

LECTURE 17: Radiation from Apertures

(The uniqueness theorem. The equivalence principle. The application of the equivalence principle to aperture problem. The uniform rectangular aperture. The tapered rectangular aperture.)

1. Introduction

Aperture antennas constitute a large class of antennas, which emit EM waves through an opening (or aperture). These antennas have close analogs in acoustics: the megaphone and the parabolic microphone. The pupil of the human eye is a typical aperture receiver for optical radiation. At radio and microwave frequencies, horns, waveguide apertures, reflectors and microstrip patches are examples of aperture antennas. Aperture antennas are commonly used at UHF and above. It is because their gain increases as $\sim f^2$. For an aperture antenna to be efficient and have high directivity, it has to have an area comparable or larger than λ^2 . Thus, these antennas are impractical at low frequencies.

To facilitate the analysis of these antennas, the equivalence principle is applied. This allows carrying out the far-field analysis in the outer (unbounded) region only, which is external to the radiating aperture and the antenna. This requires the knowledge of the tangential field components at the aperture.

2. Uniqueness theorem

A solution is said to be unique if it is the only one possible among a given class of solutions.

The EM field in a given region $V_{[S]}$ is unique if

- all sources are given;
- either the tangential \mathbf{E}_{tan} components or the tangential \mathbf{H}_{tan} components are specified at the boundary S .

The uniqueness theorem is proven by making use of the Poynting theorem in integral form:

$$\oiint_S (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{s} + j\omega \iiint_{V_S} (\mu |\mathbf{H}|^2 - \varepsilon |\mathbf{E}|^2) dv + \iiint_{V_S} \sigma |\mathbf{E}|^2 dv = - \iiint_{V_S} (\mathbf{E} \cdot \mathbf{J}^* + \mathbf{H}^* \cdot \mathbf{M}^i) dv. \quad (17.1)$$

We start with the supposition that a given EM problem has two solutions (due to the same sources and the same boundary conditions): $(\mathbf{E}^a, \mathbf{H}^a)$ and $(\mathbf{E}^b, \mathbf{H}^b)$. The difference field is then formed:

$$\begin{cases} \delta\mathbf{E} = \mathbf{E}^a - \mathbf{E}^b, \\ \delta\mathbf{H} = \mathbf{H}^a - \mathbf{H}^b. \end{cases} \quad (17.2)$$

Since the difference field has no sources, it satisfies the source-free form of (17.1):

$$\oint_S (\delta\mathbf{E} \times \delta\mathbf{H}^*) \cdot d\mathbf{s} + j\omega \iiint_{V_S} (\mu |\delta\mathbf{H}|^2 - \varepsilon |\delta\mathbf{E}|^2) dv + \iiint_{V_S} \sigma |\delta\mathbf{E}|^2 dv = 0. \quad (17.3)$$

Since both fields satisfy the same boundary conditions, then over S $\delta\mathbf{E}_{\text{tan}} = 0$ or $\delta\mathbf{H}_{\text{tan}} = 0$ which makes the surface integral in (17.3) zero. This results in

$$j\omega \iiint_{V_S} (\mu |\delta\mathbf{H}|^2 - \varepsilon |\delta\mathbf{E}|^2) dv + \iiint_{V_S} \sigma |\delta\mathbf{E}|^2 dv = 0, \quad (17.4)$$

which is true only if

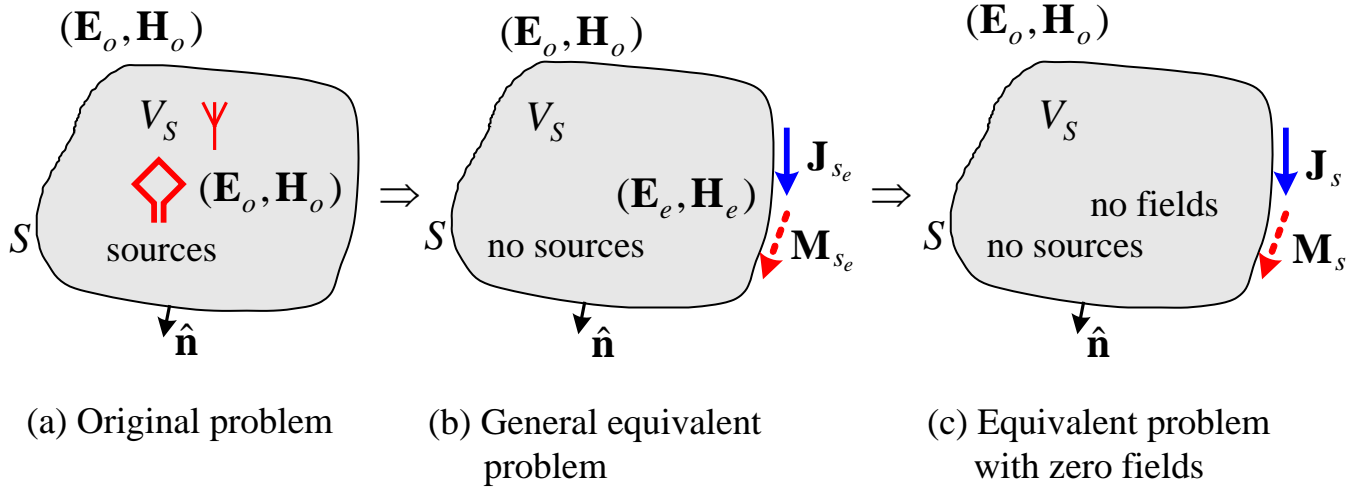
$$\begin{cases} \omega \iiint_{V_S} (\mu |\delta\mathbf{H}|^2 - \varepsilon |\delta\mathbf{E}|^2) dv = 0, \\ \iiint_{V_S} \sigma |\delta\mathbf{E}|^2 dv = 0. \end{cases} \quad (17.5)$$

If we assume some dissipation, however slight, equations (17.5) are satisfied only if $\delta\mathbf{E} = \delta\mathbf{H} = 0$ everywhere in the volume V_S . This implies the uniqueness of the solution. If $\sigma = 0$, which is a common approximation, multiple solutions $(\delta\mathbf{E}, \delta\mathbf{H})$ may exist in the form of self-resonant modes of the structure. However, these resonant modes can be derived using eigenvalue analysis and they are not considered as the particular solution for the given sources. The particular unique solution for the loss-free case can be obtained from a problem where σ is assumed nonzero and then the limit is found as $\sigma \rightarrow 0$.

Note that the surface integral in (17.3) is zero if any one of the following conditions is true on any part of the boundary: (1) $\delta\mathbf{E}_{\text{tan}} = 0$, or (2) $\delta\mathbf{H}_{\text{tan}} = 0$, or (3) $\delta E_{\text{tan1}} = \delta H_{\text{tan1}} = 0$, or (4) $\delta E_{\text{tan2}} = \delta H_{\text{tan2}} = 0$. Here, $\delta\mathbf{E}_{\text{tan}} = \delta\mathbf{E} - \delta\mathbf{E} \cdot \hat{\mathbf{n}}$ is the tangential component of $\delta\mathbf{E}$ at the surface S while δE_{tan1} and δE_{tan2} are its components. The same notations hold for $\delta\mathbf{H}$, too.

3. Equivalence principles

The equivalence principle follows from the uniqueness theorem. It allows for the simplification of certain EM problems. As long as a problem preserves the boundary conditions of a given (original) problem for the field at S , it is going to produce the only one possible solution for the region outside V_S . Such a problem is referred to as an equivalent problem.



For the equivalent problem in (b),

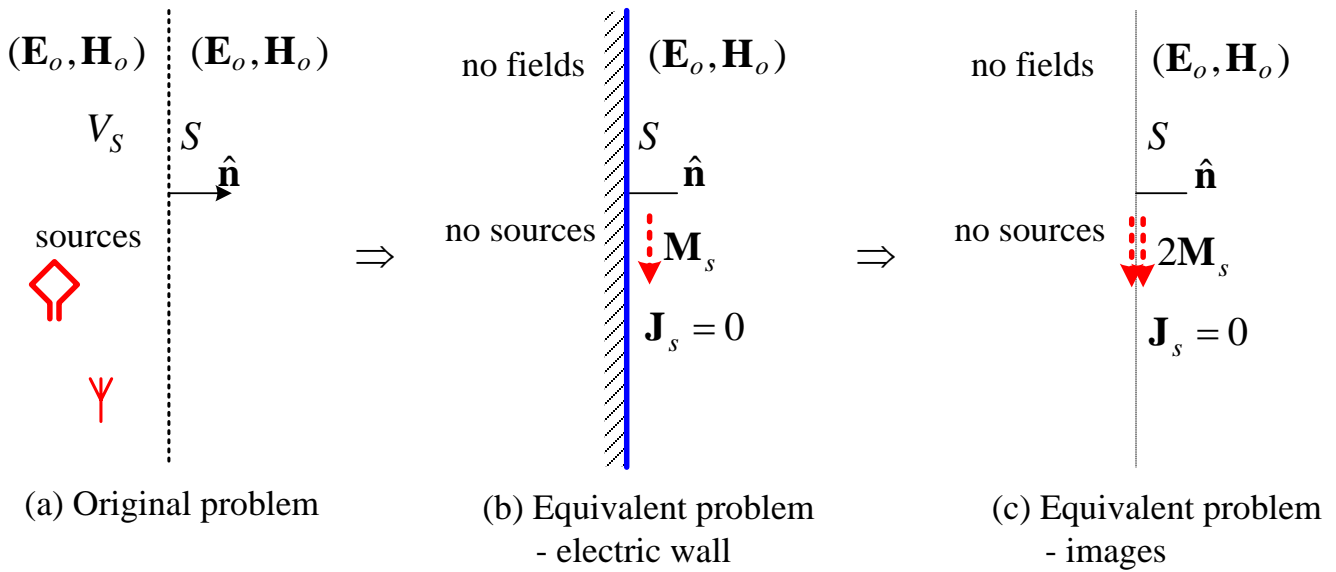
$$\begin{cases} \mathbf{J}_{s_e} = \hat{\mathbf{n}} \times (\mathbf{H}_o - \mathbf{H}_e), \\ \mathbf{M}_{s_e} = (\mathbf{E}_o - \mathbf{E}_e) \times \hat{\mathbf{n}}. \end{cases} \quad (17.6)$$

