LECTURE 2: Introduction into the Theory of Radiation

(*Maxwell's equations – revision. Power density and Poynting vector – revision. Radiated power – definition. Basic principle of radiation. Vector and scalar potentials – revision. Far fields and vector potentials.*)

1. Maxwell's Equations – Revision

(a) the law of induction (Faraday's law):

$$-\nabla\times\mathbf{E}=\frac{\partial\mathbf{B}}{\partial t}+\mathbf{M}^{♣} \qquad (2.1)$$

$$\oint_c \mathbf{E}\cdot d\mathbf{c}=-\frac{\partial}{\partial t}\iint_{S_{[c]}}\mathbf{B}\cdot d\mathbf{s} \quad \Leftrightarrow \quad e=-\frac{\partial\Psi}{\partial t} \qquad (2.1\text{-i})$$

E (V/m)	electric field (electric field intensity)
B (T=Wb/m^2)	magnetic flux density
M (V/m^2)	magnetic current density♣
Ψ (Wb=$V\cdot s$)	magnetic flux
e (V)	electromotive force

(b) Ampere's law, generalized by Maxwell to include the displacement current $\partial\mathbf{D}/\partial t$:

$$\nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t}+\mathbf{J} \qquad (2.2)$$

$$\oint_c \mathbf{H}\cdot d\mathbf{c}=\iint_{S_{[c]}}\left(\frac{\partial\mathbf{D}}{\partial t}+\mathbf{J}\right)\cdot d\mathbf{s} \quad \Leftrightarrow \quad I=\oint_c \mathbf{H}\cdot d\mathbf{c} \qquad (2.2\text{-i})$$

H (A/m)	magnetic field (magnetic field intensity)
D (C/m^2)	electric flux density (electric displacement)
J (A/m^2)	electric current density
I (A)	electric current

♣ **M** is a fictitious quantity, which renders Maxwell's equations symmetrical and which proves a useful mathematical tool when solving EM boundary value problems applying equivalence theorem.

(c) Gauss' law of electricity:

$$\nabla \cdot \mathbf{D} = \rho \tag{2.3}$$

$$\oiint_S \mathbf{D} \cdot d\mathbf{s} = \iiint_{V_{[S]}} \rho dv = Q \tag{2.3-i}$$

ρ (C/m^3) electric charge density

Q (C) electric charge

Equation (2.3) follows from equation (2.2) and the continuity relation:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}. \tag{2.4}$$

Hint: Take the divergence of both sides of (2.2).

(d) Gauss' magnetic law:

$$\nabla \cdot \mathbf{B} = \rho_m \text{♣♣} \tag{2.5}$$

The equation $\nabla \cdot \mathbf{B} = 0$ follows from equation (2.1), provided that $\mathbf{M} = 0$.

Maxwell's equations alone are insufficient to solve for the four vector quantities: **E**, **D**, **H**, and **B** (twelve scalar quantities). Two additional vector equations are needed.

(e) Constitutive relationships

The constitutive relationships describe the properties of matter with respect to electric and magnetic forces.

$$\mathbf{D} = \vec{\vec{\boldsymbol{\varepsilon}}} \cdot \mathbf{E} \tag{2.6}$$

$$\mathbf{B} = \vec{\vec{\boldsymbol{\mu}}} \cdot \mathbf{H}. \tag{2.7}$$

In an anisotropic medium, the dielectric permittivity and the magnetic permeability are *tensors*. In vacuum, which is isotropic, the permittivity and the permeability are constants (or tensors whose *diagonal elements only* are non-zero and are the same): $\varepsilon_0 \approx 8.854187817 \times 10^{-12}$ F/m, $\mu_0 = 4\pi \times 10^{-7}$ H/m. In an isotropic medium, **D** and **E** are collinear, and so are **B** and **H**.

The dielectric properties relate to the electric field (electric force). Dielectric materials with relative permittivity ε_r > 1 are built of atomic/molecular sub-

♣♣ ρ_m is a fictitious quantity introduced via the continuity relation $\nabla \cdot \mathbf{M} = -\partial \rho_m / \partial t$. As per experimental evidence, $\nabla \cdot \mathbf{B} = 0$.

domains, which have the properties of dipoles. In an external electric field, the dipoles tend to orient in such a way that their own fields have a cancellation effect on the external field. The electric force $\mathbf{F}_e = Q\mathbf{E}$ exerted on a test point charge Q_t from a source Q_s in such medium is ε_r times weaker than the electric force of the same source in vacuum.

On the contrary, magnetic materials with relative permeability μ_r > 1 are made of sub-domains, which tend to orient in the external magnetic field in such a way that their own magnetic fields align with the external field. The magnetic force $\mathbf{F}_m = Q_t \mathbf{v} \times \mathbf{B}$ exerted on a moving (with velocity $\mathbf{v}$) test point charge Q_t in such a medium is μ_r times stronger than the force that this same source (e.g. electric currents) would create in vacuum.

We are mostly concerned with isotropic media, i.e., media where the equations $\mathbf{B} = \mu_0 \mu_r \mathbf{H}$ and $\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E}$ hold.

