Snapshots of the **E**-field lines around a dipole

4. Radiation boundary condition

With few exceptions, antennas are assumed to radiate in open (free) space. This is a crucial factor determining the field behavior. Often, the EM sources (currents and charges on the antenna) are more or less accurately known. These sources are then assumed to radiate in unbounded space and it is required to determine the resulting EM field. Such problems, where the sources are known, and the reaction (result) is to be determined are called *analysis* (*forward*, *direct*) problems.[1] To ensure the uniqueness of the solution in an open (unbounded) problem, one has to impose the radiation boundary condition (RBC) on the EM field vectors, i.e., for distances far away from the source ($r \to \infty$),

$$\left| \begin{aligned} & r(\mathbf{E} - \eta \mathbf{H} \times \hat{\mathbf{r}}) \to 0, \\ & r(\mathbf{H} - \frac{1}{\eta} \hat{\mathbf{r}} \times \mathbf{E}) \to 0 \ . \end{aligned} \right. \tag{2.29}$$

The above RBC is known as the Sommerfeld RBC. Here, η is the intrinsic impedance of the medium; $\eta = \sqrt{\mu_0 / \varepsilon_0} \approx 377\ \Omega$ in vacuum.

The specifics of the antenna problems lead to the introduction of auxiliary vector potential functions, which allow simpler and compact solutions.

It is customary to perform the EM analysis for the case of time-harmonic fields, i.e., in terms of phasors. This course will adhere to the tradition. Therefore, from now on, all field quantities (vectors and scalars) are to be understood as ***complex phasor quantities, whose magnitudes correspond to the magnitudes of the respective sine waves***.

5. Vector and scalar potentials – review

In radiation theory, the potential functions are almost exclusively in the form of retarded potentials, i.e., the magnetic vector potential **A** and its scalar counterpart Φ form a 4-potential. We next introduce the retarded potentials.

5.1. The magnetic vector potential **A**

We first consider only electric sources (**J** and ρ, $\nabla \cdot \mathbf{J} = -j\omega\rho$).

$$\left| \begin{aligned} & \nabla \times \mathbf{E} = -j\omega\mu \mathbf{H}, \\ & \nabla \times \mathbf{H} = j\omega\varepsilon \mathbf{E} + \mathbf{J}. \end{aligned} \right. \tag{2.30}$$

Since $\nabla \cdot \mathbf{B} = 0$, we can assume that

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{2.31}$$

Substituting (2.31) in (2.30) yields

$$\left| \begin{aligned} & \mathbf{E} = -j\omega \mathbf{A} - \nabla \Phi, \\ & j\omega\varepsilon \mathbf{E} = \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) - \mathbf{J}. \end{aligned} \right. \tag{2.32}$$

From (2.32), a single equation can be written for **A**:

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[1] The inverse (*design*) problem of finding the sources of a known result (reaction) are much more difficult and we shall not consider them here.

$$\nabla\times\nabla\times\mathbf{A}+j\omega\mu\varepsilon(j\omega\mathbf{A}+\nabla\Phi)=\mu\mathbf{J}\,. \tag{2.33}$$

Here, Φ denotes the electric scalar potential, which plays an essential role in the analysis of electrostatic fields. To uniquely define the vector field **A**, we need to define not only its curl, but also its divergence. There are no restrictions in defining $\nabla\cdot\mathbf{A}$. Since $\nabla\times\nabla\times=\nabla\nabla\cdot-\nabla^2$, equation (2.33) can be simplified by assuming that

$$\nabla\cdot\mathbf{A}=-j\omega\mu\varepsilon\Phi\,. \tag{2.34}$$

Equation (2.34) is known as the Lorenz gauge. It reduces (2.33) to

$$\nabla^2\mathbf{A}+\omega^2\mu\varepsilon\mathbf{A}=-\mu\mathbf{J}\,. \tag{2.35}$$