For the equivalent problem (c),

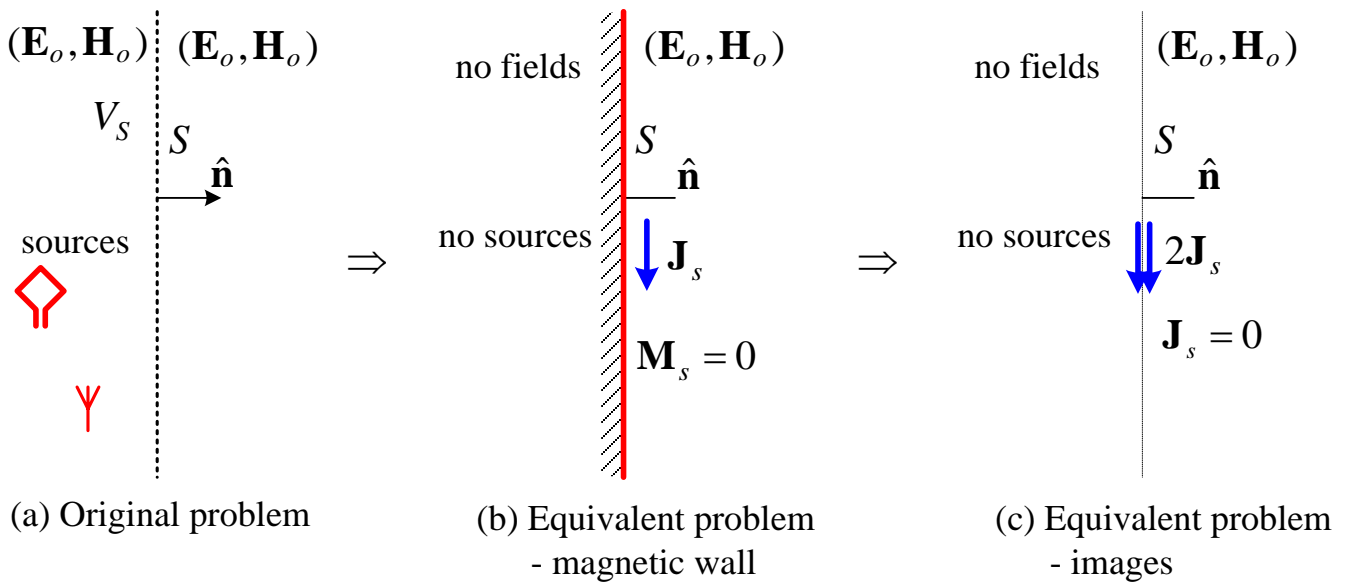
$$\begin{cases} \mathbf{J}_s = \hat{\mathbf{n}} \times \mathbf{H}_o, \\ \mathbf{M}_s = \mathbf{E}_o \times \hat{\mathbf{n}}. \end{cases} \quad (17.7)$$

The zero-field formulation is often referred to as *Love's equivalence principle*. We can apply Love's equivalence principle in three different ways.

- (a) We can assume that the boundary S is a perfect conductor. As per image theory, in an equivalent open problem, this eliminates the surface electric currents, i.e., $\mathbf{J}_s = 0$, and leaves just surface magnetic currents of double strength $2\mathbf{M}_s$. Such an equivalent problem is illustrated below.



(b) We can assume that the boundary S is a perfect magnetic conductor. As per image theory, in an equivalent open problem, this eliminates the surface magnetic currents, i.e., $\mathbf{M}_s = 0$, and leaves just surface electric currents of double strength $2\mathbf{J}_s$. This approach is illustrated below.



(c) Make no assumptions about the materials inside S , and define both \mathbf{J}_s and \mathbf{M}_s currents, which radiate in free space (no fictitious conductors behind them). It can be shown that these equivalent currents create zero fields inside V_S . [Ewald-Oseen extinction theorem: A. Ishimaru, *Electromagnetic Wave Propagation, Radiation, and Scattering*, Prentice Hall, 1991, p. 173]

The first two approaches are not very accurate in the general case of a curved boundary surface S . However, in the case of flat infinite planes (walls), the image theory holds exactly and all three approaches should produce the same external field according to the uniqueness theorem. The image theory can be applied to curved surfaces provided the curvature radius is large compared to the wavelength.

The above approaches are used to compute fields in half-space as excited by apertures. The field behind S is assumed known and are used to define the equivalent surface currents. The open-region far-zone solutions for the vector potentials \mathbf{A} (resulting from \mathbf{J}_s) and \mathbf{F} (resulting from \mathbf{M}_s) are found from

$$\mathbf{A}(P) = \mu \frac{e^{-j\beta r}}{4\pi r} \iint_S \mathbf{J}_s(\mathbf{r}') e^{j\beta \hat{\mathbf{r}} \cdot \mathbf{r}'} ds', \quad (17.8)$$

$$\mathbf{F}(P) = \varepsilon \frac{e^{-j\beta r}}{4\pi r} \iint_S \mathbf{M}_s(\mathbf{r}') e^{j\beta \hat{\mathbf{r}} \cdot \mathbf{r}'} ds'. \quad (17.9)$$

Here, $\hat{\mathbf{r}}$ denotes the unit vector pointing from the origin of the coordinate system to the point of observation P . The integration point Q is specified through the position vector \mathbf{r}' . In the far zone, it is assumed that the field propagates radially away from the antenna. It is convenient to introduce the *propagation vector* or *wave vector*,

$$\boldsymbol{\beta} = \beta \hat{\mathbf{r}}, \quad (17.10)$$

which characterizes both the phase constant and the direction of propagation of the wave. The vector potentials can then be written as

$$\mathbf{A}(P) = \mu \frac{e^{-j\beta r}}{4\pi r} \iint_S \mathbf{J}_s(\mathbf{r}') e^{j\boldsymbol{\beta} \cdot \mathbf{r}'} ds', \quad (17.11)$$

$$\mathbf{F}(P) = \varepsilon \frac{e^{-j\beta r}}{4\pi r} \iint_S \mathbf{M}_s(\mathbf{r}') e^{j\boldsymbol{\beta} \cdot \mathbf{r}'} ds'. \quad (17.12)$$

The relations between the far-zone fields and vector potentials are

$$\mathbf{E}_A^{far} = -j\omega(A_\theta \hat{\boldsymbol{\theta}} + A_\varphi \hat{\boldsymbol{\phi}}), \quad (17.13)$$

$$\mathbf{H}_F^{far} = -j\omega(F_\theta \hat{\boldsymbol{\theta}} + F_\varphi \hat{\boldsymbol{\phi}}). \quad (17.14)$$

Since

$$\mathbf{E}_F^{far} = \eta \mathbf{H}_F^{far} \times \hat{\mathbf{r}}, \quad (17.15)$$

the total far-zone electric field (due to both \mathbf{A} and \mathbf{F}) is found as

$$\mathbf{E}^{far} = \mathbf{E}_A^{far} + \mathbf{E}_F^{far} = -j\omega \left[(A_\theta + \eta F_\varphi) \hat{\boldsymbol{\theta}} + (A_\varphi - \eta F_\theta) \hat{\boldsymbol{\phi}} \right]. \quad (17.16)$$

Equation (17.16) involves both vector potentials as arising from both types of surface currents. Computations are reduced in half if image theory is used in conjunction with an electric or magnetic wall assumption.

4. Application of the equivalence principle to aperture problems

The equivalence principle is widely used in the analysis of aperture antennas. To calculate exactly the far field, the exact field distribution at the (infinite) aperture is needed. In the case of exact knowledge of the aperture field distribution, all three approaches given above produce the same results. However, the aperture field distribution is usually not known exactly and approximations are used. Then, the three equivalence-principle approaches produce slightly different results, the consistency being dependent on how accurate our knowledge about the aperture field is. Usually, it is assumed that the field is to be determined in half-space, leaving the feed and the antenna behind an infinite wall S (electric or magnetic). The aperture of the antenna S_A is this portion of S where we have an approximate knowledge of the field distribution based on the type of the feed line or the incident wave illuminating the aperture. This is the so-called *physical optics* approximation, which is more accurate than the *geometrical optics* approach of ray tracing. The larger the aperture (as compared to the wavelength), the more accurate the approximation based on the incident wave.