(f) Time-harmonic field analysis

In harmonic analysis of EM fields, the field phasors are introduced:

$$\begin{aligned} \mathbf{e}(x,y,z,t) &= \mathrm{Re}\left\{\mathbf{E}(x,y,z)e^{j\omega t}\right\} \\ \mathbf{h}(x,y,z,t) &= \mathrm{Re}\left\{\mathbf{H}(x,y,z)e^{j\omega t}\right\}. \end{aligned} \tag{2.8}$$

For example, the phasor of $e(x,y,z,t) = E_m(x,y,z)\cos(\omega t + \varphi_E)$ is $E(x,y,z) = E_m e^{j\varphi_E}$. For clarity, from this point on, we will denote time-dependent field quantities with lower-case letters (bold for vectors), while their phasors will be denoted with upper-case letters. Complex-conjugate phasors will be denoted with an asterisk *.

The frequency-domain Maxwell equations are obtained from the time-dependent equations using the following correspondences:

$$f(x,y,z,t) \doteq F(x,y,z)$$

$$\frac{\partial f_{(x,y,z,t)}}{\partial t} \doteq j\omega F(x,y,z)$$

$$\frac{\partial f}{\partial \xi} \doteq \frac{\partial F}{\partial \xi} \quad , \quad \xi = x,y,z \ .$$

Thus, Maxwell's equations in phasor form are:

$$\nabla \times \mathbf{H} = j\omega\tilde{\varepsilon}\mathbf{E} + \mathbf{J}\,, \quad \tilde{\varepsilon} = \varepsilon' - j\left(\varepsilon'' + \sigma/\omega\right) \tag{2.9}$$

$$-\nabla \times \mathbf{E} = j\omega\tilde{\mu}\mathbf{H} + \mathbf{M}\,, \ \tilde{\mu} = \mu' - j\mu''. \tag{2.10}$$

These equations include the equivalent (fictitious) magnetic currents **M**. The imaginary part of the *complex dielectric permittivity* $\tilde{\varepsilon}$ describes loss. Often, the dielectric loss is represented by the dielectric loss angle δ_{d}:

$$\tilde{\varepsilon} = \varepsilon'\left[1 - j\left(\frac{\varepsilon''}{\varepsilon'} + \frac{\sigma}{\omega\varepsilon'}\right)\right] = \varepsilon'\left[1 - j\left(\tan\delta_{\mathrm{d}} + \frac{\sigma}{\omega\varepsilon'}\right)\right]. \quad (2.11)$$

Since it is difficult to separate conduction loss (σ) from polarization loss (ε''), usually the high-frequency loss is represented with only one *effective* loss parameter: σ_{eff}, $\varepsilon''_{\mathrm{eff}}$, or $\tan\delta_{\mathrm{d}}$. We can switch between these parameters using:

$$\sigma_{\mathrm{eff}} = \omega\varepsilon'' = \omega\varepsilon_0\varepsilon''_{\mathrm{r}} = \omega\varepsilon_0\varepsilon'_{\mathrm{r}}\tan\delta_{\mathrm{d}} \quad (2.12)$$

or

$$\varepsilon''_{\mathrm{eff}} = \varepsilon_0\varepsilon''_{\mathrm{r,eff}} = \sigma / \omega. \quad (2.13)$$

Similarly, the magnetic loss is described by the imaginary part of the *complex magnetic permeability* $\tilde{\mu}$ or by the magnetic loss angle δ_{m}:

$$\tilde{\mu} = \mu' - j\mu'' = \mu'\left(1 - j\frac{\mu''}{\mu'}\right) = \mu'(1 - j\tan\delta_{\mathrm{m}}). \quad (2.14)$$

In antenna theory, we are mostly concerned with *isotropic*, *homogeneous* and *loss-free* propagation media.

The complex permittivity and permeability determine the intrinsic *propagation constant* $\gamma = \alpha + j\beta$ of a medium since

$$\gamma = \alpha + j\beta = j\omega\sqrt{\tilde{\mu}\tilde{\varepsilon}}. \quad (2.15)$$

It is also customary to describe the medium through the (complex) wavenumber *k*, which relates to the propagation constant as $\gamma = jk$, and, thus $k = \omega\sqrt{\tilde{\mu}\tilde{\varepsilon}}$.

The penetration of the high-frequency waves into conductive (or lossy) media is often described in terms of the penetration (or skin) depth δ_{s}, which is defined by

$$\delta_{\mathrm{s}} = 1/\alpha = 1/\operatorname{Re}\gamma. \quad (2.16)$$

This is the depth at which the field strength is *e* (2.71828…) times weaker compared to its value upon entering the medium. In the case of very good conductors, e.g., metals, for which $\sigma \gg \omega\varepsilon'$, there is a much simpler (but approximate) formula, namely,

$$\delta_{\mathrm{s}} = \frac{1}{\sqrt{\pi f \mu\sigma}}. \quad (2.17)$$

2. Power Density, Poynting Vector, Radiated Power

2.1. Poynting vector – revision

In the time-domain analysis, the Poynting vector is defined as

$$\mathbf{p}(t)=\mathbf{e}(t)\times\mathbf{h}(t),\ \text{W/m}^2. \tag{2.18}$$

As follows from *Poynting's theorem*, **p** is a vector representing the **density** and the **direction** of the **EM power flow**. Thus, the total power leaving certain volume V is obtained as

