Introduction to Microwave Imaging Part II: Linear Inversion Methods

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COURSE OVERVIEW

Day 1: Introduction & Forward Models of Microwave Imaging
- Field-based Integral Solutions of the Scattering Problem in Time and Frequency
- Born and Rytov Approximations of the Forward Model of Scattering
- Scattering Parameters and Integral Solutions in terms of S-parameters
- 2D Model of Tomography in Microwave Scattering

Day 2: Linear Inversion Methods
- Deconvolution Methods
  - Microwave Holography (MH)
  - Scattered Power Mapping (SPM)
- Image Reconstruction of Pulsed-radar Data
- Synthetic Focusing: Delay and Sum (DAS)

Day 3: Performance Metrics & Hardware
- Spatial Resolution
- Dynamic Range
- Data Signal-to-noise Ratio

Select Topics
- Overview of Nonlinear Inversion Methods
  - Direct Iterative Methods
  - Model-based Optimization Methods
- Tissue Imaging – Challenges and Advancements
THE KERNEL OF THE DATA EQUATION: POINT-SPREAD FUNCTION (PSF)
• microwave measurements involve scanning over large acquisition surfaces – each response being function of the observation position \( r \)
• at each observation position \( r \) several responses may be acquired

example of antenna array measuring 9 responses at each \( r \)

[Image credit: Justin J. McCombe]
THE REALISTIC MEASUREMENT SCENARIO: PLANAR SCANNING

- example of planar scanning for microwave imaging

- reflected signals: $S_{11}, S_{22}$
- transmitted signals: $S_{21}, S_{12}$ ($S_{21} = S_{12}$ in reciprocal systems)

<table>
<thead>
<tr>
<th>type of response</th>
<th>number of values $S_{ik}(x,y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>co-pol X-X</td>
<td>$4 \times N_\omega$</td>
</tr>
<tr>
<td>co-pol Y-Y</td>
<td>$4 \times N_\omega$</td>
</tr>
<tr>
<td>cross-pol X-Y</td>
<td>$4 \times N_\omega$</td>
</tr>
<tr>
<td>cross-pol Y-X</td>
<td>$4 \times N_\omega$</td>
</tr>
<tr>
<td>TOTAL</td>
<td>$16 \times N_\omega$</td>
</tr>
</tbody>
</table>

Number of possible responses acquired at each position
FORWARD MODEL WITH PLANAR SCANNING

- $S$-parameter forward model in planar scanning

\[
S_{s_e}^{sc}(x, y, z; \omega) \approx \frac{i\omega \varepsilon_0}{2a_i a_k} \int \int \int_{V_s} \Delta e_r(x', y', z') \left[ E_{\xi, Rx}^{inc} \cdot E_{\xi, Tx}^{inc} \right]_{(x', y', z'; x, y, z; \omega)} \, dx' \, dy' \, dz', \quad \xi = 1, \ldots, N_T
\]

- $\xi$ (response type) replaces $(i,k)$

EXAMPLE: RESPONSE TYPES WITH 2-PORT MEASUREMENT

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$(i, k)$</th>
<th>response type</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1,1)</td>
<td>reflection $S_{11}$</td>
</tr>
<tr>
<td>2</td>
<td>(2,2)</td>
<td>reflection $S_{22}$</td>
</tr>
<tr>
<td>3</td>
<td>(1,2) or (2,1)</td>
<td>transmission $S_{12} = S_{21}$</td>
</tr>
</tbody>
</table>

$N_T = 3$
FORWARD MODEL WITH PLANAR SCANNING – 2

- S-parameter forward model in planar scanning

\[
S^{sc}_{\xi}(x, y, z; \omega) \approx \kappa \iiint_{V_s} \Delta \varepsilon_r(x', y', z') \left[ E_{\xi, Rx}^{inc} \cdot E_{\xi, Tx}^{inc} \right] (x', y', z', x, y, z; \omega) \, dx' \, dy' \, dz', \ \xi = 1, \ldots, N_T
\]

- position of Tx/Rx antenna pair given by \( r_{Rx} \)

\[
r_{Rx} \equiv (x, y, \bar{z}) \text{ and } r_{Tx} \equiv (x, y, \bar{z} - D)
\]
resolvent kernel of forward model – re-visiting last lecture

\[ S_{\xi}^{sc}(x, y, \bar{z}; \omega) \approx \kappa \int \int \int_{V_s} \Delta \varepsilon_r(x', y', z') \left[ \mathbf{E}_{\xi, \text{Rx}}^{\text{inc}} \cdot \mathbf{E}_{\xi, \text{Tx}}^{\text{inc}} \right]_{(x', y', z'; x, y, \bar{z}; \omega)} dx' dy' dz', \ \xi = 1, \ldots, N_T \]

approximate (Born) resolvent kernel \( \mathcal{C}_\xi(r'; r; \omega) \)

- method 1: analytical far-zone expressions well suited only for measurements in air with large stand-off distances
- method 2: simulated field distributions suffer from *modeling errors*
- method 3: *Measurements? Yes, measure the system point-spread function (PSF)*
RESOLVENT KERNEL & POINT-SPREAD FUNCTION (PSF)