Let us assume that the field at the aperture S_A is known: $\mathbf{E}_a, \mathbf{H}_a$, and it is zero everywhere on S except at S_A . The equivalent current densities are:

$$\begin{cases} \mathbf{J}_s = \hat{\mathbf{n}} \times \mathbf{H}_a, \\ \mathbf{M}_s = \mathbf{E}_a \times \hat{\mathbf{n}}. \end{cases} \quad (17.17)$$

The substitution of (17.17) into (17.11) and (17.12) produces

$$\mathbf{A}(P) = \mu \frac{e^{-j\beta r}}{4\pi r} \hat{\mathbf{n}} \times \iint_{S_A} \mathbf{H}_a e^{j\beta \cdot \mathbf{r}'} ds', \quad (17.18)$$

$$\mathbf{F}(P) = -\varepsilon \frac{e^{-j\beta r}}{4\pi r} \hat{\mathbf{n}} \times \iint_{S_A} \mathbf{E}_a e^{j\beta \cdot \mathbf{r}'} ds'. \quad (17.19)$$

The radiation integrals in (17.18) and (17.19) are denoted as

$$\mathbf{I}^H = \iint_S \mathbf{H}_a e^{j\beta \cdot \mathbf{r}'} ds', \quad (17.20)$$

$$\mathbf{I}^E = \iint_S \mathbf{E}_a e^{j\beta \cdot \mathbf{r}'} ds', \quad (17.21)$$

for brevity.

We can find the general vector expression for the far field \mathbf{E} making use of (17.16) written as

$$\mathbf{E}^{far} = -j\omega\mathbf{A} - j\omega\eta\mathbf{F} \times \hat{\mathbf{r}}, \quad (17.22)$$

where *the longitudinal A_r component is to be neglected*. Substituting (17.18) and (17.19) into (17.22) yields

$$\mathbf{E}^{far} = -j\beta \frac{e^{-j\beta r}}{4\pi r} \hat{\mathbf{r}} \times \iint_{S_A} [\hat{\mathbf{n}} \times \mathbf{E}_a - \eta \hat{\mathbf{r}} \times (\hat{\mathbf{n}} \times \mathbf{H}_a)] e^{j\beta \cdot \mathbf{r}'} ds'. \quad (17.23)$$

This is the full vector form of the radiated field resulting from the aperture field, and it is referred to as the *vector diffraction integral* (or *vector Kirchhoff integral*).

We now consider a practical case of a flat aperture lying in the xy plane with $\hat{\mathbf{n}} \equiv \hat{\mathbf{z}}$. Then,

$$\mathbf{A} = \mu \frac{e^{-j\beta r}}{4\pi r} (-I_y^H \hat{\mathbf{x}} + I_x^H \hat{\mathbf{y}}), \quad (17.24)$$

$$\mathbf{F} = -\varepsilon \frac{e^{-j\beta r}}{4\pi r} (-I_y^E \hat{\mathbf{x}} + I_x^E \hat{\mathbf{y}}). \quad (17.25)$$

The integrals in the above expressions can be explicitly written for this case, in which $\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}$:

$$I_x^E = \iint_{S_A} E_{a_x}(x', y') e^{j\beta(x' \sin \theta \cos \varphi + y' \sin \theta \sin \varphi)} dx' dy', \quad (17.26)$$

$$I_y^E = \iint_{S_A} E_{a_y}(x', y') e^{j\beta(x' \sin \theta \cos \varphi + y' \sin \theta \sin \varphi)} dx' dy', \quad (17.27)$$

$$I_x^H = \iint_{S_A} H_{a_x}(x', y') e^{j\beta(x' \sin \theta \cos \varphi + y' \sin \theta \sin \varphi)} dx' dy', \quad (17.28)$$

$$I_y^H = \iint_{S_A} H_{a_y}(x', y') e^{j\beta(x' \sin \theta \cos \varphi + y' \sin \theta \sin \varphi)} dx' dy'. \quad (17.29)$$

Note that the above integrals are the double inverse Fourier transforms of the aperture field components where x transforms into $\beta_x = \beta \sin \theta \cos \varphi$ and y transforms into $\beta_y = \beta \sin \theta \sin \varphi$.

The transverse components of the magnetic vector potential in spherical terms are obtained from (17.24) as

$$A_\theta = \mu \frac{e^{-j\beta r}}{4\pi r} (-I_y^H \cdot \cos \theta \cos \varphi + I_x^H \cdot \cos \theta \sin \varphi), \quad (17.30)$$

$$A_\varphi = \mu \frac{e^{-j\beta r}}{4\pi r} (I_y^H \cdot \sin \varphi + I_x^H \cdot \cos \varphi), \quad (17.31)$$

which can also be written in the vector form

$$\mathbf{A}_\perp = \mu \frac{e^{-j\beta r}}{4\pi r} \left[\hat{\boldsymbol{\theta}} \cos \theta (I_x^H \sin \varphi - I_y^H \cos \varphi) + \hat{\boldsymbol{\phi}} (I_x^H \cos \varphi + I_y^H \sin \varphi) \right]. \quad (17.32)$$

Analogously,

$$\mathbf{F}_\perp = -\varepsilon \frac{e^{-j\beta r}}{4\pi r} \left[\hat{\boldsymbol{\theta}} \cos \theta (I_x^E \sin \varphi - I_y^E \cos \varphi) + \hat{\boldsymbol{\phi}} (I_x^E \cos \varphi + I_y^E \sin \varphi) \right]. \quad (17.33)$$

By substituting the above expressions in (17.16), we obtain the far \mathbf{E} field as

$$E_\theta = j\beta \frac{e^{-j\beta r}}{4\pi r} \left[I_x^E \cos \varphi + I_y^E \sin \varphi + \eta \cos \theta \cdot (I_y^H \cos \varphi - I_x^H \sin \varphi) \right], \quad (17.34)$$

$$E_\varphi = j\beta \frac{e^{-j\beta r}}{4\pi r} \left[-\eta (I_x^H \cos \varphi + I_y^H \sin \varphi) + \cos \theta \cdot (I_y^E \cos \varphi - I_x^E \sin \varphi) \right]. \quad (17.35)$$

For apertures mounted on a conducting plane, the preferred equivalent model is the one using an electric wall with doubled magnetic current density

$$\mathbf{M}_s = 2 \cdot (\mathbf{E}_a \times \hat{\mathbf{n}}) \quad (17.36)$$

radiating in open space. The solution, of course, is valid only for $z \geq 0$. In this case, $\mathbf{I}^H = 0$ and the field is given by

$$E_\theta = j\beta \frac{e^{-j\beta r}}{4\pi r} (I_x^E \cos \varphi + I_y^E \sin \varphi), \quad (17.37)$$

$$E_\varphi = j\beta \frac{e^{-j\beta r}}{4\pi r} \cos \theta (I_y^E \cos \varphi - I_x^E \sin \varphi). \quad (17.38)$$

For apertures in open space, the dual current formulation is used. Then, the usual assumption is that the aperture field resembles that of a locally-TEM wave, i.e.,

$$\mathbf{H}_a = \hat{\mathbf{z}} \times \mathbf{E}_a / \eta. \quad (17.39)$$

This implies that

$$\mathbf{I}^H = \frac{1}{\eta} \hat{\mathbf{z}} \times \mathbf{I}^E \text{ or } I_x^H = -\frac{I_y^E}{\eta}, I_y^H = \frac{I_x^E}{\eta}. \quad (17.40)$$

This assumption is valid for moderate and high-gain apertures; therefore, the apertures should be at least a couple of wavelengths in extent. The above assumptions reduce (17.34)-(17.35) to

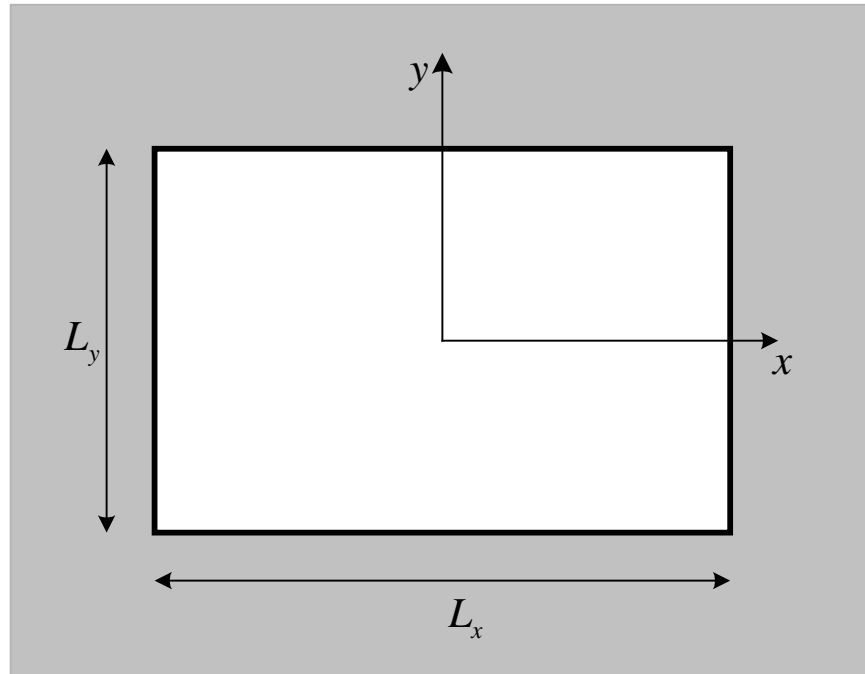
$$E_\theta = j\beta \frac{e^{-j\beta r}}{4\pi r} \frac{(1 + \cos \theta)}{2} (I_x^E \cos \varphi + I_y^E \sin \varphi), \quad (17.41)$$

$$E_\varphi = j\beta \frac{e^{-j\beta r}}{4\pi r} \frac{(1 + \cos \theta)}{2} (I_y^E \cos \varphi - I_x^E \sin \varphi). \quad (17.42)$$

Compare (17.41)-(17.42) to (17.37)-(17.38). If the aperture has high gain, the factors containing $\cos \theta$ are not going to affect the pattern significantly and the two sets of formulas are going to be nearly equivalent.