$$\Pi(t)=\oiint_{S_{[V]}}\mathbf{p}(t)\cdot d\mathbf{s},\ \text{W}. \tag{2.19}$$

Since

$$\mathbf{e}(t)=\text{Re}\left\{\mathbf{E}e^{j\omega t}\right\}=\frac{1}{2}\left(\mathbf{E}e^{j\omega t}+\mathbf{E}^*e^{-j\omega t}\right), \tag{2.20}$$

$$\mathbf{h}(t)=\text{Re}\left\{\mathbf{H}e^{j\omega t}\right\}=\frac{1}{2}\left(\mathbf{H}e^{j\omega t}+\mathbf{H}^*e^{-j\omega t}\right), \tag{2.21}$$

the instantaneous power-flow density can be represented as

$$\mathbf{p}(t)=\underbrace{0.5\,\text{Re}\left\{\mathbf{E}\times\mathbf{H}^*\right\}}_{\mathbf{p}_{av}}+0.5\,\text{Re}\left\{\mathbf{E}\times\mathbf{H}\cdot e^{2\cdot j\omega t}\right\}\ \text{W/m}^2. \tag{2.22}$$

In a time-domain form, this can be written as:

$$\mathbf{p}(t)=\underbrace{0.5(\hat{\mathbf{e}}\times\hat{\mathbf{h}})E_0H_0[\cos\Delta\varphi}_{p_{av}}-\cos(2\omega t+\Delta\varphi)] \tag{2.23}$$

where $\Delta\varphi=\varphi_E-\varphi_H$ is the phase difference between the electric and magnetic fields.

The first term in (2.22) and (2.23), $\mathbf{p}_{av}$, has no time dependence. It is the time-average value, about which the power flux density fluctuates in time with double frequency 2ω. The time-average Poynting vector $\mathbf{p}_{av}$ is a vector of constant value and direction because it is the time-averaged flow of EM power density. It is used to calculate the *active (or time-average) power flow* as

$$\Pi_{av}=\oiint_{S_{[V]}}\mathbf{p}_{av}\cdot d\mathbf{s},\ \text{W}. \tag{2.24}$$

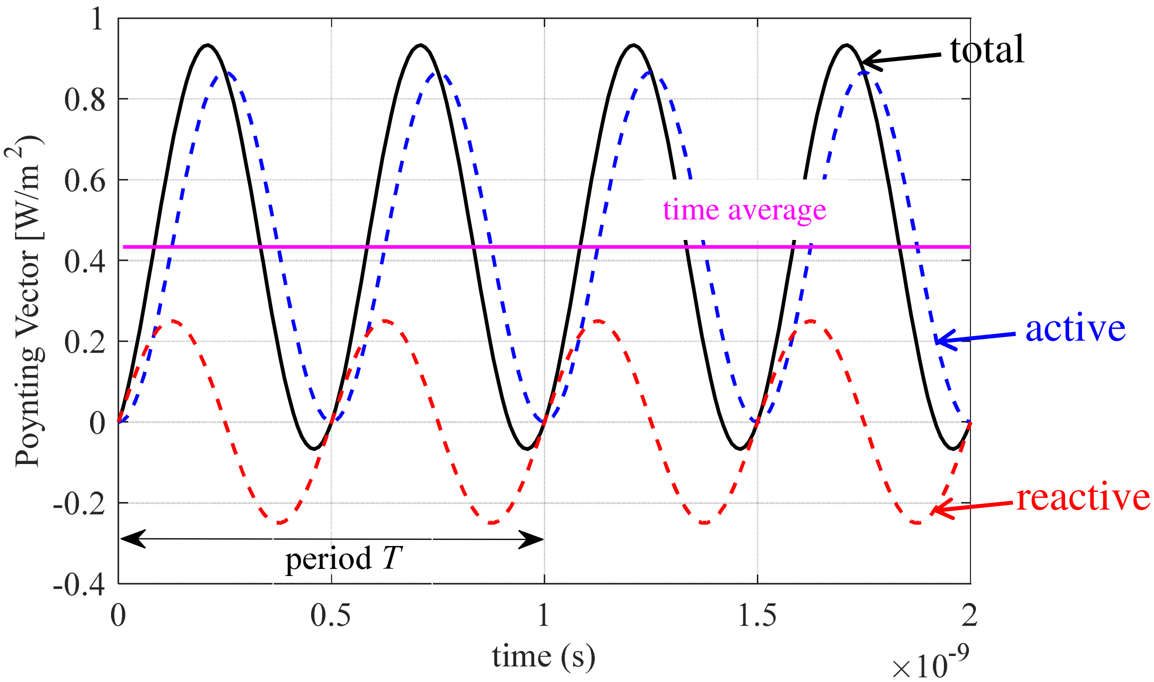
The second term in (2.22) and (2.23) is a vector changing its value and direction with a double frequency 2ω. It describes power flow, which fluctuates in space (propagates *to and fro*) without contribution to the power transport.