- PSF is data measured with point scatterer (electrically small object)

\[
S_{\xi}^{\text{PSF}}(x, y, \bar{z}; \omega) \approx \kappa \iiint_{V_{\xi}} \Delta \epsilon_{r, \text{sp}} \delta(x' - x, y - y', z - z') \cdot K_{\xi}(x', y', z'; x, y, \bar{z}; \omega) dx' dy' dz'
\]

- relating PSF to kernel

\[
\Rightarrow S_{\xi}^{\text{PSF}}(x, y, \bar{z}; x', y', z'; \omega) \approx \left( \kappa \Delta \epsilon_{r, \text{sp}} \Omega_{\text{sp}} \right) \cdot K_{\xi}(x', y', z'; x, y, \bar{z}; \omega), \quad \xi = 1, \ldots, N_{T}
\]
\[ K_\xi(x', y', z'; x, y, \bar{z}; \omega) = \frac{S_{\xi}^{\text{PSF}}(x, y, \bar{z}; x', y', z'; \omega)}{\kappa \Delta \varepsilon_{r,sp} \Omega_{sp}}, \quad \xi = 1, \ldots, N_T \]

- notice the simple exchange of positions allowing to obtain the kernel from the PSF

\[ K_\xi(r'; r; \omega) = \frac{S_{\xi}^{\text{PSF}}(r; r'; \omega)}{\kappa \Delta \varepsilon_{r,sp} \Omega_{sp}} \]

observation: \( r \equiv (x, y, z) \)

integration (probe location): \( r' \equiv (x', y', z') \)

- if medium in \( V_\xi \) is uniform (or layered for planar acquisition) – probe needs to be measured only at the center of an imaged plane \( z' = \text{const} \)

measured centered PSF: \( S_{\xi,0}^{\text{PSF}}(x, y, \bar{z}; z'; \omega) \equiv S_{\xi}^{\text{PSF}}(x, y, \bar{z}; 0, 0, z'; \omega) \)
• since medium is uniform or layered

\[ S_{\xi}^{\text{PSF}}(x, y, \bar{z}; x', y', z'; \omega) = S_{\xi,0}^{\text{PSF}}(x - x', y - y', \bar{z}; z'; \omega) \quad \text{probe moves left/right} \rightarrow \text{response moves left/right} \]

\[ \mathcal{K}_\xi(x', y', z'; x, y, \bar{z}; \omega) = \frac{S_{\xi,0}^{\text{PSF}}(x - x', y - y', \bar{z}; z'; \omega)}{\kappa \Delta \epsilon_{r,\text{sp}} \Omega_{\text{sp}}} \], \, \xi = 1, \ldots, N_T \]

\[ \Rightarrow S_{\xi}^{\text{sc}}(x, y, \bar{z}; \omega) \approx \frac{1}{\Delta \epsilon_{r,\text{sp}} \Omega_{\text{sp}}} \iiint_{\mathcal{V}_s} \Delta \epsilon_r(x', y', z') S_{\xi,0}^{\text{PSF}}(x - x', y - y', \bar{z}; z'; \omega) \, dx' \, dy' \, dz', \, \xi = 1, \ldots, N_T \]

• the measured response is a convolution of the contrast and the system PSF

• the image reconstruction can then be viewed as a de-convolution process!
CALIBRATION PROCEDURE: PSF EXTRACTION

• system calibration involves two measurements:
  (i) reference object (RO) – incident-field data
  (ii) calibration object (CO) – scattering-probe data

• PSF extraction

  ➢ Born Method

  \[ S_{\xi}^{PSF} (\cdot) \approx \left( S_{\xi}^{PSF} (\cdot) \right)_B = S_{\xi}^{CO} (\cdot) - S_{\xi}^{RO} (\cdot) \]

  ➢ Rytov Method

  \[ S_{\xi}^{PSF} (\cdot) \approx \left( S_{\xi}^{PSF} (\cdot) \right)_R = S_{\xi}^{RO} (\cdot) \ln \left( \frac{S_{\xi}^{CO} (\cdot)}{S_{\xi}^{RO} (\cdot)} \right) \]

  \( \cdot \equiv (x, y, z; \omega), \quad \xi = 1, \ldots, N_T \)
EXAMPLE: PSF

• typical noise-free PSF obtained from simulated RO and CO data

\[ f = 6.5 \text{ GHz} \]


• typical noisy PSF obtained from measurements (magnitude shown)  [Tajik et al., EuCAP 2019]
MEASUREMENT PROCEDURE: DATA EXTRACTION

- data is extracted with the same method as that for PSF – being consistent is important!

- Born Method

\[ S_{\xi}^{\text{sc}}(\cdot) \approx \left( S_{\xi}^{\text{sc}}(\cdot) \right)_B = S_{\xi}^{\text{OUT}}(\cdot) - S_{\xi}^{\text{RO}}(\cdot) \]

- Rytov Method

\[ S_{\xi}^{\text{sc}}(\cdot) \approx \left( S_{\xi}^{\text{sc}}(\cdot) \right)_R = S_{\xi}^{\text{RO}}(\cdot) \ln \left( \frac{S_{\xi}^{\text{OUT}}(\cdot)}{S_{\xi}^{\text{RO}}(\cdot)} \right) \]