5. The uniform rectangular aperture on an infinite ground plane

A rectangular aperture is defined in the xy plane as shown below.



If the field is uniform in amplitude and phase across the aperture, it is referred to as a *uniform rectangular aperture*. Let us assume that the aperture field is y -polarized.

$$\mathbf{E}_a = E_0 \hat{\mathbf{y}}, \text{ for } |x| \leq \frac{L_x}{2} \text{ and } |y| \leq \frac{L_y}{2}, \quad (17.43)$$

$$\mathbf{E}_a = 0, \text{ elsewhere.}$$

According to the equivalence principle, we assume an electric wall at $z = 0$, where the equivalent magnetic current density is given by $\mathbf{M}_{s_e} = \mathbf{E}_0 \times \hat{\mathbf{n}}$. Applying image theory, we can find the equivalent sources radiating in open space as

$$\mathbf{M}_s = 2\mathbf{M}_{s_e} = 2E_0 \hat{\mathbf{y}} \times \hat{\mathbf{z}} = 2E_0 \hat{\mathbf{x}}. \quad (17.44)$$

The only non-zero radiation integral is

$$I_y^E = 2E_0 \int_{-L_x/2}^{L_x/2} e^{j\beta x' \sin \theta \cos \varphi} dx' \cdot \int_{-L_y/2}^{L_y/2} e^{j\beta y' \sin \theta \sin \varphi} dy', \quad (17.45)$$

$$\Rightarrow I_y^E = 2E_0L_xL_y \frac{\sin\left(\frac{\beta L_x}{2} \sin\theta \cos\varphi\right)}{\frac{\beta L_x}{2} \sin\theta \cos\varphi} \cdot \frac{\sin\left(\frac{\beta L_y}{2} \sin\theta \sin\varphi\right)}{\frac{\beta L_y}{2} \sin\theta \sin\varphi}. \quad (17.46)$$

To shorten the notations, let us introduce the pattern variables:

$$\begin{cases} u = \frac{\beta L_x}{2} \sin\theta \cos\varphi, \\ v = \frac{\beta L_y}{2} \sin\theta \sin\varphi. \end{cases} \quad (17.47)$$

The complete radiation field is found by substituting (17.46) into (17.37)-(17.38):

$$\begin{cases} E_\theta = j\beta \frac{e^{-j\beta r}}{2\pi r} E_0L_xL_y \sin\varphi \frac{\sin u}{u} \frac{\sin v}{v}, \\ E_\varphi = j\beta \frac{e^{-j\beta r}}{2\pi r} E_0L_xL_y \cos\theta \cos\varphi \frac{\sin u}{u} \frac{\sin v}{v}. \end{cases} \quad (17.48)$$

The total-field amplitude pattern is, therefore,

$$\begin{aligned} |\bar{E}| = F(\theta, \varphi) &= \sqrt{\sin^2\varphi + \cos^2\theta \cos^2\varphi} \cdot \frac{\sin u}{u} \frac{\sin v}{v} = \\ &= \sqrt{1 - \sin^2\theta \cos^2\varphi} \cdot \frac{\sin u}{u} \frac{\sin v}{v}. \end{aligned} \quad (17.49)$$

The principal plane patterns are:

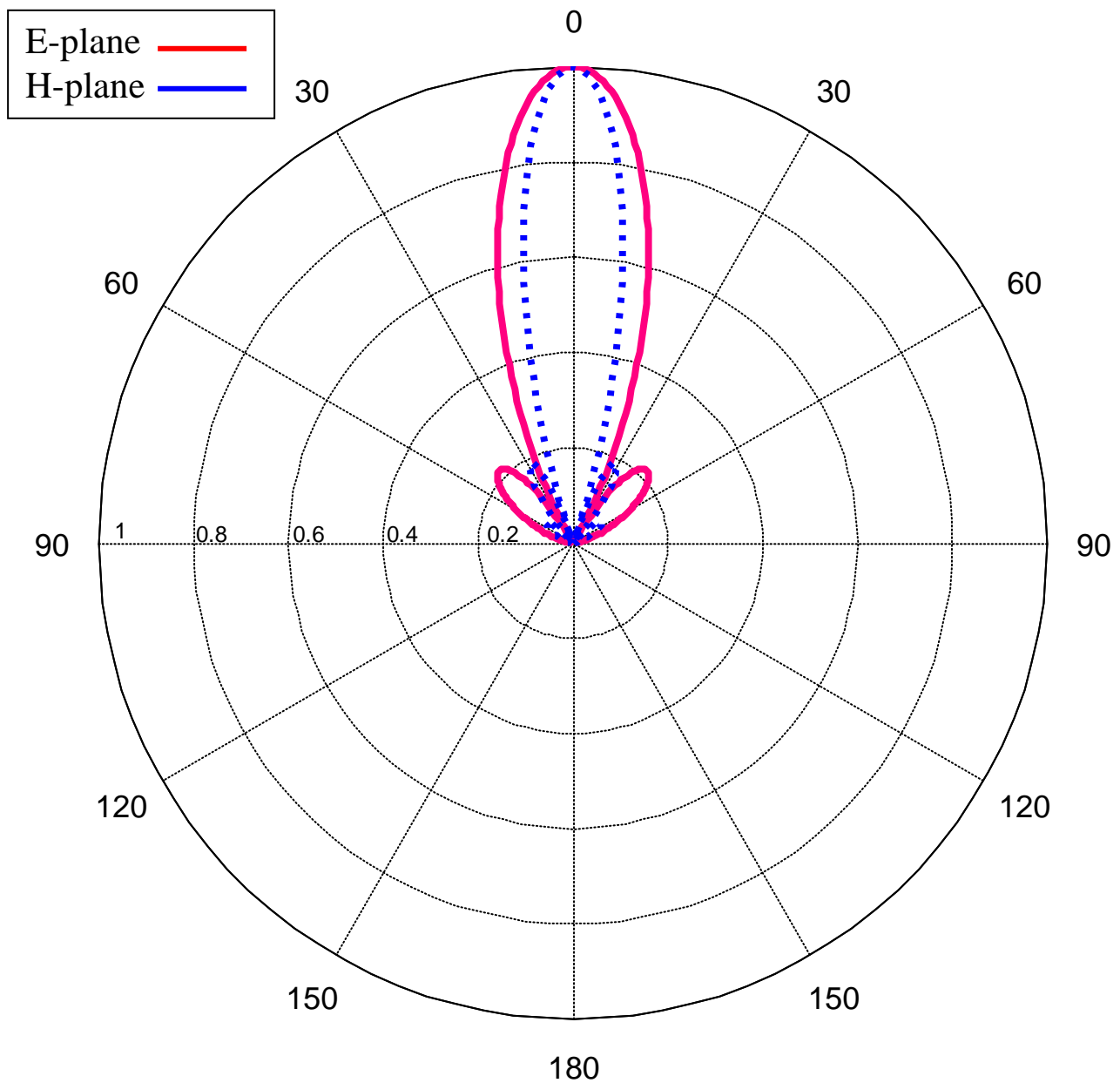
E-plane pattern ($\varphi = \pi/2$)

$$\bar{E}_\theta = \frac{\sin\left(\frac{\beta L_y}{2} \sin\theta\right)}{\left(\frac{\beta L_y}{2} \sin\theta\right)} \quad (17.50)$$

H-plane pattern ($\varphi = 0$)

$$\bar{E}_\varphi = \cos \theta \cdot \frac{\sin\left(\frac{\beta L_x}{2} \sin \theta\right)}{\left(\frac{\beta L_x}{2} \sin \theta\right)} \quad (17.51)$$