If the region is lossless, then μ and ε are real numbers, and (2.35) can be written as

$$\nabla^2\mathbf{A}+\beta^2\mathbf{A}=-\mu\mathbf{J}\,, \tag{2.36}$$

where $\beta=\omega\sqrt{\mu\varepsilon}$ is the phase constant of the medium. If the region is lossy (which is rarely the case in antenna problems), complex permittivity $\bar{\varepsilon}$ and complex permeability $\bar{\mu}$ are introduced. Then, (2.35) becomes

$$\nabla^2\mathbf{A}-\gamma^2\mathbf{A}=-\mu\mathbf{J}\,. \tag{2.37}$$

Here, $\gamma=\alpha+j\beta=j\omega\sqrt{\bar{\mu}\bar{\varepsilon}}$ is the propagation constant, and α is the attenuation constant.

5.2. The electric vector potential F

The magnetic field is a solenoidal field, i.e., $\nabla\cdot\mathbf{B}=0$, because there are no physically existing magnetic charges. Therefore, there are no physically existing magnetic currents either. However, the fictitious (equivalent) magnetic currents **M** are a useful tool when applied with the equivalence principle. These currents are introduced in Maxwell's equations in a manner dual to that of the electric currents **J**. Now, we consider the field due to *magnetic sources only*, i.e., we set $\mathbf{J}=0$ and $\rho=0$, and therefore, $\nabla\cdot\mathbf{D}=0$. Then, the system of Maxwell's equations is

$$\left|\begin{aligned}&\nabla\times\mathbf{E}=-j\omega\mu\mathbf{H}-\mathbf{M},\\&\nabla\times\mathbf{H}=j\omega\varepsilon\mathbf{E}\,.\end{aligned}\right. \tag{2.38}$$

Since **D** is solenoidal, it can be expressed as the curl of a vector, namely the electric vector potential **F**:

$$\mathbf{D}=-\nabla\times\mathbf{F}\,. \tag{2.39}$$

Equation (2.39) is substituted into (2.38). All mathematical transformations are analogous to those made in Section 5.1. Finally, it is shown that a field due to magnetic sources is described by the vector **F** alone, where **F** satisfies

$$\nabla^2\mathbf{F} + \omega^2\mu\varepsilon\mathbf{F} = -\varepsilon\mathbf{M} \tag{2.40}$$

provided that the Lorenz gauge is imposed in the form

$$\nabla\cdot\mathbf{F} = -j\omega\mu\varepsilon\Psi\,. \tag{2.41}$$

Here, Ψ is the magnetic scalar potential.

In a linear medium, a field due to both types of sources (magnetic and electric) can be found by superimposing the partial field due to the electric sources only and the one due to the magnetic sources only.

TABLE 2.1: FIELD VECTORS IN TERMS OF VECTOR POTENTIALS

Magnetic vector-potential **A** (electric sources only)	Electric vector-potential **F** (magnetic sources only)
$\mathbf{B} = \nabla\times\mathbf{A},\ \mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A}$ $\mathbf{E} = -j\omega\mathbf{A} - \frac{j}{\omega\mu\varepsilon}\nabla\nabla\cdot\mathbf{A}$ or $\mathbf{E} = \frac{1}{j\omega\mu\varepsilon}\nabla\times\nabla\times\mathbf{A} - \frac{\mathbf{J}}{j\omega\varepsilon}$	$\mathbf{D} = -\nabla\times\mathbf{F},\ \mathbf{E} = -\frac{1}{\varepsilon}\nabla\times\mathbf{F}$ $\mathbf{H} = -j\omega\mathbf{F} - \frac{j}{\omega\mu\varepsilon}\nabla\nabla\cdot\mathbf{F}$ or $\mathbf{H} = \frac{1}{j\omega\mu\varepsilon}\nabla\times\nabla\times\mathbf{F} - \frac{\mathbf{M}}{j\omega\mu}$

6. Retarded potentials – review

Retarded potential is a term usually used to denote the solution of the inhomogeneous Helmholtz' equation (in the frequency domain) or that of the inhomogeneous wave equation (in the time domain) in an unbounded region.