\( (\cdot) \equiv (x, y, \bar{z}; \omega), \quad \xi = 1, \ldots, N_T \)
INVERSION WITH MICROWAVE HOLOGRAPHY
\[
S_{\xi}^{\text{sc}}(x, y, \bar{z}; \omega) \approx \frac{1}{\Delta \varepsilon_{r,sp} \Omega_{sp}} \int \int \int \Delta \varepsilon_r(x', y', z') S_{\xi,0}^{\text{PSF}}(x - x', y - y', \bar{z}; z'; \omega) \, dx'dy'dz', \quad \xi = 1, \ldots, N_T
\]

- in Fourier (or \(k\)) space

\[
\tilde{S}_{\xi}(k_x, k_y; \bar{z}; \omega) \approx \frac{\Delta x' \Delta y'}{\Delta \varepsilon_{r,sp} \Omega_{sp}} \int_{z'} \tilde{F}(k_x, k_y; z') \cdot \tilde{S}_{\xi,0z'}^{\text{PSF}}(k_x, k_y; z'; \omega) \, dz'
\]

- discretize integral along \(z'\) into a sum

- we now have a system of equations to solve at each spectral position \(\kappa = (k_x, k_y)\)

\[
\tilde{S}_{\xi}^{(m)}(\kappa) \approx \sum_{n=1}^{N_z} \tilde{f}(\kappa; z_n') \left[ \tilde{S}_{\xi,0}^{\text{PSF}}(\kappa; z_n') \right]^{(m)} \quad m = 1, \ldots, N_{\omega} \quad \xi = 1, \ldots, N_T
\]

\[
\tilde{f}(\kappa; z_n') = \frac{\Delta x' \Delta y' \Delta z'_n}{\Delta \varepsilon_{r,sp} \Omega_{sp}} \cdot \tilde{F}(\kappa; z_n')
\]
INVERSION IN FOURIER SPACE

\[
\tilde{S}^{(m)}_{\zeta}(\kappa) \approx \sum_{n=1}^{N_z} \tilde{f}(\kappa; z'_n) \left[ \tilde{S}^{\text{PSF}}_{\zeta,0z'}(\kappa; z'_n) \right]^{(m)}
\]

\[
m = 1, \ldots, N_\omega
\]

\[
\zeta = 1, \ldots, N_T
\]

• at each discrete point in Fourier space, a small system of equations is solved for a total of \(N_xN_y\) such systems

\[
A(\kappa_{ij}) \cdot f(\kappa_{ij}) = d(\kappa_{ij}) \quad i = 1, \ldots, N_x, \quad j = 1, \ldots, N_y
\]

\[
\begin{bmatrix}
\tilde{S}_{0}^{\text{PSF}}(\kappa_{ij}; z'_1) \\
\vdots \\
\tilde{S}_{0}^{\text{PSF}}(\kappa_{ij}; z'_{N_z})
\end{bmatrix}^{(1)} \cdots 
\begin{bmatrix}
\tilde{S}_{0}^{\text{PSF}}(\kappa_{ij}; z'_1) \\
\vdots \\
\tilde{S}_{0}^{\text{PSF}}(\kappa_{ij}; z'_{N_z})
\end{bmatrix}^{(N_\omega)} 
\cdot 
\begin{bmatrix}
\tilde{f}(\kappa_{ij}; z'_1) \\
\vdots \\
\tilde{f}(\kappa_{ij}; z'_{N_z})
\end{bmatrix}^{(N_\omega)} 
= 
\begin{bmatrix}
\tilde{S}^{(1)}(\kappa_{ij}) \\
\vdots \\
\tilde{S}^{(N_\omega)}(\kappa_{ij})
\end{bmatrix}_{N_\omegaN_T \times 1}
\]

in discrete Fourier space

\[
\kappa_{ij} \equiv (i\Delta k_x, j\Delta k_y)
\]

vectors of response types

\[
[\tilde{S}^{(m)}(\cdot)]^T = [\tilde{S}^{(1)}(\cdot) \cdots \tilde{S}^{(N_\omega)}(\cdot)]
\]

\[
\text{data } d(\kappa) \quad \text{contrast } f(\kappa) \quad \text{system matrix } A(\kappa)
\]
• at each plane along range \((z'_n, n = 1, \ldots, N_z)\)

\[
\Delta \varepsilon_r(x', y', z'_n) = \frac{\Delta \varepsilon_{r,sp} \Omega_{sp}}{\Omega_v} \mathcal{F}_{2D}^{-1} \left\{ \tilde{f}(k; z'_n) \right\}, \quad n = 1, \ldots, N_z
\]

\[
\varepsilon_r(x', y', z'_n) = \varepsilon_{r,b} + \Delta \varepsilon_r(x', y', z'_n)
\]
ADVANTAGES OF SOLVING IN FOURIER SPACE: *DIVIDE AND CONQUER*

- we solve \((N_x \cdot N_y)\) small systems of equations
  - number of solved systems on the order of \(10^4\) to \(10^5\)
- size of each system is small: \(N_T N_\omega \times N_z\) (e.g. \(60 \times 5\))
- typical execution times: 2 to 3 seconds on a laptop using Matlab
- **solution is orders of magnitude faster than solving in real space** where one very large system of equations needs to be solved of size

\[
N_D \times N_v \quad \text{with} \quad N_D = N_x N_y N_\omega N_T \sim 10^7 \text{ to } 10^8 \\
N_v = N_x N_y N_z \sim 10^6 \text{ to } 10^7
\]
### 3-D SIMULATION EXAMPLE: C-shape and 3 Cubes