PRINCIPLE PATTERNS FOR APERTURE OF SIZE: $L_x = 3\lambda$, $L_y = 2\lambda$

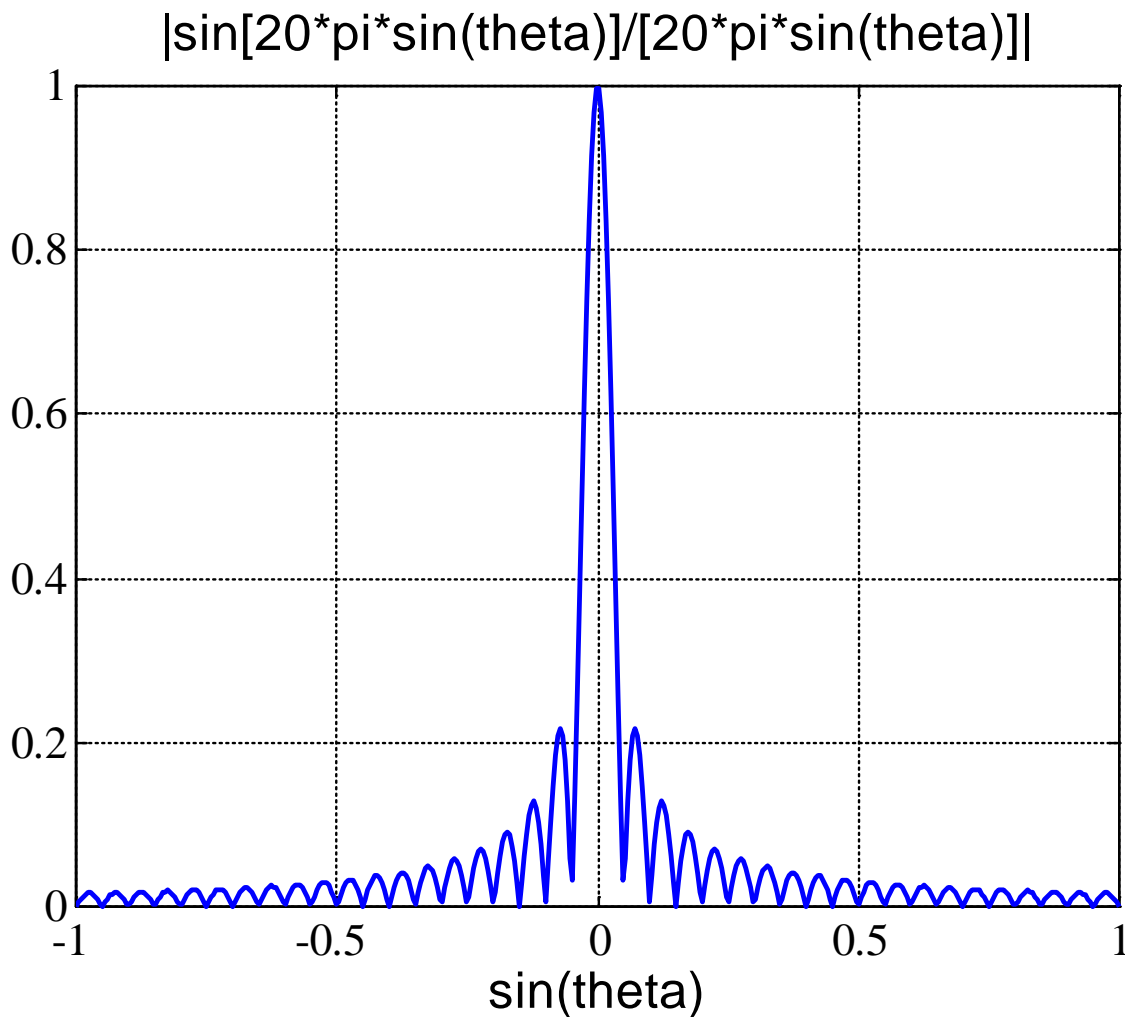


For electrically large apertures, the main beam is narrow and the $(1 - \sin^2 \theta \cos^2 \varphi)^{1/2}$ in (17.49) is negligible, i.e., it is roughly equal to 1 for all observation angles within the main beam. That is why, in the theory of large apertures and arrays, it is assumed that the amplitude pattern is

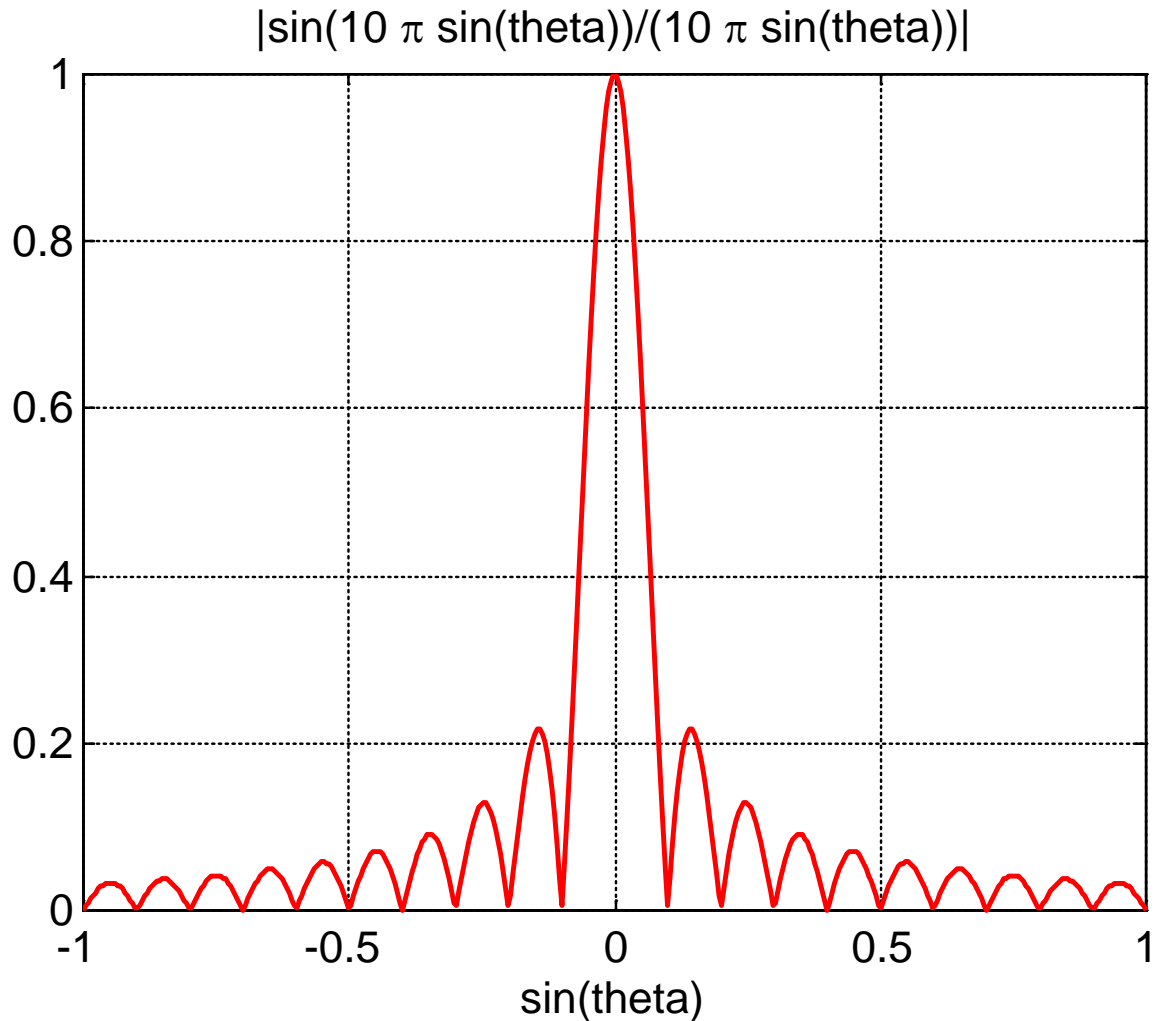
$$f(u, v) \approx \left| \frac{\sin u}{u} \cdot \frac{\sin v}{v} \right|, \quad (17.52)$$

where $u = \frac{\beta L_x}{2} \sin \theta \cos \varphi$ and $v = \frac{\beta L_y}{2} \sin \theta \sin \varphi$.

Below is a view of the $|(\sin u)/u|$ function for $L_x = 20\lambda$ and $\varphi = 0^\circ$ (H-plane pattern):



Here is a view of the $|\sin v/v|$ function for $L_y = 10\lambda$ and $\varphi = 90^\circ$ (E-plane pattern):



Beamwidths

(a) First-null beamwidth (FNBW)

We need the locations of the first nulls in the pattern in order to calculate the FNBW. The nulls of the E-plane pattern are determined from (17.50) as

$$\frac{\beta L_y}{2} \sin \theta_{\theta=\theta_n} = n\pi, \quad n = 1, 2, \dots, \quad (17.53)$$

$$\Rightarrow \theta_n = \arcsin\left(\frac{n\lambda}{L_y}\right), \text{ rad.} \quad (17.54)$$

The first null occurs at $n = 1$.

$$\Rightarrow FNBW_E = 2\theta_n = 2\arcsin\left(\frac{\lambda}{L_y}\right), \text{ rad.} \quad (17.55)$$

In a similar fashion, $FNBW_H$ is determined to be

$$FNBW_H = 2\arcsin\left(\frac{\lambda}{L_x}\right), \text{ rad.} \quad (17.56)$$

(b) Half-power beamwidth (HPBW)

The half-power point in the E-plane occurs when

$$\frac{\sin\left(\frac{\beta L_y}{2} \sin \theta\right)}{\left(\frac{\beta L_y}{2} \sin \theta\right)} = \frac{1}{\sqrt{2}}, \quad (17.57)$$

or

$$\frac{\beta L_y}{2} \sin \theta_{\theta=\theta_h} = 1.391, \quad (17.58)$$

$$\Rightarrow \theta_h = \arcsin\left(\frac{0.443\lambda}{L_y}\right), \text{ rad,} \quad (17.59)$$

$$HPBW_E = 2\arcsin\left(\frac{0.443\lambda}{L_y}\right). \quad (17.60)$$

A first-order approximation is possible for very small arguments in (17.60), i.e., when $L_y \gg 0.443\lambda$ (large aperture):

$$HPBW_E \approx 0.886 \frac{\lambda}{L_y}. \quad (17.61)$$

The half-power beamwidth in the H-plane is analogous:

$$HPBW_H = 2 \arcsin \left(\frac{0.443\lambda}{L_x} \right). \quad (17.62)$$

Side-lobe level

It is obvious from the properties of the $|\sin x / x|$ function that the first side lobe has the largest maximum of all side lobes, and it is

$$|E_\theta(\theta = \theta_s)| \approx \left| \frac{\sin 4.494}{4.494} \right| \approx 0.217 \approx -13.26, \text{ dB}. \quad (17.63)$$

When evaluating side-lobe levels and beamwidths in the H-plane, one has to include the $\cos \theta$ factor, too. The larger the aperture, the less important this factor is.