If there is no phase difference between $\mathbf{E}$ and $\mathbf{H}$, $\mathbf{p}(t)$ always maintains the same direction (the direction of the ***out***going wave relative to the antenna) although it changes in intensity. This is because the 2nd term in (2.22) does not exceed in magnitude the first term, $\mathbf{p}_{av}$. This indicates that the power moves away from the source at every instant of time, with the Poynting vector never directed toward the source. However, if **E** and **H** are out of phase ($\Delta\varphi = \varphi_E - \varphi_H \neq 0$), during certain time periods the Poynting vector does reverse its direction toward the source.

In fact, the time-dependent Poynting vector can be decomposed into two parts: (i) a positive (**active**) part fluctuating with double frequency about the time-average value $\mathbf{p}_{av} = 0.5 \cdot \mathrm{Re}\{\mathbf{E} \times \mathbf{H}^*\}$, swinging between zero and $E_0 H_0 \cos\Delta\varphi$, and (ii) **reactive** part of magnitude $0.5 \cdot \mathrm{Im}\{\mathbf{E} \times \mathbf{H}^*\}$, which also fluctuates with double frequency and has zero time-average value:

$$\mathbf{p}(t) = 0.5(\hat{\mathbf{e}} \times \hat{\mathbf{h}}) E_0 H_0 \cdot \left\{ \cos\Delta\varphi \cdot [1 - \cos(2\omega t)] + \sin\Delta\varphi \cdot \sin(2\omega t) \right\} \text{ W/m}^2. \quad (2.25)$$

Example for the time-domain Poynting vector: $f = 1$ GHz, $\Delta\varphi = 30°$, $E_0 H_0 = 1$



Definition: The complex Poynting vector is the vector

$$\mathbf{P} = 0.5\mathbf{E} \times \mathbf{H}^*, \quad (2.26)$$

the real part of which is equal to the time-average power flux density: $\mathbf{p}_{av} = \mathrm{Re}\,\mathbf{P}$.

2.2. Radiated power

Definition: Radiated power is the time-average power radiated by the antenna:

$$\Pi_{rad} = \oiint_{S_{[V]}} \mathbf{p}_{av} \cdot d\mathbf{s} = \oiint_{S_{[V]}} \mathrm{Re}\,\mathbf{P} \cdot d\mathbf{s} = \frac{1}{2} \oiint_{S_{[V]}} \mathrm{Re}\left\{\mathbf{E} \times \mathbf{H}^*\right\} \cdot d\mathbf{s}. \quad (2.27)$$

3. Basic Principle of Radiation

Radiation is produced by accelerated or decelerated charge (time-varying current element, infinitesimal dipole, Hertzian dipole).

3.1. Current element

Definition: A current element ($I\Delta l$), $\mathrm{A} \times \mathrm{m}$, is a filament of length Δl carrying current I. It is a fairly abstract concept as it features constant magnitude of the current along Δl.

The time-varying current element is the elementary source of EM radiation. It has fundamental significance in radiation theory similar to the fundamental concept of a point charge in electrostatics. The field radiated by a complex antenna in a linear medium can be analyzed using the superposition principle after decomposing the antenna into elementary sources, i.e., current elements.

The time-dependent current density vector $\mathbf{j}$ depends on the charge density ρ and its velocity $\mathbf{v}$ as

$$\mathbf{j} = \rho \cdot \mathbf{v}, \quad \mathrm{A/m^2}. \quad (2.28)$$

If the current flows along a wire of cross-section ΔS, then the product $\rho_l = \rho \cdot \Delta S$ [C/m] is the charge per unit length (charge line density) along the wire. Thus, for the current $i = \mathbf{j} \cdot \Delta\mathbf{S}$ it follows that

$$i = v \cdot \rho_l, \ \mathrm{A}. \quad (2.29)$$

Then

$$\frac{di}{dt} = \rho_l \frac{dv}{dt} = \rho_l \cdot a, \ \mathrm{A/s}, \quad (2.30)$$

where a (m/s^2) is the acceleration of the charge. The time-derivative of a current element $i\Delta l$ is then proportional to the amount of charge q enclosed in the volume of the current element and to its acceleration:

$$\Delta l \frac{di}{dt} = \Delta l \cdot \rho_l \cdot a = q \cdot a, \ \mathrm{A \times m/s}. \quad (2.31)$$

3.2. Mathematical description of the accelerated charge as a radiation source

It is not immediately obvious from Maxwell's equations that the time-varying current is the source of radiation. A transformation of the time-dependent Maxwell equations,

$$\left|\begin{aligned} &-\nabla\times\mathbf{e}=\mu\frac{\partial\mathbf{h}}{\partial t} \\ &\nabla\times\mathbf{h}=\varepsilon\frac{\partial\mathbf{e}}{\partial t}+\mathbf{j}\ , \end{aligned}\right. \tag{2.32}$$

into a single second-order equation either for **E** or for the **H** field proves this statement. By taking the curl of both sides of the first equation in (2.32) and by making use of the second equation in (2.32), we obtain

$$\nabla\times\nabla\times\mathbf{e}+\mu\varepsilon\frac{\partial^2\mathbf{e}}{\partial t^2}=-\mu\frac{\partial\mathbf{j}}{\partial t}. \tag{2.33}$$