<table>
<thead>
<tr>
<th>Object</th>
<th>$\varepsilon_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>C-shape</td>
<td>1.5 - i0</td>
</tr>
<tr>
<td>Cubes</td>
<td>1.1 - i0</td>
</tr>
</tbody>
</table>

- 2 dipole antennas aligned along boresight, separated by 10 cm
- Reflection and transmission coefficients acquired

- C-shape is 4 cm from lower dipole
- Cubes are 4 cm, 5 cm, and 6 cm from lower dipole
- Acquisition plane of area 30 cm by 30 cm with 1 cm sampling interval along $x$ and $y$
- Frequency range from 3 GHz to 8 GHz, $\Delta f = 1$ GHz
- Scattering probe in calibration: 1 cm³ cube of $\varepsilon_{r,sp} = 1.1 - i0$

[Tajik et al., JPIER-B 2017]
3-D SIMULATION EXAMPLE: IMAGES OF C-shape and 3 Cubes

Born Approximation

Rytov Approximation

- due to the targets’ low contrast, both approximation yield practically the same images
- quantitative estimate of permittivity distribution is very good
MEASUREMENT EXAMPLE: Teddy Bear (2-D Image)

- scanned area: 29 cm by 29 cm
- sampling step along $x$ and $y$: 5 mm
- only transmission coefficient acquired
- frequency range from 8 GHz to 12 GHz (41 samples)
- two open-end waveguides (WR-90) aligned along boresight
  - measurement 1: just teddy bear
  - measurement 2: two objects inserted in teddy bear

1) dielectric cross: $\varepsilon_r = 12 - i0$ (thickness 1 cm, cross arm 3 cm) inserted in bear’s tummy
2) dielectric L-shaped object: $\varepsilon_r = 10 - i5$ (thickness 1 cm, L arm 2 cm) inserted in right arm
MEASUREMENT EXAMPLE: 2-D IMAGES OF Teddy Bear

normalized contrast

Magnitude

empty

with hidden objects
MEASUREMENT EXAMPLE: 2-D QUANTITATIVE ESTIMATES (Teddy Bear)

permittivity

Real Part of the Permittivity

Imaginary Part of the Permittivity

empty

with hidden objects

with physical constraints
EXAMPLE: 3-D NEAR-ZONE IMAGING OF METALLIC OBJECTS (RE-VISITED)

[Amineh et al., Trans. IM, 2015]

Δx = Δy = 5 mm
Δf = 250 MHz

<table>
<thead>
<tr>
<th>f (GHz)</th>
<th>λ (mm)</th>
<th>D_{far} (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>100</td>
<td>12.5</td>
</tr>
<tr>
<td>8.2</td>
<td>37</td>
<td>34</td>
</tr>
<tr>
<td>20</td>
<td>15</td>
<td>83</td>
</tr>
</tbody>
</table>

X-band (WR90) open-end waveguides (f_c ≈ 6.56 GHz)

scattering probe
metallic targets

[photo credit: Justin McCombe]
EXAMPLE: 3-D NEAR-ZONE IMAGING OF METALLIC OBJECTS (RE-VISITED)

expected spatial resolution

depth: \( \delta_z \approx 10 \text{ mm} \)

lateral: \( \delta_{x,y} \approx 4 \text{ mm} \)
INVERSION WITH SCATTERED POWER MAPPING (SPM)
QUALITATIVE IMAGING WITH SENSITIVITY MAPS

- SPM is rooted in early work on the use of response sensitivities in image reconstruction
  
  

- **response sensitivities**: response derivatives with respect to some system parameters

- in imaging – **derivatives with respect to permittivity** at each voxel (Fréchet derivative)

- assume we want to minimize the $\ell_2$-norm error between the measured OUT (total-field) and RO (incident-field) data

\[
F[\varepsilon(r')] = 0.5 \sum_{\xi=1}^{N_T} \int_{\omega} \int_{r \in S_a} \left[ \left| S_{\xi}^{\text{OUT}}(r, \omega) - S_{\xi}^{\text{RO}}[r, \omega; \varepsilon(r')] \right|^2 \right] d\omega d\omega
\]

- for that we need to know how to properly change the permittivity in the RO $\varepsilon_b(r') \rightarrow \varepsilon(r')$

  ➢ we need to know how the error function $F$ would change when $\varepsilon$ changes at $\forall r_n' \in V_s$

\[
\frac{\partial F}{\partial \varepsilon(r_n')} = ? \quad n = 1, \ldots, N_v
\]
• Fréchet derivative with respect to $\varepsilon(r') = \varepsilon'(r') - i\varepsilon''(r')$

\[
\text{Re}\{D(r')\} = \frac{\partial F}{\partial \varepsilon'(r')}
\]
→ indicates where contrast in $\varepsilon'$ exists in the OUT

\[
\text{Im}\{D(r')\} = -\frac{\partial F}{\partial \varepsilon''(r')}
\]
→ indicates where contrast in $\varepsilon''$ exists in the OUT

where

\[
D(r') = \sum_{\xi=1}^{N_T} \int_\omega \int_{r \in S} \left[ S^{RO}_{\xi}(r, \omega) - S^{OUT}_{\xi}(r, \omega) \right] \cdot \left[ \frac{\partial S^{RO}_{\xi}(r, \omega)}{\partial \varepsilon(r')} \right]^* \, dr \, d\omega
\]