Directivity

In a general approach to the calculation of the directivity, the total radiated power Π has to be calculated first using the far-field pattern expression (17.49)

$$D_0 = \frac{4\pi}{\Omega_A} = 4\pi \frac{U_{\max}}{\Pi}. \quad (17.64)$$

Here,

$$U(\theta, \varphi) = \frac{1}{2\eta} [|E_\theta|^2 + |E_\varphi|^2] r^2 = U_{\max} \cdot |F(\theta, \varphi)|^2, \quad (17.65)$$

$$\Omega_A = \int_0^{2\pi} \int_0^{\pi/2} |F(\theta, \varphi)|^2 \sin \theta d\theta d\varphi. \quad (17.66)$$

However, in the case of an aperture illuminated by a TEM wave, we can use a simpler approach. Generally, for all aperture antennas, the assumption of a uniform TEM wave at the aperture ($\mathbf{E} = \hat{\mathbf{y}}E_0$),

$$\mathbf{H}_a = -\hat{\mathbf{x}}E_0 / \eta, \quad (17.67)$$

is quite accurate (although η is not necessarily the intrinsic impedance of vacuum). The far-field components in this case were already derived in (17.41) and (17.42). They lead to the following expression for the radiation intensity,

$$U(\theta, \varphi) = \frac{\beta^2}{32\pi^2\eta} (1 + \cos\theta)^2 (|I_x^E|^2 + |I_y^E|^2). \quad (17.68)$$

The maximum value of the function in (17.68) is derived after substituting the radiation integrals from (17.26) and (17.27):

$$U_{\max} = \frac{\beta^2}{8\pi^2\eta} \left| \iint_{S_A} \mathbf{E}_a ds' \right|^2. \quad (17.69)$$

The integration of the radiation intensity (17.68) over a closed sphere is in general not easy. It can be avoided by observing that the total power reaching the far zone must have passed through the aperture in the first place. In an aperture, where the field obeys (17.67), this power is determined as

$$\Pi = \oiint_S \mathbf{P}_{av} \cdot d\mathbf{s} = \frac{1}{2\eta} \iint_{S_A} |\mathbf{E}_a|^2 ds'. \quad (17.70)$$

Substituting (17.69) and (17.70) into (17.64) finally yields

$$D_0 = \frac{4\pi}{\lambda^2} \times \frac{\left| \iint_{S_A} \mathbf{E}_a ds' \right|^2}{\iint_{S_A} |\mathbf{E}_a|^2 ds'}. \quad (17.71)$$

← A_{eff}

In the case of a **uniform rectangular aperture**,

$$\Pi = L_x L_y \frac{|E_0|^2}{2\eta}, \quad (17.72)$$

$$U_{\max} = \left(\frac{L_x L_y}{\lambda} \right)^2 \frac{|E_0|^2}{2\eta}. \quad (17.73)$$

Thus, the directivity is found to be

$$D_0 = 4\pi \frac{U_{\max}}{\Pi} = \frac{4\pi}{\lambda^2} L_x L_y = \frac{4\pi}{\lambda^2} A_p = \frac{4\pi}{\lambda^2} A_{eff}. \quad (17.74)$$

Note that **the physical and effective areas of a uniform aperture are equal**.

6. The uniform rectangular aperture in open space

Now the rectangular aperture is *not* mounted on a ground plane. The field distribution is the same as in (17.43) but now the \mathbf{H} field must be defined, too, in order to apply the equivalence principle with both types of surface currents,

$$\left. \begin{aligned} \mathbf{E}_a &= \hat{\mathbf{y}}E_0 \\ \mathbf{H}_a &= -\hat{\mathbf{x}}E_0 / \eta \end{aligned} \right\} \begin{aligned} -L_x / 2 \leq x' \leq L_x / 2 \\ -L_y / 2 \leq y' \leq L_y / 2. \end{aligned} \quad (17.75)$$

Again, an assumption was made that there is a simple relation between the electric and the magnetic field components through the impedance η .

To form the equivalent problem, an infinite surface is chosen to extend in the $z=0$ plane. Over the entire surface, the equivalent \mathbf{J}_s and \mathbf{M}_s surface currents must be defined. Both \mathbf{J}_s and \mathbf{M}_s are not zero outside the aperture in the $z=0$ plane because the tangential field is not zero there. Moreover, the field is *not known a priori* outside the aperture. Thus, the exact equivalent problem cannot be built in practice (at least not in the infinite plane model).

The usual assumption is that \mathbf{E}_a and \mathbf{H}_a are zero outside the aperture in the $z=0$ plane, and, therefore, so are the equivalent currents \mathbf{J}_s and \mathbf{M}_s ,

$$\left. \begin{aligned} \mathbf{M}_s &= -\hat{\mathbf{n}} \times \mathbf{E}_a = \underbrace{-\hat{\mathbf{z}} \times \hat{\mathbf{y}}}_{\hat{\mathbf{x}}} E_0 \\ \mathbf{J}_s &= \hat{\mathbf{n}} \times \mathbf{H}_a = \underbrace{\hat{\mathbf{z}} \times (-\hat{\mathbf{x}})}_{-\hat{\mathbf{y}}} \frac{E_0}{\eta} \end{aligned} \right\} \text{for } \begin{cases} -L_x / 2 \leq x' \leq L_x / 2 \\ -L_y / 2 \leq y' \leq L_y / 2. \end{cases} \quad (17.76)$$

$$\mathbf{J}_s = \mathbf{M}_s = 0 \text{ for } |x'| > L_x / 2, \quad |y'| > L_y$$

Since the equivalent currents are related via the impedance assumption (17.75), only the integral I_y^E is needed for substitution in the far-field expressions (17.41)-(17.42). It is the same as in (17.46):

$$\begin{aligned} I_y^E &= 2E_0 \int_{-L_x/2}^{L_x/2} e^{j\beta x' \sin \theta \cos \varphi} dx' \cdot \int_{-L_y/2}^{L_y/2} e^{j\beta y' \sin \theta \sin \varphi} dy' = \\ &= 2E_0 L_x L_y \frac{\sin\left(\frac{\beta L_x}{2} \sin \theta \cos \varphi\right)}{\frac{\beta L_x}{2} \sin \theta \cos \varphi} \cdot \frac{\sin\left(\frac{\beta L_y}{2} \sin \theta \sin \varphi\right)}{\frac{\beta L_y}{2} \sin \theta \sin \varphi}. \end{aligned} \quad (17.77)$$

The far-field components are obtained by substituting (17.77) into (17.41) and (17.42):

$$\begin{aligned} E_{\theta} &= C \sin \varphi \frac{(1 + \cos \theta)}{2} \frac{\sin u}{u} \frac{\sin v}{v}, \\ E_{\varphi} &= C \cos \varphi \frac{(1 + \cos \theta)}{2} \frac{\sin u}{u} \frac{\sin v}{v}, \end{aligned} \quad (17.78)$$

where

$$\begin{aligned} C &= j\beta L_x L_y E_0 \frac{e^{-j\beta r}}{2\pi r}, \\ u &= \frac{\beta L_x}{2} \sin \theta \cos \varphi, \\ v &= \frac{\beta L_y}{2} \sin \theta \sin \varphi. \end{aligned}$$

The far-field expressions in (17.78) are very similar to those of the aperture mounted on a ground plane, see (17.48). For small values of θ , the patterns of both apertures are practically identical.

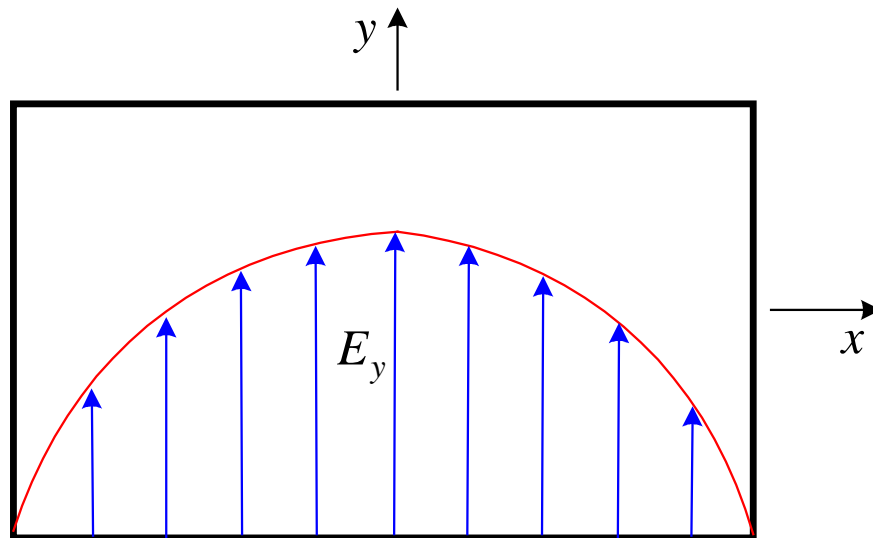
An exact analytical evaluation of the directivity is difficult. However, according to the approximations made, the directivity formula derived in (17.71) should provide accurate enough value. According to (17.71), the directivity is the same as in the case of the aperture mounted on a ground plane.