Consider the *z*-directed electric current density $\mathbf{J} = \hat{\mathbf{z}}\ J_z$. Then, according to (2.36), the magnetic vector potential **A** also has only a *z*-component governed by the following equation in a lossless medium:

$$\nabla^2 A_z + \beta^2 A_z = -\mu J_z\,. \tag{2.42}$$

Eq. (2.42) is a Helmholtz equation and its solution in open space is determined by the integral

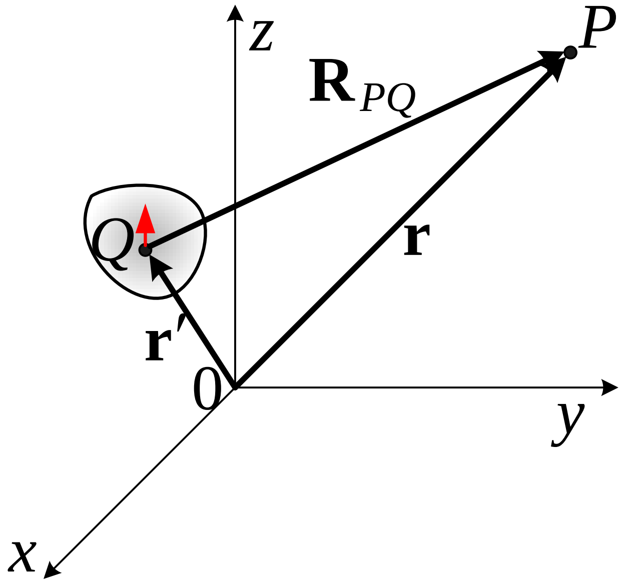
$$A_z(P) = \iiint_{v_Q} G(P,Q)\cdot\left[-\mu J_z(Q)\right]dv_Q \tag{2.43}$$

where $G(P,Q)$ is the open-space Green's function of the Helmholtz equation (see Appendix), *P* is the observation point, and *Q* is the source point.

Substituting (2.81) in (2.43) gives

$$A_z(P) = \iiint_{v_Q} \mu J_z(Q) \cdot \frac{e^{-j\beta R_{PQ}}}{4\pi R_{PQ}} dv_Q \,. \tag{2.44}$$

To further generalize the above formula, one should assume the existence of source currents of arbitrary directions, which would produce partial magnetic vector potentials in any direction. Note that a current element in the $\hat{\xi}$ direction results in a vector potential $\mathbf{A} = A_\xi \hat{\xi}$ in the same direction (unless the medium is inhomogeneous and/or anisotropic). Thus,



$$\boxed{\mathbf{A}(P) = \iiint_{v_Q} \mu \mathbf{J}(Q) \frac{e^{-j\beta R_{PQ}}}{4\pi R_{PQ}} dv_Q} \,. \tag{2.45}$$

The solution for the electric vector potential due to magnetic current sources $\mathbf{M}(Q)$ is analogous:

$$\boxed{\mathbf{F}(P) = \iiint_{v_Q} \varepsilon \mathbf{M}(Q) \frac{e^{-j\beta R_{PQ}}}{4\pi R_{PQ}} dv_Q} \,. \tag{2.46}$$

Finally, we recall that not only *volume* sources are used to model current distributions. A useful approximation, especially for currents on a conductor surface, is the *surface* current density (or simply surface current):

$$\mathbf{J}_s(x,y) = \lim_{\delta \to 0} \int_{-\delta/2}^{\delta/2} \mathbf{J}(x,y,z) dz \,, \text{ A/m.} \tag{2.47}$$

The magnetic vector potential **A** produced by distributed surface currents is then expressed as