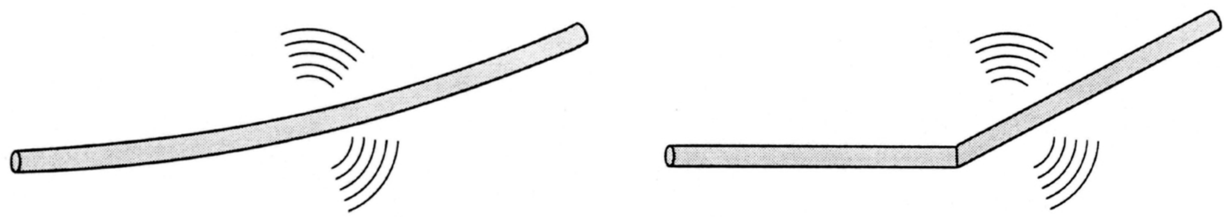
From (2.33), it is obvious that the time derivative of the electric current is the source for the wave-like vector $\mathbf{e}(\mathbf{r},t)$. Time-constant currents do not radiate. In an analogous way, one can obtain the wave equation for the magnetic field **H** and its sources:

$$\nabla\times\nabla\times\mathbf{h}+\mu\varepsilon\frac{\partial^2\mathbf{h}}{\partial t^2}=\nabla\times\mathbf{j}. \tag{2.34}$$

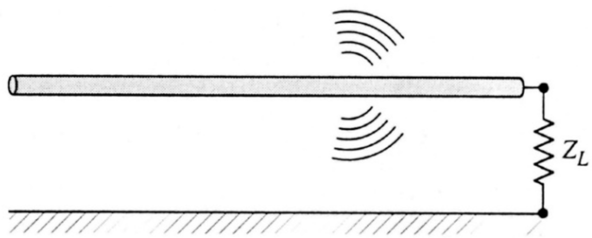
Notice that curl-free currents (e.g., $\mathbf{j}=\nabla\psi$) do not radiate either.

To accelerate/decelerate the charges, one needs sources of electromotive force and/or discontinuities of the medium in which the charges move. Such discontinuities can be bends or open ends of wires, change in the electrical properties of the region, etc. In summary:

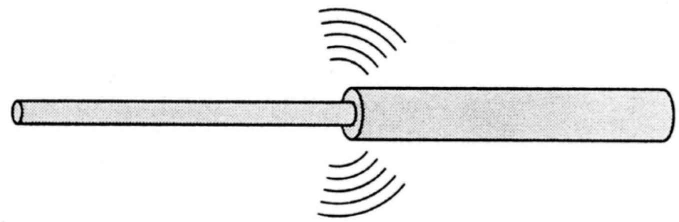
- If charge is not moving, current is zero ⇒ no radiation.
- If charge is moving with a uniform velocity (DC) ⇒ no radiation.
- If charge is accelerated due to electromotive force or due to discontinuities, such as terminations, bends, curvatures ⇒ radiation occurs.



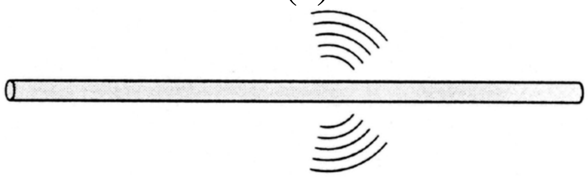
(a) (b)



(c)

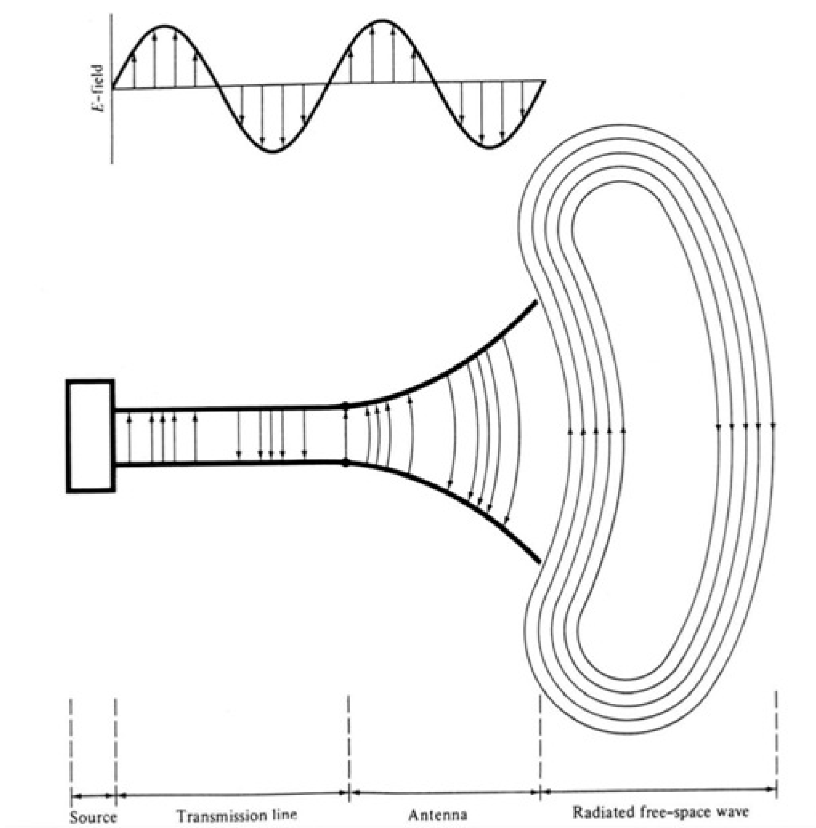


(d)