7. The tapered rectangular aperture on a ground plane

The uniform rectangular aperture has the maximum possible effective area (for an aperture-type antenna) equal to its physical area. This also implies that it has the highest possible directivity for all constant-phase excitations of a rectangular aperture. However, the directivity is not the only important factor in the design of an antenna. A factor that frequently comes into conflict with the directivity is the side-lobe level (SLL). The uniform distribution excitation produces the highest SLL of all constant-phase excitations of a rectangular aperture. It is shown below that a reduction of the SLL can be achieved by tapering the equivalent sources distribution from a maximum at the aperture's center to zero values at its edges.

One practical aperture of tapered source distribution is the open rectangular waveguide. The dominant TE₁₀ mode has the following distribution:

$$\mathbf{E}_a = \hat{\mathbf{y}}E_0 \cos\left(\frac{\pi}{L_x}x'\right), \quad \begin{cases} -L_x/2 \leq x' \leq L_x/2 \\ -L_y/2 \leq y' \leq L_y/2 \end{cases} \quad (17.79)$$



The general procedure for the far-field analysis is the same as before (Sections 5 and 6). The only difference is in the field distribution. Again, only the integral I_y^E is evaluated:

$$I_y^E = 2E_0 \int_{-L_x/2}^{L_x/2} \cos\left(\frac{\pi}{L_x}x'\right) e^{j\beta x' \sin\theta \cos\varphi} dx' \cdot \int_{-L_y/2}^{L_y/2} e^{j\beta y' \sin\theta \sin\varphi} dy'. \quad (17.80)$$

The integral of the y' variable was already encountered in (17.45)-(17.46):

$$I_y = \int_{-L_y/2}^{L_y/2} e^{j\beta y' \sin\theta \sin\varphi} dy' = L_y \frac{\sin\left[\frac{\beta L_y}{2} \sin\theta \sin\varphi\right]}{\frac{\beta L_y}{2} \sin\theta \sin\varphi}. \quad (17.81)$$

The integral of the x' variable is also easily solved:

$$I_x = \int_{-L_x/2}^{L_x/2} \cos\left(\frac{\pi}{L_x}x'\right) e^{j\beta x' \sin\theta \cos\varphi} dx' =$$

$$\begin{aligned}
&= \int_{-L_x/2}^{L_x/2} \cos\left(\frac{\pi}{L_x} x'\right) \left[\cos(\beta x' \sin \theta \cos \varphi) + j \sin(\beta x' \sin \theta \cos \varphi) \right] dx' = \\
&= \frac{1}{2} \int_{-L_x/2}^{L_x/2} \left\{ \cos\left[\left(\frac{\pi}{L_x} - \beta \sin \theta \cos \varphi\right) x'\right] + \cos\left[\left(\frac{\pi}{L_x} + \beta \sin \theta \cos \varphi\right) x'\right] \right\} dx' + \\
&+ \frac{j}{2} \int_{-L_x/2}^{L_x/2} \left\{ \sin\left[\left(\beta \sin \theta \cos \varphi - \frac{\pi}{L_x}\right) x'\right] + \sin\left[\left(\beta \sin \theta \cos \varphi + \frac{\pi}{L_x}\right) x'\right] \right\} dx'
\end{aligned}$$

$$\Rightarrow I(x) = \frac{\pi L_x \cos\left(\frac{\beta L_x}{2} \sin \theta \cos \varphi\right)}{2 \left[\left(\frac{\pi}{2}\right)^2 - \frac{\beta L_x}{2} \sin \theta \cos \varphi \right]}$$

$$\Rightarrow I_y^E = \pi E_0 L_x L_y \frac{\cos\left(\frac{\beta L_x}{2} \sin \theta \cos \varphi\right) \sin\left[\frac{\beta L_y}{2} \sin \theta \sin \varphi\right]}{\left[\left(\frac{\pi}{2}\right)^2 - \frac{\beta L_x}{2} \sin \theta \cos \varphi \right] \underbrace{\left(\frac{\beta L_y}{2} \sin \theta \sin \varphi\right)}_v} \quad (17.82)$$

To derive the far-field components, (17.82) is substituted in (17.34)-(17.35):

$$\begin{aligned}
E_\theta &= -\frac{\pi}{2} C \sin \varphi \cdot \frac{\cos u}{\left[u^2 - \left(\frac{\pi}{2}\right)^2 \right]} \cdot \frac{\sin v}{v} \\
E_\varphi &= -\frac{\pi}{2} C \cos \theta \cos \varphi \cdot \frac{\cos u}{\left[u^2 - \left(\frac{\pi}{2}\right)^2 \right]} \cdot \frac{\sin v}{v}
\end{aligned} \quad (17.83)$$

where

$$C = j\beta L_x L_y E_0 \frac{e^{-j\beta r}}{2\pi r},$$

$$u = \frac{\beta L_x}{2} \sin \theta \cos \varphi,$$

$$v = \frac{\beta L_y}{2} \sin \theta \sin \varphi.$$

Principle plane patterns

In the E-plane, the aperture is not tapered. As expected, the E-plane principal pattern is the same as that of a uniform aperture.

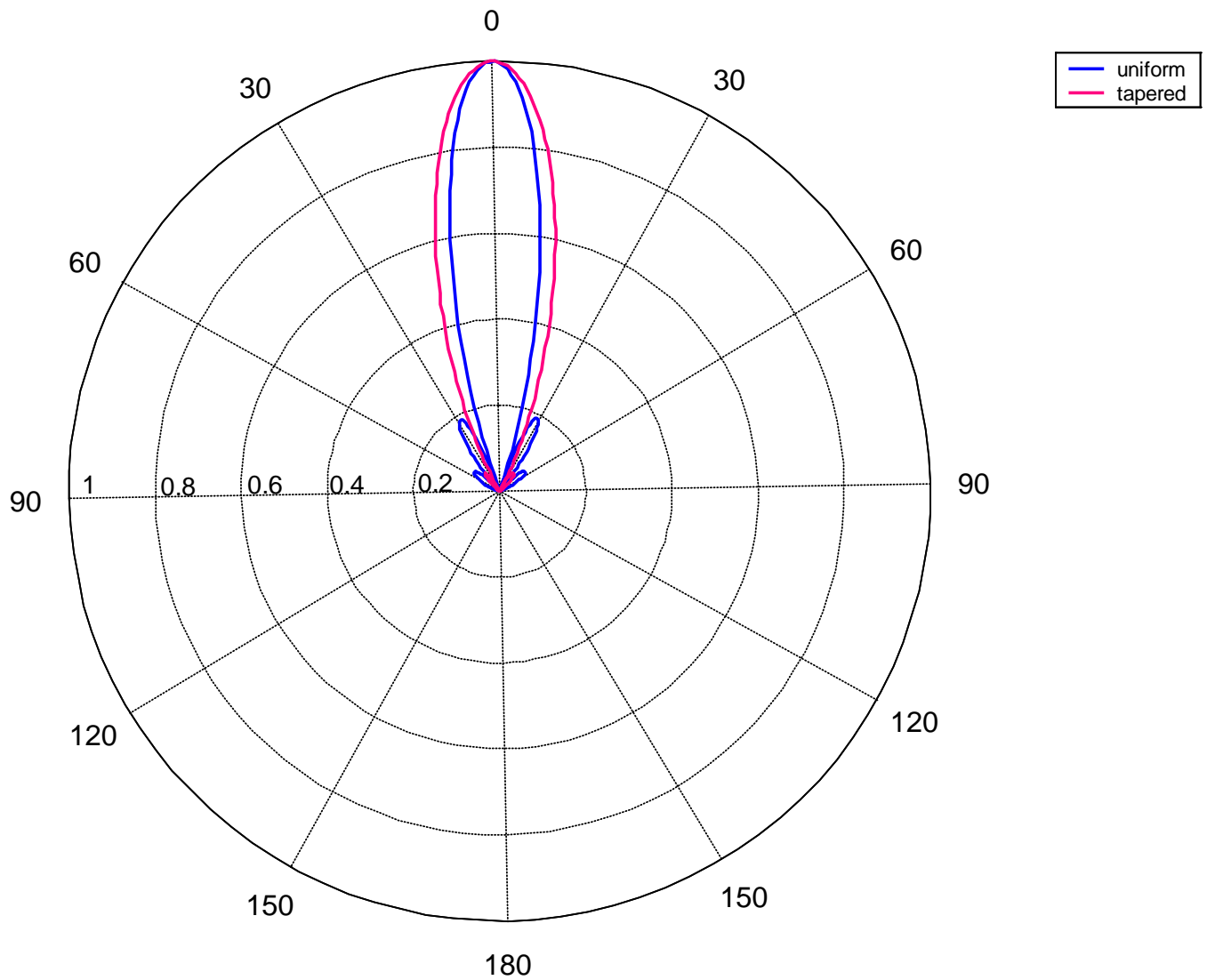
E-plane ($\varphi = 90^\circ$):

$$\bar{E}_\theta = \frac{\sin\left(\frac{\beta L_y}{2} \sin \theta\right)}{\left(\frac{\beta L_y}{2} \sin \theta\right)} \quad (17.84)$$