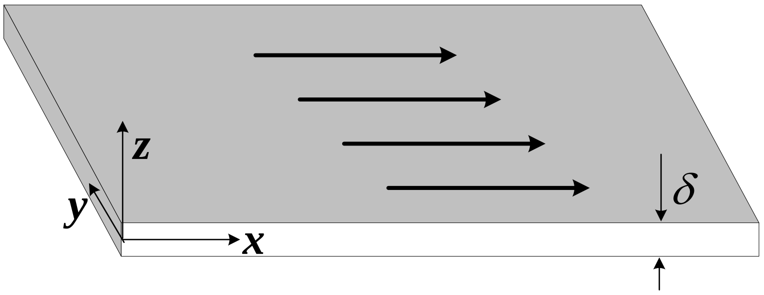
$$\mathbf{A}(P) = \iint_S \mu \mathbf{J}_s(Q) \frac{e^{-j\beta R_{PQ}}}{4\pi R_{PQ}} ds_Q . \tag{2.48}$$

Currents on a very thin wire are usually approximated by a linear source, which is the current I flowing through the wire:

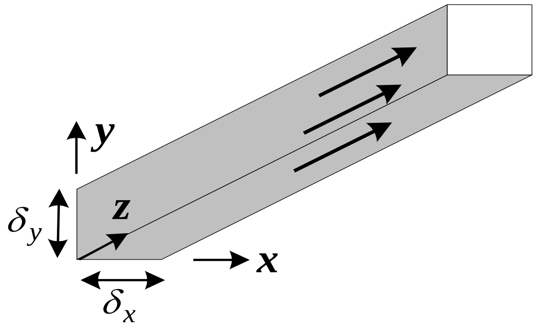
$$\mathbf{I}(z) = \lim_{\substack{\delta_x \to 0 \\ \delta_y \to 0}} \iint_{\delta_x \delta_y} \mathbf{J}(x,y,z) dxdy . \tag{2.49}$$

The potential of line currents is

$$\mathbf{A}(P) = \int_L \mu I(Q) \frac{e^{-j\beta R_{PQ}}}{4\pi R_{PQ}} d\mathbf{l}_Q . \tag{2.50}$$



(a) surface current on a sheet



(b) linear current on a thin wire

7. Far fields and vector potentials

7.1. Potentials

Antennas are sources of finite physical dimensions. The further away from the antenna the observation point is, the more the wave looks like a spherical wave and the more the antenna looks like a point source regardless of its actual shape. In such cases, we talk about *far field* and *far zone*. The exact meaning of these terms will be discussed later. For now, we will simply accept that the vector potentials behave like spherical waves, when the observation point is far from the source:

$$\mathbf{A} \simeq \left[\hat{\mathbf{r}} \cdot A_r(\theta,\varphi) + \hat{\boldsymbol{\theta}} \cdot A_\theta(\theta,\varphi) + \hat{\boldsymbol{\varphi}} \cdot A_\varphi(\theta,\varphi) \right] \frac{e^{-jkr}}{r}, \quad r \to \infty . \tag{2.51}$$

Here, $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}})$ are the unit vectors of the spherical coordinate system centered on the antenna and $k = \omega\sqrt{\mu_0 \varepsilon_0}$ is the wave number (the phase constant β in vacuum). The term e^{-jkr} shows propagation along $\hat{\mathbf{r}}$ away from the antenna at the speed of light. The term $1/r$ shows the spherical spread of the potential in space, which results in a decrease of its magnitude as the radius of the sphere increases.

