Lecture 17

Boundary Conditions at Dielectric Interfaces

Sections: 5.8

Homework: See homework file
BCs for the Tangential Field Components – 1

• we consider interfaces between two perfect ($\sigma = 0$) dielectric regions

• use conservative property of field $\oint_C \mathbf{E} \cdot d\mathbf{L} = 0$

• choose contour across interface

• contour is small enough to consider field constant along its line segments

\[ -E_{\text{tan}1}l_1 + 0.5(E_n^{(1)} + E_n^{(2)})l_2 + E_{\text{tan}1}l_3 - 0.5(E_n^{(1)} + E_n^{(2)})l_4 = 0 \]
• take limit when $\Delta h \rightarrow 0$

$$\Rightarrow -E_{\text{tan}1}^{(1)} \Delta w + E_{\text{tan}1}^{(2)} \Delta w = 0 \Rightarrow E_{\text{tan}1}^{(1)} = E_{\text{tan}1}^{(2)}$$

• the same is proven for the other pair of tangential field components with a contour along $a_{\text{tan}2}$ and $a_n$

$$E_{\text{tan}2}^{(1)} = E_{\text{tan}2}^{(2)}$$

• boundary condition for $E_{\text{tan}}$ in vector form

$$E_{\text{tan}}^{(1,2)} = E_{\text{tan}1}^{(1,2)} a_{\text{tan}1} + E_{\text{tan}2}^{(1,2)} a_{\text{tan}2} \Rightarrow E_{\text{tan}}^{(1)} = E_{\text{tan}}^{(2)}$$

the tangential $E$ component is continuous across dielectric interface
• equivalent vector formulation

\[ \mathbf{a}_n \times \mathbf{E} = \mathbf{a}_n \times (E_n \mathbf{a}_n + \mathbf{E}_{\text{tan}}) = \mathbf{a}_n \times \mathbf{E}_{\text{tan}} \]

\[ \mathbf{E}_{\text{tan}}^{(1)} = \mathbf{E}_{\text{tan}}^{(2)} \Rightarrow \mathbf{a}_n \times \mathbf{E}_{\text{tan}}^{(1)} = \mathbf{a}_n \times \mathbf{E}_{\text{tan}}^{(2)} \]

\[ \mathbf{a}_n \times (\mathbf{E}^{(2)} - \mathbf{E}^{(1)}) = 0 \]

• the tangential components of the flux density

\[ D_{\text{tan}}^{(1)} = \frac{D_{\text{tan}}^{(2)}}{\varepsilon_1} \]
\[ D_{\text{tan}}^{(1)} = \frac{D_{\text{tan}}^{(2)}}{\varepsilon_2} \]

the tangential D component is discontinuous

• when medium 1 is a perfect conductor (particular case)

\[ \mathbf{E}_{\text{tan}}^{(2)} = \mathbf{E}_{\text{tan}}^{(1)} = 0, \quad \mathbf{D}_{\text{tan}}^{(2)} = \mathbf{D}_{\text{tan}}^{(1)} = 0 \]
BCs for the Normal Field Components – 1

- apply Gauss’ law over a closed surface centered around the interface

\[ \lim_{{\Delta h \to 0}} \int_S \mathbf{D} \cdot d\mathbf{s} = (D_n^{(2)} - D_n^{(1)}) \cdot A = Q_f = 0 \]

\[ D_n^{(2)} = D_n^{(1)} \Rightarrow \varepsilon_2 E_n^{(2)} = \varepsilon_1 E_n^{(1)} \]

\[ \frac{E_n^{(2)}}{E_n^{(1)}} = \frac{\varepsilon_1}{\varepsilon_2} \]

the normal \( \mathbf{D} \) component is continuous while the normal \( \mathbf{E} \) component is discontinuous across dielectric interfaces
Medium 1 of $\varepsilon_r^{(1)} = 4$ is located in the region $x \leq 0$. Medium 2 of $\varepsilon_r^{(2)} = 1$ is located in the region $x \geq 0$. The field components in medium 1 are: $E_x^{(1)} = 4 \text{ V/m}$, $E_y^{(1)} = 3 \text{ V/m}$, and $E_z^{(1)} = 1 \text{ V/m}$. What are the field components in medium 2?

$E_x^{(2)} =$

$E_y^{(2)} =$

$E_z^{(2)} =$
• when medium 1 is a perfect conductor

\[
\lim_{\Delta h \to 0} \int_S \mathbf{D} \cdot d\mathbf{s} = (D_n^{(2)} - D_n^{(1)}) \cdot A = Q_f = \rho_{sf} \cdot A
\]

\[
D_n^{(2)} = \rho_{sf}, \quad E_n^{(2)} = \frac{\rho_{sf}}{\varepsilon_0 \varepsilon_{r2}}
\]

\[
D_n^{(1)} = E_n^{(1)} = 0
\]

• we have already derived this BC in Lecture 15
Consider the flat interface between region 1 \((z \leq 0)\) with \(\epsilon_{r_1} = 9\) and region 2 \((z \geq 0)\) with \(\epsilon_{r_2} = 1\). In region 1,

\[
\mathbf{E}^{(1)} = 3\mathbf{a}_x + 18\mathbf{a}_y + 6\mathbf{a}_z \text{ V/m}.
\]

At the interface \(z = 0\), find: (a) \(\mathbf{E}_{\text{tan}}^{(2)}, \mathbf{D}_{\text{tan}}^{(2)}\); (b) \(\mathbf{E}_n^{(2)}, \mathbf{D}_n^{(2)}\); (c) \(\rho_{sb}\).
Dielectric Interfaces: Orientation of Field Vectors

- the tangential $E$ component is continuous while the normal one is not
- the $E$ field vector along with the field lines and equipotential lines changes its orientation abruptly

\[
\frac{\tan \alpha_2}{\tan \alpha_1} = \frac{\varepsilon_2}{\varepsilon_1}
\]

\[
E_n^{(2)} = 2E_n^{(1)}
\]

\[
\varepsilon_{r2} = 2\varepsilon_{r1}
\]
Dielectric Interfaces: Orientation of Field Vectors - Illustration

[www.falstad.com]
You have learned:

- that the tangential $E$ component is continuous across interfaces, both dielectric-to-dielectric and PEC-to-dielectric.

- that the normal $D$ component is continuous across dielectric interfaces; it is discontinuous across PEC-to-dielectric interfaces due to the presence of free surface charge.

- how to interpret field maps at interfaces.