• sensitivity map: 3-D plot of the real and imaginary parts of $D(r')$
SENSITIVITY MAPS: FROM DERIVATIVES TO FINITE DIFFERENCES

\[ D(\mathbf{r}') = \sum_{\xi=1}^{N_T} \int_0^\infty \int_{r \in S_a} \left[ S_{\xi}^{RO} (\mathbf{r}, \omega) - S_{\xi}^{OUT} (\mathbf{r}, \omega) \right] \cdot \left[ \frac{\partial S_{\xi}^{RO} (\mathbf{r}, \omega)}{\partial \varepsilon(\mathbf{r}')} \right]^* \, d\mathbf{r} \, d\omega \]

\[ -S_{\xi}^{sc} (\mathbf{r}, \omega) \quad \text{(Born approximation)} \]

- approximating the response derivative with the PSF

\[ \frac{\partial S_{\xi}^{RO} (\mathbf{r}, \omega)}{\partial \varepsilon(\mathbf{r}')} \approx \frac{\Delta S_{\xi}^{RO} (\mathbf{r}, \omega)}{\Delta \varepsilon(\mathbf{r}')}, \quad \frac{\Delta S_{\xi}^{CO} (\mathbf{r}, \mathbf{r}'; \omega) - S_{\xi}^{RO} (\mathbf{r}, \omega)}{\Delta \varepsilon_{sp} (\mathbf{r}')} = \frac{S_{\xi}^{PSF} (\mathbf{r}, \mathbf{r}'; \omega)}{\Delta \varepsilon_{sp}} \]

\[ \Rightarrow -\Delta \varepsilon_{sp}^* \cdot D(\mathbf{r}') = M(\mathbf{r}') = \sum_{\xi=1}^{N_T} \int_0^\infty \int_{r \in S_a} S_{\xi}^{sc} (\mathbf{r}, \omega) \cdot \left[ S_{\xi}^{PSF} (\mathbf{r}, \mathbf{r}'; \omega) \right]^* \, d\mathbf{r} \, d\omega \]

- scattered power map

\[ \text{position of scattering probe} \]
SCATTERED-POWER MAPS: RECONSTRUCTION FORMULA

\[ M(r') = \sum_{\xi=1}^{N_r} \int_{\omega} \int_{r \in S_a} S_{\xi}^{\text{sc}}(r, \omega) \cdot \left[ S_{\xi}^{\text{PSF}}(r, r'; \omega) \right]^* \, dr \, d\omega \]

- **scattered-power map**: 3-D qualitative image of the OUT contrast relative to the RO

SOME IMPORTANT ADVANTAGES

- reconstruction is practically instantaneous – no systems of equations are solved!
  … reconstruction formula is a simple summation of response products
- reconstruction can be carried out with ANY set of observation points (no need to have acquisition surfaces of canonical shapes (planar, cylindrical, spherical)
  … as long as the PSF is available analytically or from measurements

[Tu et al., Inv. Problems, 2015][Shumakov et al., IEEE Trans. MTT, 2018]
SCATTERED-POWER MAPS AND TEMPORAL CROSS-CORRELATION

- SPM image $M(r')$ can be viewed as a plot of the **aggregate measure of similarity** between the OUT responses and the respective PSF responses due to point scatterer at $r'$

$$M(r') = \sum_{\xi=1}^{N_T} \int \int S_{\xi}^{sc}(r, \omega) \cdot \left[ S_{\xi}^{PSF}(r, r'; \omega) \right]^* \, dr \, d\omega = \sum_{\xi=1}^{N_T} \int \int \mathcal{F}_t \left\{ S_{\xi}^{sc}(r, t) \otimes S_{\xi}^{PSF}(r, r'; t) \right\} \, dr \, d\omega$$

- reminder: cross-correlation is a measure of similarity between 2 waveforms as a function of their mutual time-shift

- it can be shown that (with infinite bandwidth)

$$M(r') \sim \sum_{\xi=1}^{N_T} \int \int X_\xi(r, r'; \tau = 0) \, dr$$

**no shift**
• consider planar scanning and assume lateral translational invariance of the PSF

\[ S_{\xi}^{\text{PSF}} (x, y, \bar{z}; x', y', z'; \omega) \approx S_{\xi 0}^{\text{PSF}} (x - x', y - y', \bar{z}; z'; \omega) \]

\[ M(x', y', z') = \sum_{\xi=1}^{N_T} \int_{\omega} \left( \int_{S_{\xi}} S_{\xi}^{\text{sc}} (x, y, \bar{z}, \omega) \cdot \left[ S_{\xi 0}^{\text{PSF}} (x - x', y - y', \bar{z}; z'; \omega) \right]^* \right) \; dx \; dy \; d\omega \]

cross-correlation in (x, y)

• image reconstruction formula

\[ M(x', y', z') = \mathcal{F}_{2D}^{-1} \left\{ \sum_{\xi=1}^{N_T} \sum_{m=1}^{N_\omega} \tilde{S}_{\xi}^{\text{sc}} (k_x, k_y, \bar{z}, \omega) \cdot \left[ \tilde{S}_{\xi 0}^{\text{PSF}} (k_x, k_y, \bar{z}; z'; \omega) \right]^* \right\} \]
SIMULATION EXAMPLE: QUALITATIVE SPM IMAGE OF F-SHAPE

sample PSF: \( S_{11} \) at 4 GHz MAG/PHASE

Altair FEKO

\[
\varepsilon_{r,b} = 1.0 \\
\varepsilon_{r,sp} = 1.1 \\
\varepsilon_{r,OUT} = 1.2
\]