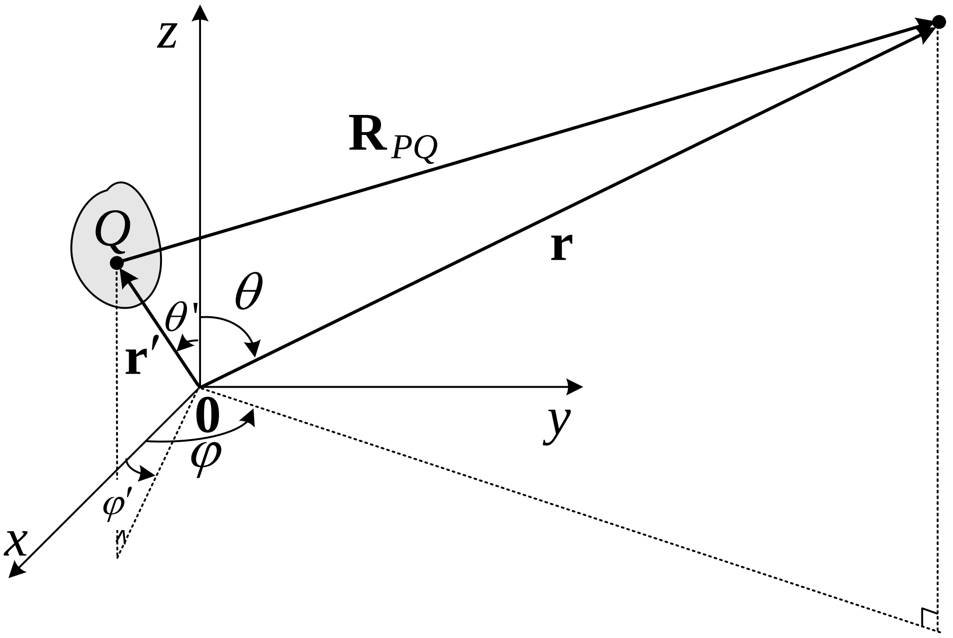
Notice an important feature of the far-field potential: the dependence on the distance r is separable from the dependence on the observation angle (θ,φ), and it is the same for any antenna: e^{-jkr}/r.

Formula (2.51) is a *far-field approximation* of the vector potential at distant points. We arrive at it starting from the integral in (2.45). When the observation point P is very far from the source, the distance R_{PQ} between P and the integration points varies only slightly as Q sweeps the volume of the source. It is almost the same as the distance r from the origin to P since we usually position the origin of the coordinate system close to the source center. The following first-order approximation (attributed to Kirchhoff) is made for the integrand:

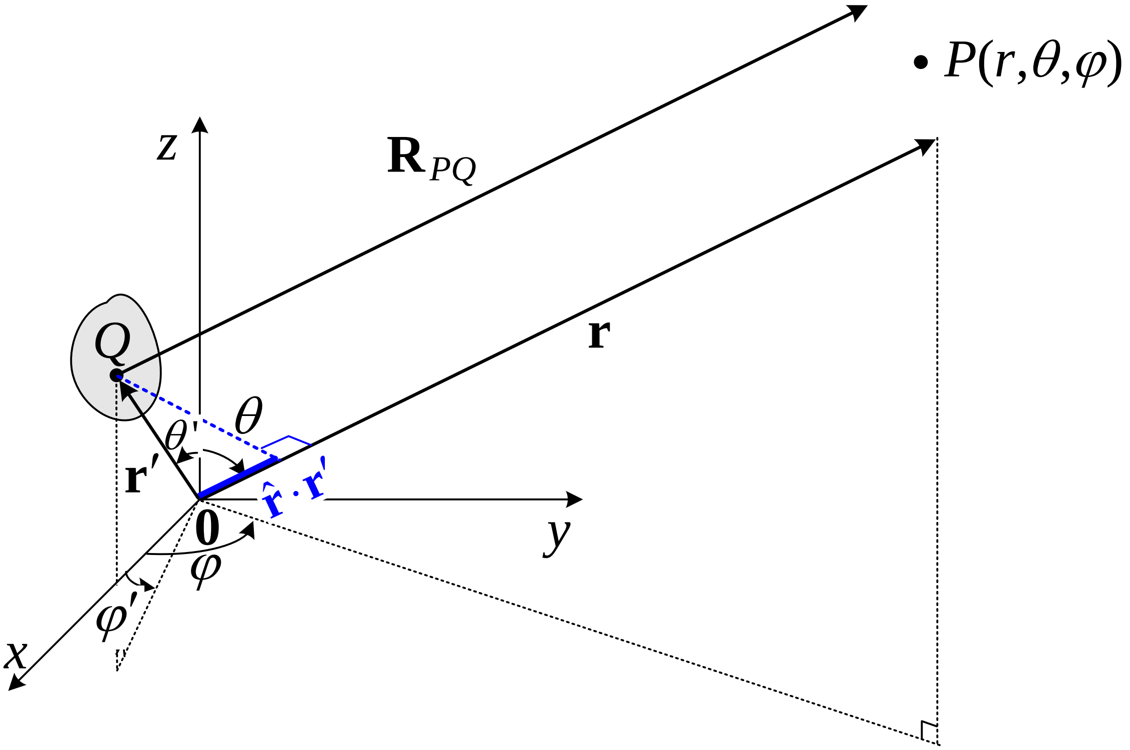
$$\frac{e^{-jkR_{PQ}}}{R_{PQ}} \approx \frac{e^{-jk(r-\hat{\mathbf{r}}\cdot\mathbf{r}')}}{r}. \tag{2.52}$$