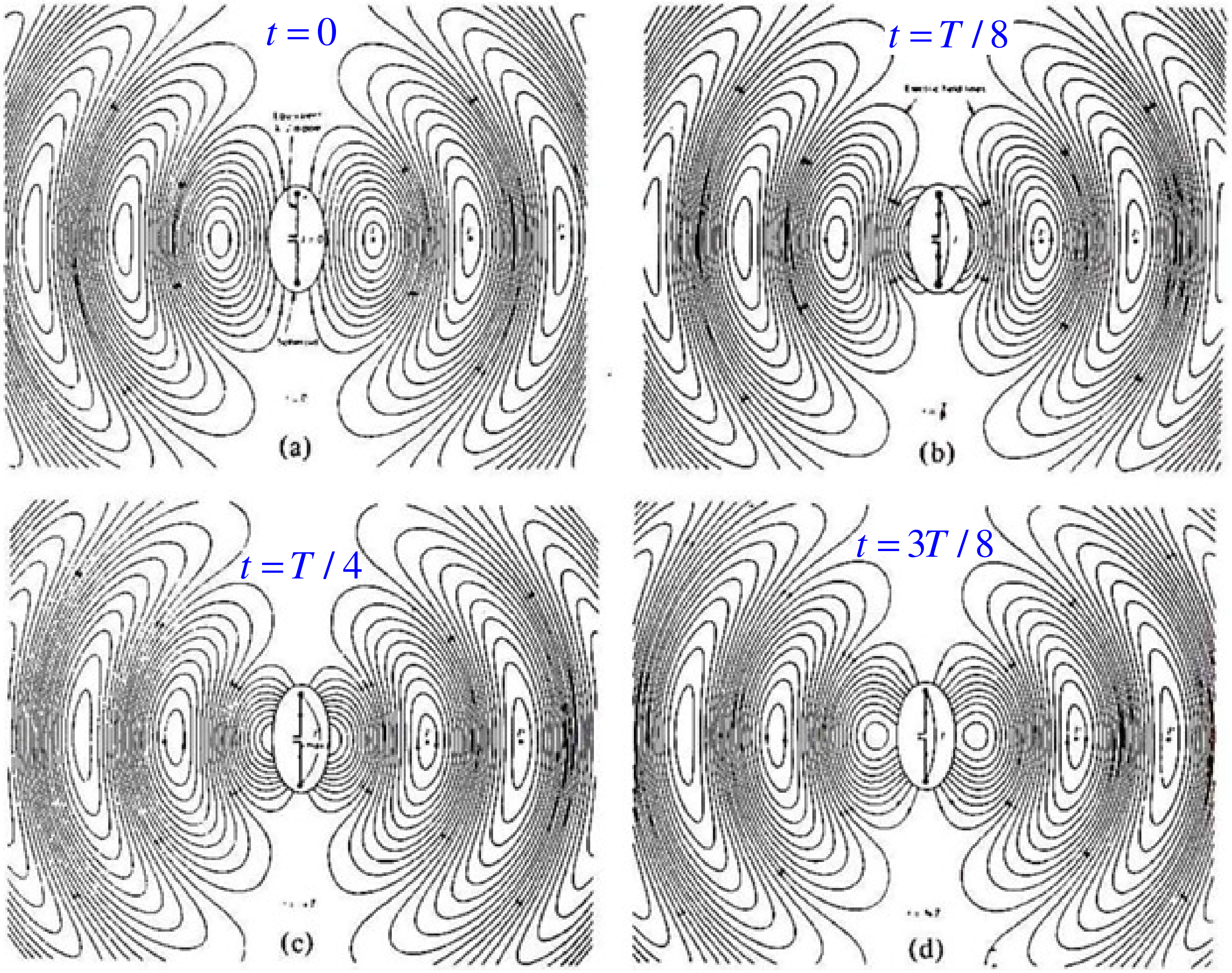


(e)

3.1. Intuitive representation of radiation from simple sources



(a) Illustration of the **E**-field lines in a transmission (feed) line and at the antenna aperture [Balanis, 3rd ed.]