\( f_{\text{min}} = 3 \text{ GHz} \) \\
\( f_{\text{max}} = 16 \text{ GHz} \) \\
\( \Delta f = 1 \text{ GHz} \)
• blurring typical for cross-correlation methods
  ➢ limited number of responses
  ➢ diffraction limit on resolution
QUANTITATIVE SPM WITH MEASURED PSFs

- quantitative SPM uses the qualitative maps to improve the image quality significantly

\[
M(x', y', z') = \sum_{\xi=1}^{N_T} \int_{\omega} \int_{S_a} S_{\xi}^{sc}(x, y, \bar{z}, \omega) \cdot \left[ S_{\xi 0}^{PSF} (x-x', y-y', \bar{z}; z'; \omega) \right]^* dx dy d\omega
\]

\[
S_{\xi}^{sc}(x, y, \bar{z}; \omega) \approx \frac{1}{\Delta \xi_{r,sp} \Delta \Omega_{sp}} \int_{x'} \int_{y'} \int_{z'} \Delta \varepsilon_r(x', y', z') S_{\xi 0}^{PSF} (x-x', y-y', \bar{z}; z'; \omega) dx' dy' dz' , \quad \xi = 1, \ldots, N_T
\]

\[
M(x', y', z') = \frac{1}{\Delta \xi_{r,sp} \Omega_{sp}} \int_{x''} \int_{y''} \int_{z''} \Delta \varepsilon_r(x'', y'', z'') \cdot M_{sp@(x'',y'',z'')}(x', y', z') dx'' dy'' dz''
\]

qualitative SPM image of OUT
unknown contrast qualitative SPM image of scattering probe when it is at (x'', y'', z'')
• quantitative SPM solves the linear problem

\[
M(x', y', z') = \frac{1}{\Delta \epsilon_{r,sp} \Omega_{sp}} \iiint_{V_s} \Delta \epsilon_r(x'', y'', z'') \cdot M_{sp@(x'', y'', z'')} (x', y', z') \, dx'' \, dy'' \, dz''
\]

• linear system of equations can be quite large in real space: square system of size \(N_v \times N_v\)
• solving in Fourier (\(k\)-) space is much faster (similar to holography)
• assumption of medium \textit{lateral uniformity}: a shift in the position of the scattering probe leads to a corresponding shift in its qualitative map obtained with the central PSF

\[
M_{sp@(x'', y'', z'')} (x', y', z') = M_{sp@(0,0,z''')} (x' - x'', y' - y'', z')
\]

\(\Rightarrow\)

\[
M(x', y', z') = \frac{1}{\Delta \epsilon_{r,sp} \Omega_{sp}} \int \int \int_{z''} \Delta \epsilon_r(x'', y'', z'') \cdot M_{sp@(0,0,z''')} (x' - x'', y' - y'', z') \, dx'' \, dy'' \, dz''
\]

convolution in \((x,y)\)
\[\tilde{M}(k_x, k_y, z_p) = \frac{\Omega_v}{\Delta \varepsilon_{r,sp} \Omega_{sp}} \sum_{q=1}^{N_z} \tilde{f}(k_x, k_y, z_q) \cdot \tilde{M}_{sp@(0,0,z_q)}(k_x, k_y, z_p), \ p = 1, \ldots, N_z\]

- small square system of equations is solved at each spectral position \(\kappa = (k_x, k_y)\)

\[M(\kappa)x(\kappa) = m(\kappa)\]

\[x(\kappa) = \begin{bmatrix} \tilde{f}(\kappa, z_1) & \cdots & \tilde{f}(\kappa, z_{N_z}) \end{bmatrix}^T \]

\[m(\kappa) = \begin{bmatrix} \tilde{M}(\kappa, z_1) & \cdots & \tilde{M}(\kappa, z_{N_z}) \end{bmatrix}^T \]

- final step: back to \((x,y)\) space

\[\Delta \varepsilon_r(x', y', z_n') = \frac{\Delta \varepsilon_{r,sp} \Omega_{sp}}{\Omega_v} \mathcal{F}_{2D}^{-1}\{ \tilde{f}(\kappa; z_n') \}, \ n = 1, \ldots, N_z\]
SIMULATION EXAMPLE: QUANTITATIVE SPM IMAGE OF F-SHAPE

[Shumakov, in Nikolova, *Introduction to Microwave Imaging*, 2017]
EXPERIMENTAL EXAMPLE: QUANTITATIVE SPM

[Shumakov et al., IEEE Trans. MTT, 2018]

- 5 cm thick carbon-rubber sample $\varepsilon_{r,b} \approx 10 - i5$
- frequency: from 3 GHz to 9 GHz (61 samples) $\varepsilon_{r,sp} \approx 15 - i0.003$
- scattering probe
- all embedded objects are 1 cm thick
- reflection and transmission coefficients on two TEM horn antennas aligned along boresight
- imaged area 13 cm by 13 cm (2 mm sampling step)
EXPERIMENTAL EXAMPLE: QUANTITATIVE SPM – IMAGES