Here, $\mathbf{r}$ is the position vector of the observation point P and $r=|\mathbf{r}|$ is its length. Its direction is given by the unit vector $\hat{\mathbf{r}}$, so that $\mathbf{r}=\hat{\mathbf{r}}\cdot r$. The position vector of the integration point Q is $\mathbf{r}'$. Equation (2.52) is called the *far-field approximation*.

The approximation in the phase term is illustrated in the figures below. The first figure shows the real problem. The second one shows the approximated problem, where it is in effect assumed that the vectors $\mathbf{R}_{PQ}$ and $\mathbf{r}$ are parallel.



(a) original problem



(b) far-field approximation of the original problem

We now apply the far-field approximation to the vector potential in (2.45):

$$\mathbf{A}(P) = \mu \frac{e^{-jkr}}{4\pi r} \iiint\limits_{v_Q} \mathbf{J}(Q) e^{jk\hat{\mathbf{r}}\cdot\mathbf{r}'} dv_Q . \tag{2.53}$$

The integrand in (2.53) does not depend on the distance between source and observation point. It depends only on the current distribution in the source volume and the angle between the position vector of the integration point $\mathbf{r}'$ and the unit position vector of the observation point $\hat{\mathbf{r}}$. This finally explains the general equation for the *far-field vector potential* in (2.51).

7.2. Far-zone field

The far-field approximation of the vector potential leads to much simpler equations for the far-field vectors. Assume that there are only electrical currents. Then the EM field is fully described only by the magnetic vector potential **A**. We have to substitute (2.51) into the equations of Table 2.1, where $\mathbf{F} = 0$:

$$\mathbf{E} = -j\omega\mathbf{A} - \frac{j}{\omega\mu\varepsilon} \nabla\nabla\cdot\mathbf{A}, \tag{2.54}$$

$$\mathbf{H} = \frac{1}{\mu} \nabla\times\mathbf{A} \ . \tag{2.55}$$

The differential operators $\nabla\times$ and $\nabla\nabla\cdot$ have to be expressed in spherical coordinates. All terms decreasing with the distance as $1/r^2$ and faster are neglected. What remains is

$$\mathbf{E} = \frac{1}{r}\left\{ -j\omega e^{-jkr} \left[\hat{\boldsymbol{\theta}} A_\theta(\theta,\varphi) + \hat{\boldsymbol{\varphi}} A_\varphi(\theta,\varphi) \right] \right\} + \frac{1}{r^2}\{ \ \} + \cdots, \quad r\to\infty, \tag{2.56}$$

$$\mathbf{H} = \frac{1}{r}\left\{ j\frac{\omega}{\eta} e^{-jkr} \left[\hat{\boldsymbol{\theta}} A_\varphi(\theta,\varphi) - \hat{\boldsymbol{\varphi}} A_\theta(\theta,\varphi) \right] \right\} + \frac{1}{r^2}\{ \ \} + \cdots, \quad r\to\infty. \tag{2.57}$$