(b) Snapshots of the **E**-field lines around a dipole

(c) animations of the E-field lines of infinitesimal dipole
[https://www.en.didaktik.physik.uni-muenchen.de/multimedia/dipolstrahlung/index.html]

4. Radiation Boundary Condition

With few exceptions, antennas are assumed to radiate in open (unbounded) space. This is a critical factor determining the field behavior. Often, the EM sources (currents and charges on the antenna) are more or less accurately known. These sources are then assumed to radiate in unbounded space and the resulting EM field is determined from integrals over the currents on the antenna. Such problems, where the field sources are known and the resulting field is to be determined are called *analysis* (*forward*, *direct*) problems.[1] To ensure the uniqueness of the solution in an unbounded analysis problem, we have to impose

[1] The inverse (or design) problem is the problem of finding the sources of a known field.

the radiation boundary condition (RBC) on the EM field vectors, i.e., for distances far away from the source ($r \to \infty$),

$$\begin{aligned} & r(\mathbf{E} - \eta \mathbf{H} \times \hat{\mathbf{r}}) \to 0, \\ & r(\mathbf{H} - \frac{1}{\eta} \hat{\mathbf{r}} \times \mathbf{E}) \to 0 \ . \end{aligned} \quad (2.35)$$

The above RBC is known as the *Sommerfeld* vector RBC or the *Silver-Müller* RBC. Here, η is the intrinsic impedance of the medium; $\eta = \sqrt{\mu_0 / \varepsilon_0} \approx 377\ \Omega$ in vacuum.

Antenna analysis benefits from the introduction of auxiliary ***vector potential functions***, which allow simpler and more compact solutions.

It is customary to perform the EM analysis for the case of time-harmonic fields, i.e., in terms of phasors. This course adheres to this tradition. Therefore, from now on, all field quantities (vectors and scalars) are to be understood as complex phasor quantities, the absolute values of which correspond to the ***magnitudes*** (not the RMS value!) of the respective sine waves.

5. Vector and Scalar Potentials – Review

In radiation theory, the potential functions are almost exclusively in the form of retarded potentials, i.e., the magnetic vector potential **A** and its scalar counterpart Φ form a 4-potential in space-time and they are related through the Lorenz gauge. We next introduce the retarded potentials.

5.1. <u>The magnetic vector potential **A**</u>

We first consider only electric sources (**J** and ρ, $\nabla \cdot \mathbf{J} = -j\omega\rho$).

$$\begin{aligned} & \nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}, \\ & \nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J}. \end{aligned} \quad (2.36)$$

Since $\nabla \cdot \mathbf{B} = 0$, we can assume that

$$\mathbf{B} = \nabla \times \mathbf{A} \,. \quad (2.37)$$

Substituting (2.37) in (2.36) yields

$$\begin{aligned} & \mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi, \\ & j\omega\varepsilon\mathbf{E} = \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) - \mathbf{J}. \end{aligned} \quad (2.38)$$

From (2.38), a single equation can be written for **A** in a uniform medium:

$$\nabla \times \nabla \times \mathbf{A} + j\omega\mu\varepsilon(j\omega\mathbf{A} + \nabla\Phi) = \mu\mathbf{J} \,. \quad (2.39)$$

Here, Φ denotes the electric scalar potential, which plays an essential role in the analysis of electrostatic fields. To uniquely define **A**, we need to define not only its curl, but also its divergence. There are no restrictions in defining $\nabla \cdot \mathbf{A}$. Since $\nabla\times\nabla\times = \nabla\nabla\cdot - \nabla^2$, equation (2.39) can be simplified by assuming that

$$\nabla \cdot \mathbf{A} = -j\omega\mu\varepsilon\Phi . \tag{2.40}$$

Equation (2.40) is known as the *Lorenz gauge*. It reduces (2.39) to

$$\nabla^2\mathbf{A} + \omega^2\mu\varepsilon\mathbf{A} = -\mu\mathbf{J} . \tag{2.41}$$

If the region is lossless, then μ and ε are real, and (2.41) can be written as

$$\nabla^2\mathbf{A} + \beta^2\mathbf{A} = -\mu\mathbf{J} , \tag{2.42}$$

where $\beta = \omega\sqrt{\mu\varepsilon}$ is the *phase constant* of the medium. If the region is lossy, the complex permittivity $\tilde{\varepsilon}$ and permeability $\tilde{\mu}$ are introduced. Then, (2.41) becomes

$$\nabla^2\mathbf{A} - \gamma^2\mathbf{A} = -\mu\mathbf{J} . \tag{2.43}$$

Here, $\gamma = \alpha + j\beta = j\omega\sqrt{\tilde{\mu}\tilde{\varepsilon}}$ is the *propagation constant* and α is the *attenuation constant*.

5.2. <u>The electric vector potential **F**</u>

The magnetic field is a solenoidal field, i.e., $\nabla \cdot \mathbf{B} = 0$, because there are no magnetic charges. Therefore, there are no physically existing magnetic currents either. However, the fictitious (equivalent) magnetic currents (density is denoted as **M**) are a useful tool for antenna analysis when applied with the equivalence principle. These currents are introduced in Maxwell's equations in a manner dual to that of the electric currents **J**. Now, we consider the field due to *magnetic sources only*, i.e., we set $\mathbf{J} = 0$ and $\rho = 0$, and therefore, $\nabla \cdot \mathbf{D} = 0$. Then, the system of Maxwell's equations is

$$\left| \begin{array}{l} \nabla\times\mathbf{E} = -j\omega\mu\mathbf{H} - \mathbf{M}, \\ \nabla\times\mathbf{H} = j\omega\varepsilon\mathbf{E} \;. \end{array} \right. \tag{2.44}$$