[Shumakov et al., IEEE Trans. MTT, 2018]
EXPERIMENTAL EXAMPLE: TEDDY BEAR 2-D IMAGE

normalized permittivity contrast

empty

with hidden objects
INVERSION WITH SYNTHETIC FOCUSING
SYNHETIC FOCUSING: MATCHED FILTERING

• synthetic focusing is the process of cross-correlating each measured signal with a signal, which represents the radar response to a point scatterer at $r'$ (this is just the system PSF)

$$x_\xi (r, r'; t) = S_{sc}^\xi (r, t) \otimes S_{\xi PSF}^r (r, r'; t)$$

• this computation is known as matched filtering – it checks how well a signal “matches” the PSF at $r'$ for all time shifts

• Why is this called “filtering”? in the frequency domain:

$$X(r', \omega) = \sum_{\xi=1}^{N_T} \iint_{r \in S_n} S_{sc}^\xi (r, \omega) \cdot \left[ S_{\xi PSF}^r (r, r'; \omega) \right]^* dr$$

$$S_{sc}^\xi (r, \omega) \cdot H_\xi (r, r'; \omega) = X_\xi (r, r'; \omega)$$

input filter output

transfer function of the matched filter (aka steering filter)
SYNTHETIC FOCUSING: SIGNAL PROCESSING STAGES

- **STAGE 1:** matched filtering of all measured responses with “focus” on \( r' \)
  \[
  x_\xi(r, r'; t) = S^{sc}_\xi(r, t) \otimes S^{PSF}_\xi(r, r'; t) = \hat{h}_\xi(r, r'; t)
  \]

- **STAGE 2:** summing up all \( r' \)-focused responses at each time instant
  \[
  y(r'; t_n) = \sum_{\xi=1}^{N_T} \int_{r \in S_a} x_\xi(r, r'; t_n)dr = \sum_{\xi=1}^{N_T} \int_{r \in S_a} \left[ S^{sc}_\xi(r, t) \otimes S^{PSF}_\xi(r, r'; t) \right]_{(t_n)} dr , \ n = 0,1,\ldots,N_t
  \]

- **STAGE 3:** windowing focused output to suppress radar clutter
  \[
  y_w(r'; t_n) = y(r'; t_n) \cdot w(t_n) , \ n = 0,1,\ldots,N_t
  \]

- **STAGE 4:** calculating scattering intensity at \( r' \)
  \[
  I(r') = \int_t y_w^2(r', t)dt \]
  PLOT as function of \( r' \)
SYNTHETIC FOCUSING: SIGNAL-FLOW SCHEMATIC

[Nikolova, *Introduction to Microwave Imaging*, 2017]
• core computation in synthetic focusing

\[
y(r'; t) = \sum_{\xi=1}^{N_T} \int_{r \in S_a} \left[ S^{sc}_\xi(r) \otimes S^{PSF}_\xi(r, r') \right]_{(t)} \, dr
\]

• time-domain linear model of scattering (inverse FT of the frequency-domain model)

\[
S^{sc}_\xi(r; \omega) \sim \int_{V_s} \Delta \varepsilon_r(r') \cdot S^{PSF}_\xi(r; r'; \omega) \, dr' \Rightarrow S^{sc}_\xi(r; t) \sim \int_{V_s} \Delta \varepsilon_r(r') \cdot S^{PSF}_\xi(r; r'; t) \, dr'
\]

\[\Rightarrow y(r'; t) \sim \int_{V_s} \Delta \varepsilon_r(r'') \cdot \sum_{\xi=1}^{N_T} \int_{r \in S_a} \left[ S^{PSF}_\xi(r, r') \otimes S^{PSF}_\xi(r, r'') \right]_{(t)} \, dr \, dr''\]
\[ y(r';t) \sim \iiint_{r'' \in V_s} \Delta \varepsilon_r(r'') \cdot \sum_{\xi=1}^{N_T} \iiint_{r \in S_a} \left[ S^\text{PSF}_\xi(r, r') \otimes S^\text{PSF}_\xi(r, r'') \right] (t) \, dr \, dr'' \]

- with large number of responses, the strength of \( y(r';t) \) is proportional to the contrast \( \Delta \varepsilon_r(r') \) because the autocorrelation term dominates the integral over \( r'' \)

\[ \text{autocorrelation term: } r'' = r' \Rightarrow \Delta \varepsilon_r(r') \cdot \sum_{\xi=1}^{N_T} \iiint_{r \in S_a} \left[ S^\text{PSF}_\xi(r, r') \otimes S^\text{PSF}_\xi(r, r') \right] (t) \, dr \]

\[ \text{cross-correlation terms: } r'' \neq r' \Rightarrow \Delta \varepsilon_r(r'') \cdot \sum_{\xi=1}^{N_T} \iiint_{r \in S_a} \left[ S^\text{PSF}_\xi(r, r') \otimes S^\text{PSF}_\xi(r, r'') \right] (t) \, dr \]

integration over incoherent weak x-correlations
DELAY AND SUM (DAS) RECONSTRUCTION ALGORITHM