Here, $\eta = \sqrt{\mu/\varepsilon}$ denotes the intrinsic impedance of the medium. We write the far-field equations (2.56) and (2.57) in a more compact way as

$$\left.\begin{aligned} E_r &\simeq 0 \\ E_\theta &\simeq -j\omega A_\theta \\ E_\varphi &\simeq -j\omega A_\varphi \end{aligned}\right\} \Rightarrow \mathbf{E}^A \simeq -j\omega\mathbf{A}, \ \text{ where } E_r^A \simeq 0, \tag{2.58}$$

$$\left.\begin{array}{l} H_r \simeq 0 \\ H_\theta \simeq +j\dfrac{\omega}{\eta}A_\varphi = -\dfrac{E_\varphi}{\eta} \\ H_\varphi \simeq -j\dfrac{\omega}{\eta}A_\theta = +\dfrac{E_\theta}{\eta} \end{array}\right\} \Rightarrow \mathbf{H}^A \simeq -j\frac{\omega}{\eta}\hat{\mathbf{r}}\times\mathbf{A} = \frac{1}{\eta}\hat{\mathbf{r}}\times\mathbf{E}^A. \qquad (2.59)$$

In an analogous manner, we obtain the relations between the field vectors and the electric vector potential **F**, when only magnetic sources are present:

$$\left.\begin{array}{l} H_r \simeq 0 \\ H_\theta \simeq -j\omega F_\theta \\ H_\varphi \simeq -j\omega F_\varphi \end{array}\right\} \Rightarrow \mathbf{H}^F \simeq -j\omega\mathbf{F},\ \ H_r^F \simeq 0, \qquad (2.60)$$

$$\left.\begin{array}{l} E_r \simeq 0 \\ E_\theta \simeq -j\omega\eta F_\varphi = \eta H_\varphi \\ E_\varphi \simeq +j\omega\eta F_\theta = -\eta H_\theta \end{array}\right\} \Rightarrow \mathbf{E}^F \simeq j\omega\eta\hat{\mathbf{r}}\times\mathbf{F} = -\eta\hat{\mathbf{r}}\times\mathbf{H}^F. \qquad (2.61)$$

In summary, the far field of any antenna has the following important features, which follow from equations (2.58) through (2.61):

- The far field has no radial components, $E_r = H_r = 0$. Since the radial direction is also the direction of propagation, the far field is a typical TEM (Transverse Electro-Magnetic) wave.
- The **E** vector and the **H** vector are mutually orthogonal, both of them being also orthogonal to the direction of propagation.
- The magnitudes of the electric field and the magnetic field are related always as $|\mathbf{E}| = \eta|\mathbf{H}|$.

APPENDIX I

Green's Function for the Helmholtz' Equation

Suppose the following PDE must be solved:

$$L\Phi(\mathbf{x}) = f(\mathbf{x}) \tag{2.62}$$

where **x** denotes the set of variables, e.g., $\mathbf{x} = (x, y, z)$. Suppose also that a Green's function exists such that it allows for the integral solution

$$\Phi(\mathbf{x}) = \iiint_{v'} G(\mathbf{x}, \mathbf{x}') \cdot f(\mathbf{x}') dv' . \tag{2.63}$$

Applying the operator *L* to both sides of (2.63), leads to

$$L\Phi(\mathbf{x}) = \iiint_{v'} \left[LG(\mathbf{x}, \mathbf{x}') \right] \cdot f(\mathbf{x}') dv' = f(\mathbf{x}) . \tag{2.64}$$

From (2.64), we conclude that the Green's function must satisfy the same PDE as Φ with a point source described by Dirac's delta function:

$$LG(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') . \tag{2.65}$$

Here, $\delta(\mathbf{x} - \mathbf{x}')$ is Dirac's delta function in 3-D space, e.g., $\delta(\mathbf{x} - \mathbf{x}') = \delta(x - x')\delta(y - y')\delta(z - z')$. If the Green's function of a problem is known, the construction of an integral solution is possible via (2.63).