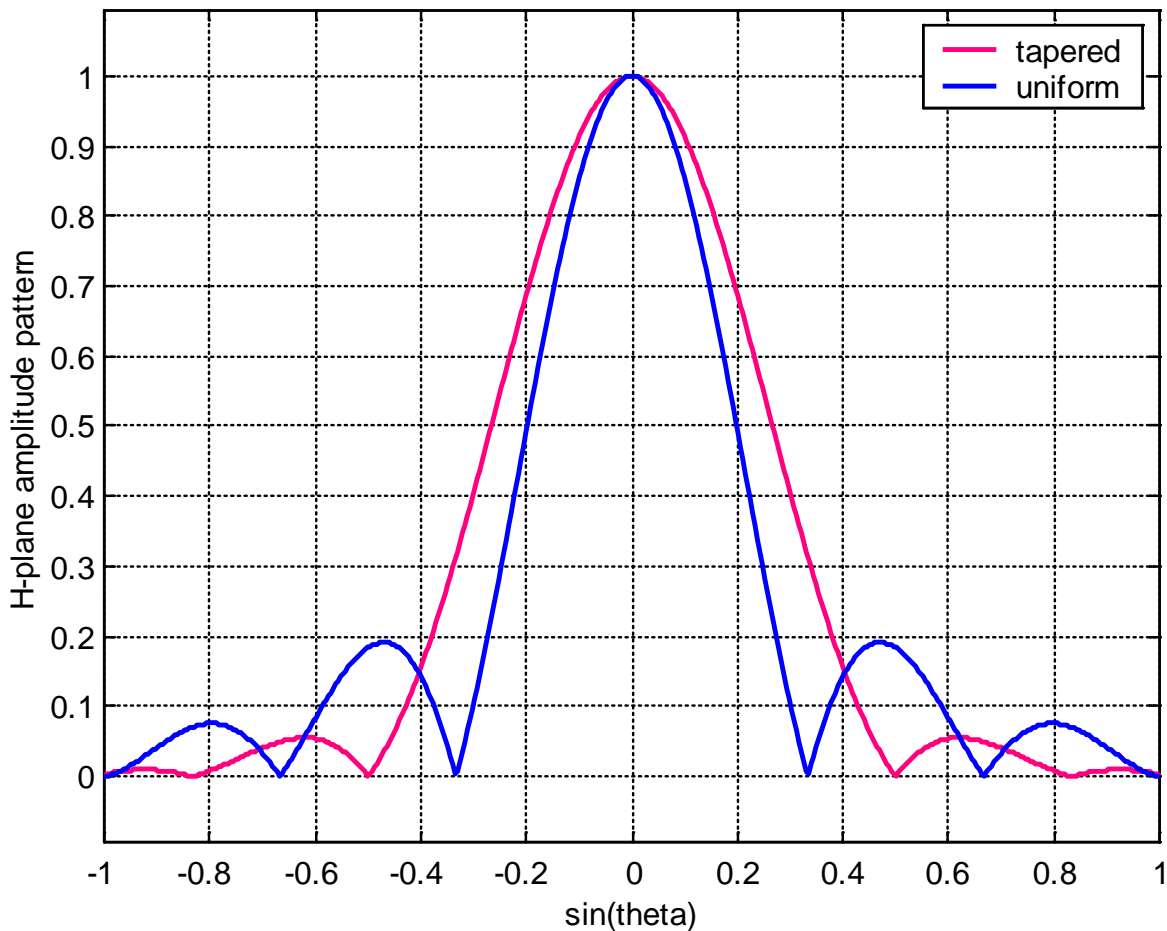
H-plane ($\varphi = 0^\circ$):

$$\bar{E}_\varphi = \cos \theta \frac{\cos\left(\frac{\beta L_x}{2} \sin \theta\right)}{\left(\frac{\beta L_x}{2} \sin \theta\right)^2 - \left(\frac{\pi}{2}\right)^2} \quad (17.85)$$

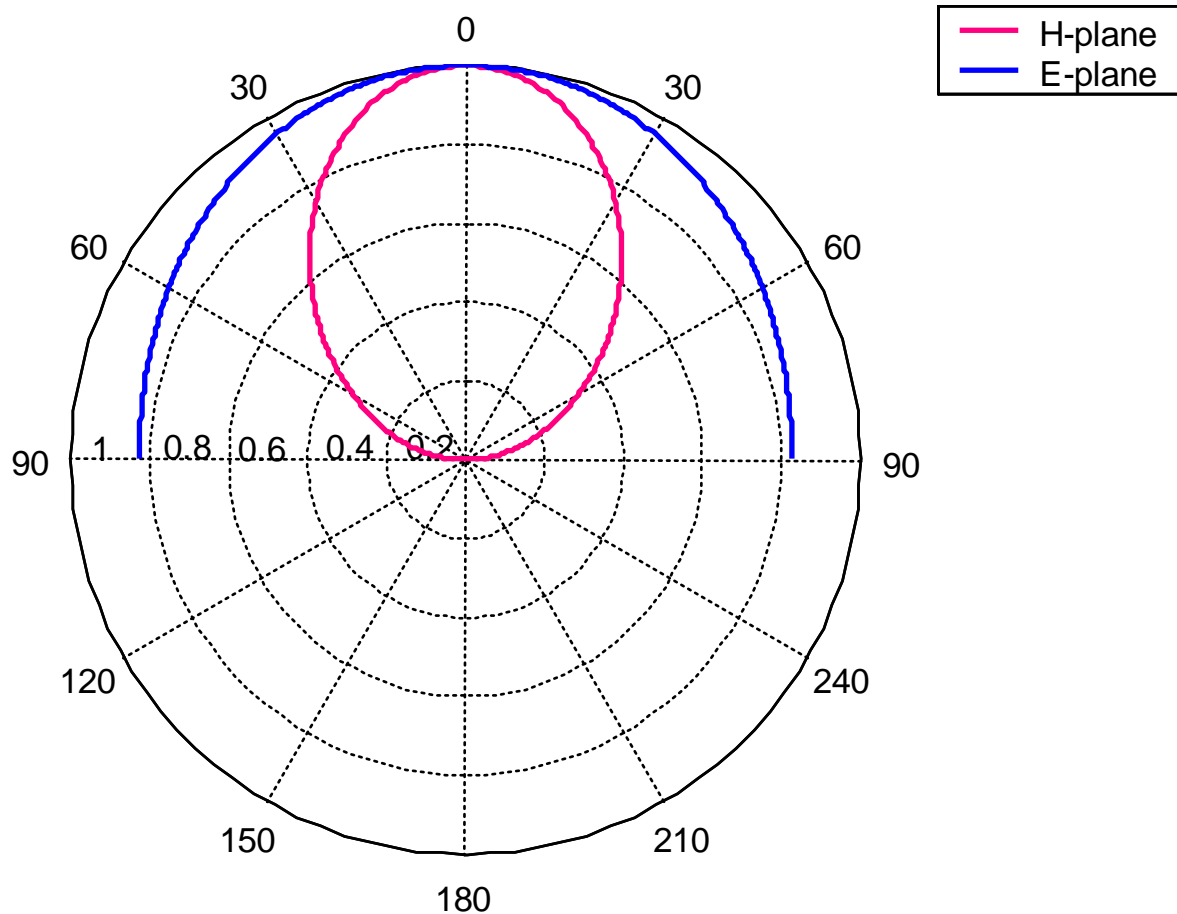
H-PLANE PATTERN – UNIFORM VS. TAPERED ILLUMINATION ($L_x = 3\lambda$):



The lower SLL of the tapered-source pattern is obvious. It is better seen in the rectangular plot given below. The price to pay for the lower SLL is the decrease in directivity (the beamwidth of the major lobe increases).



The above example of $L_x = 3\lambda$ illustrates well the effect of source distribution on the far-field pattern. However, a more practical example is the rectangular-waveguide open-end aperture, where the waveguide operates in a dominant mode, i.e. $\lambda_0 / 2 < L_x < \lambda_0$. Here, λ_0 is the wavelength in open space ($\lambda_0 = c / f_0$). Consider the case $L_x = 0.75\lambda_0$. The principal-plane patterns for an aperture on a ground plane look like this:



In the above example, a practical X-band waveguide was considered whose cross-section has the following size: $L_x = 2.286$ cm, $L_y = 1.016$ cm. ($\lambda_0 = 3.048$ cm and $f_0 = 9.84$ GHz)

The case of a dominant-mode open-end waveguide radiating in free space can be analyzed following the approaches outlined in this Section and in Section 6.

The calculation of beamwidths and directivity is analogous to the previous cases. Only the final results will be given here for the case of the x -tapered (cosine taper) aperture on a ground plane.

$$\text{Directivity: } D_0 = \frac{4\pi}{\lambda^2} \cdot \frac{8}{\pi^2} L_x L_y \quad (17.86)$$

$$\text{Effective area: } A_{\text{eff}} = \frac{8}{\pi^2} L_x L_y \approx 0.81 A_p \quad (17.87)$$

Note the decrease in the effective area compared to the uniform-aperture case.

Half-power beamwidths:

$$HPBW_E = \frac{50.6}{L_y / \lambda}, \text{ deg. } (= HPBW_E \text{ of the uniform aperture}) \quad (17.88)$$

$$HPBW_H = \frac{68.8}{L_x / \lambda}, \text{ deg. } (> HPBW_H \text{ of the uniform aperture}) \quad (17.89)$$

The above results are approximate. Better results are obtained if the following factors are taken into account:

- the phase constant of the waveguide β_g and its wave impedance Z_g are not equal to the free-space phase constant $\beta_0 = \omega\sqrt{\mu_0\epsilon_0}$ and intrinsic impedance $Z_0 = \sqrt{\mu_0 / \epsilon_0}$; they are dispersive;
- the abrupt termination at the waveguide open end introduces reflection, which affects the field at the aperture;
- there are strong fringe currents at the waveguide walls, which contribute to the overall radiation.