• DAS is the simplest synthetic-focusing algorithm
• it assumes a PSF of the form (same for all response types and antennas)

\[
S_{\text{PSF}}^{\text{DAS}}(r_{Rx}, r_{Tx}; r', t) \sim \delta \left( t + t_0 - \frac{r_{Tx}(r') + r_{Rx}(r')}{v_b} \right) / (r_{Tx}r_{Rx}), \quad \text{where } r_{Tx} = |r' - r_{Tx}|, \ r_{Rx} = |r' - r_{Rx}|
\]

reference time for scattering center at origin

• for a point scatterer at the reference point (origin)

\[
S_{\text{PSF}}^{\text{DAS}}(r_{Rx}, r_{Tx}; r' = 0; t) \equiv S_{0}^{\text{PSF}}(r_{Rx}, r_{Tx}; t)
\sim \delta \left( t + t_0 - \frac{|r_{Tx}| + |r_{Rx}|}{v_b} \right)
\]

\[
\triangleright \text{origin often chosen at furthest point and } t_0 = \frac{|r_{Tx}| + |r_{Rx}|}{v_b} \quad \text{so that } S_{0}^{\text{PSF}}(r_{Rx}, r_{Tx}; t) = \delta(t)
\]
• DAS PSF is a plane wave in the frequency domain

\[ S_{\text{PSF}}^{\text{PSF}}(\mathbf{r}_{\text{Rx}}, \mathbf{r}_{\text{Tx}}; \mathbf{r}'; \omega) = e^{-ik_b(\mathbf{r}_{\text{Tx}} + \mathbf{r}_{\text{Rx}} - \mathbf{r}_0)} \]

where

\[ k_b = \frac{\omega}{v_b} \text{ (wavenumber)} \]

\[ r_0 = |\mathbf{r}_{\text{Tx}}| + |\mathbf{r}_{\text{Rx}}| = v_b t_0 \text{ (signal path through reference point)} \]

• DAS matched filters

\[ H(\mathbf{r}_{\text{Rx}}, \mathbf{r}_{\text{Tx}}; \mathbf{r}'; \omega) = (S_{\text{PSF}}^{\text{PSF}})^* = e^{ik_b(\mathbf{r}_{\text{Tx}} + \mathbf{r}_{\text{Rx}} - \mathbf{r}_0)} \Rightarrow h(\mathbf{r}_{\text{Rx}}, \mathbf{r}_{\text{Tx}}; \mathbf{r}'; t) = \delta\left( t - t_0 + \frac{r'_{\text{Tx}} + r'_{\text{Rx}}}{v_b} \right) \]
• DAS synthetic focusing

\[
y(r'; t) = \sum_{\xi=1}^{N_T} \int_{r \in S_a} S_{\xi}^{SC}(r; t) \ast \delta \left( t - t_0 + \frac{r_{Tx} + r_{Rx}}{v_b} \right) dr
\]

\[
= \sum_{\xi=1}^{N_T} \int_{r \in S_a} \int_{\tau} S_{\xi}^{SC}(r; \tau) \delta \left( t - t_0 + \frac{r_{Tx} + r_{Rx}}{v_b} - \tau \right) d\tau dr
\]

\[
= \sum_{\xi=1}^{N_T} \int_{r \in S_a} S_{\xi}^{SC} \left( r; t - \left( t_0 - \frac{r_{Tx} + r_{Rx}}{v_b} \right) \right) dr
\]

\[
\text{response is "delayed"}
\]

\[
\text{responses collected over aperture are "delayed and sumed"}
\]

• DAS focusing is nothing but a time-shift of the response so as to virtually space-shift the scattering center from \( r' \) to the origin (reference point)
DAS: CONCEPTUAL EXAMPLE

- Target 1: $r'_1 = (-70, 80)$ mm
- Target 2: $r'_2 = (40, 130)$ mm
- Void: $r_{\text{void}} = (0, 100)$ mm
DAS: CONCEPTUAL EXAMPLE

\[ I(r') \approx 2927 \]
\[ I(r'_2) \approx 2927 \]
\[ I(r'_\text{void}) \approx 14 \]

\[ y(r', t) \]

\[ I(r') = \int_t y^2_w(r', t) dt \]

plot image
DAS: 2-D SIMULATION EXAMPLE

- 2-D simulation (TM$_z$ mode)
- point sources
- point probes recording $E_z(t)$

bandwidth at 3 dB from 1.25 GHz to 8.75 GHz
DAS: 2-D SIMULATION EXAMPLE

discussion

• Why the simple DAS algorithm (PSF is a $\delta$-function of time) performs similarly to the more sophisticated x-correlation with the actual simulated PSF?
  ➢ due to the point-like nature of the sources and probes

• Why do we have so many artifacts?
  ➢ because the forward model does not take into account mutual coupling and scattering between close-by targets
  ➢ the Tx total field is not the same as the Tx incident field (as the linearized model assumes) and these differences appear as spurious contrast (image artifacts)
  ➢ because we have only 8 probes – incomplete data!
SUMMARY OF DAY TWO

• real-time imaging methods provide an image within seconds of the data acquisition
• real-time methods substantially rely on linearized forward models
• there are 2 main types of real-time imaging methods
  ➢ direct solution of the data equation (microwave holography)
  ➢ cross-correlation with system PSF (scattered-power mapping & synthetic focusing)
• assumption of uniform or layered medium enables super-fast solution in Fourier space
• however, inversion in Fourier space has some “pitfalls” – we discuss those and the respective mitigation strategies during Day Three of this course

QMH, SPM codes available:
http://www.ece.mcmaster.ca/faculty/nikolova/IntroMicrowaveImaging/MatlabCodes/
We keep updating!
THANK YOU!