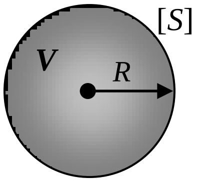
Consider the Green's function for the Helmholtz equation in open space. It must satisfy

$$\nabla^2 G + \beta^2 G = \delta(x)\delta(y)\delta(z) \tag{2.66}$$

together with the scalar radiation condition

$$\lim_{r \to \infty} r \cdot \left(\frac{\partial G}{\partial r} + j\beta G \right) = 0 \tag{2.67}$$

if the source is centered at the origin of the coordinate system, i.e., $x' = y' = z' = 0$. Integrate (2.66) within a sphere with its center at (0,0,0) and a radius *R*:



$$\iiint_v \nabla^2 G dv + \iiint_v \beta^2 G dv = 1 \tag{2.68}$$

The function *G* is due to a point source and thus has a spherical symmetry, i.e., it depends on *r* only. The Laplacian ∇^2 in spherical coordinates is reduced to derivatives with respect to *r* only:

$$\frac{d^2G}{dr^2} + \frac{2}{r}\frac{dG}{dr} + \beta^2 G = \delta(x)\delta(y)\delta(z) . \tag{2.69}$$

Everywhere except at the point (x, y, z), *G* must satisfy the homogeneous equation

$$\frac{d^2G}{dr^2} + \frac{2}{r}\frac{dG}{dr} + \beta^2 G = 0 \tag{2.70}$$

whose solution for outgoing waves is well known:

$$G(r) = C \frac{e^{-jkr}}{r} . \tag{2.71}$$

Here, *C* is a constant to be determined. Consider first the integral from (2.68):

$$I_1 = \iiint_v \beta^2 G dv . \tag{2.72}$$

$$\Rightarrow I_1 = \iiint_v \beta^2 C \frac{e^{-j\beta r}}{r} dv = \int_0^R \int_0^{2\pi} \int_0^{\pi} \beta^2 C \frac{e^{-j\beta r}}{r} r^2 \sin\theta d\theta d\varphi dr \tag{2.73}$$

$$\Rightarrow I_1(R) = j4\pi\beta C \left(R \cdot e^{-j\beta R} + \frac{e^{-j\beta R}}{j\beta} - \frac{1}{j\beta} \right). \tag{2.74}$$

To evaluate the integral in the point of singularity (0,0,0), we let $R \to 0$, i.e., we let the sphere collapse into a point. We see that

$$\lim_{R\to 0} I_1(R) = 0\,. \tag{2.75}$$

Secondly, consider the other integral in (2.68),

$$I_2 = \iiint_V \nabla^2 G dv = \iiint_V \nabla \cdot \left(\nabla G \right) dv = \oiint_S \nabla G \cdot d\mathbf{s} \tag{2.76}$$

Here, $d\mathbf{s} = R^2 \sin\theta dr d\theta d\varphi \cdot \hat{\mathbf{r}}$ is a surface element on *S*, and

$$\nabla G = \frac{\partial G}{\partial r} \hat{r} = -C \left(jk \frac{e^{-jkr}}{r} + \frac{e^{-jkr}}{r^2} \right) \hat{r} \tag{2.77}$$

$$\Rightarrow I_2(R) = -C \left(jkR \cdot e^{-jkR} + e^{-jkR} \right) \int_0^{\pi} \int_0^{2\pi} \sin\theta d\varphi d\theta \tag{2.78}$$

$$\lim_{R\to 0} I_2(R) = -4\pi C\,. \tag{2.79}$$

Substituting (2.79) and (2.75) into (2.68) and taking $\lim_{R\to 0}$, yields

$$C = -\frac{1}{4\pi}\,. \tag{2.80}$$

Finally,

$$G(r) = -\frac{e^{-jkr}}{4\pi r}\,. \tag{2.81}$$

It is not difficult to show that in the general case when the source is at a point $Q(x', y', z')$,

$$\nabla^2 G + \beta^2 G = \delta(x - x')\delta(y - y')\delta(z - z') \tag{2.82}$$

then

$$G(P,Q) = -\frac{e^{-jkR_{PQ}}}{4\pi R_{PQ}} \tag{2.83}$$

where R_{PQ} is the distance between the observation point *P* and the source point *Q*,

$$R_{PQ} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}\,. \tag{2.84}$$