Since **D** here is solenoidal (i.e. $\nabla \cdot \mathbf{D} = 0$), it can be expressed as the curl of a vector, namely, the *electric vector potential* **F**:

$$\mathbf{D} = -\nabla\times\mathbf{F} . \tag{2.45}$$

Equation (2.45) is substituted into (2.44). All mathematical transformations are analogous to those made in Section 5.1. Finally, it is shown that a field due to magnetic sources **M** is described by the vector **F** alone, where **F** satisfies

$$\nabla^2\mathbf{F} + \omega^2\mu\varepsilon\mathbf{F} = -\varepsilon\mathbf{M} \tag{2.46}$$

provided that the Lorenz gauge is imposed as

$$\nabla \cdot \mathbf{F} = -j\omega\mu\varepsilon\Psi \,. \tag{2.47}$$

Here, Ψ is the magnetic scalar potential.

In a linear medium, a field due to both types of sources (magnetic and electric) can be found by superimposing the partial field due to the electric sources only and that due to the magnetic sources only.

TABLE 2.1: FIELD VECTORS IN TERMS OF VECTOR POTENTIALS

Magnetic vector-potential **A** (electric sources only)	Electric vector-potential **F** (magnetic sources only)
$\mathbf{B} = \nabla\times\mathbf{A}$, $\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A}$ $\mathbf{E} = -j\omega\mathbf{A} - \frac{j}{\omega\mu\varepsilon}\nabla\nabla\cdot\mathbf{A}$ or $\mathbf{E} = \frac{1}{j\omega\mu\varepsilon}\nabla\times\nabla\times\mathbf{A} - \frac{\mathbf{J}}{j\omega\varepsilon}$	$\mathbf{D} = -\nabla\times\mathbf{F}$, $\mathbf{E} = -\frac{1}{\varepsilon}\nabla\times\mathbf{F}$ $\mathbf{H} = -j\omega\mathbf{F} - \frac{j}{\omega\mu\varepsilon}\nabla\nabla\cdot\mathbf{F}$ or $\mathbf{H} = \frac{1}{j\omega\mu\varepsilon}\nabla\times\nabla\times\mathbf{F} - \frac{\mathbf{M}}{j\omega\mu}$

6. Retarded Potentials – Review

Retarded potential is a term usually used to denote the solution of the inhomogeneous Helmholtz' equation (in the frequency domain) or that of the inhomogeneous wave equation (in the time domain) in an unbounded region.

Consider the z-directed electric current density $\mathbf{J} = \hat{\mathbf{z}}\ J_z$. According to (2.42) the magnetic vector potential **A** is also z-directed and is governed by the following equation in a lossless medium:

$$\nabla^2 A_z + \beta^2 A_z = -\mu J_z \,. \tag{2.48}$$

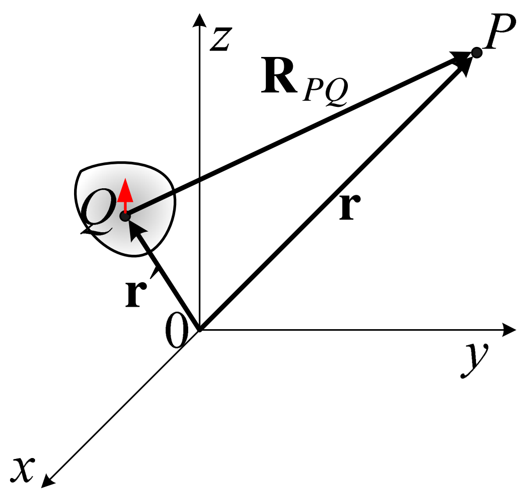
Eq. (2.48) is a Helmholtz equation and its solution in open space is determined by the integral

$$A_z(P) = \iiint_{V_Q} G(P,Q)\cdot[-\mu J_z(Q)]\,dv_Q \tag{2.49}$$

where $G(P,Q)$ is the open-space Green's function of the Helmholtz equation (see the Appendix), P is the observation point, and Q is the source point. Substituting (2.89) from the Appendix into (2.49) gives

$$A_z(P) = \iiint_{V_Q} \mu J_z(Q) \cdot \left(\frac{e^{-j\beta R_{PQ}}}{4\pi R_{PQ}} \right) dv_Q \tag{2.50}$$

where R_{PQ} is the distance between P and Q.



To further generalize the above formula, one assumes the existence of source currents of arbitrary directions, which would produce partial magnetic vector potentials in any direction. Note that a current element in the $\hat{\xi}$ direction results in a vector potential $\mathbf{A} = A_\xi \hat{\xi}$ in the same direction (unless the medium is inhomogeneous and/or anisotropic). Thus,

$$\mathbf{A}(P) = \iiint_{V_Q} \mu \mathbf{J}(Q) \left(\frac{e^{-j\beta R_{PQ}}}{4\pi R_{PQ}} \right) dv_Q . \tag{2.51}$$

The solution for the electric vector potential due to magnetic current sources $\mathbf{M}(Q)$ is analogous:

$$\mathbf{F}(P) = \iiint_{V_Q} \varepsilon \mathbf{M}(Q) \left(\frac{e^{-j\beta R_{PQ}}}{4\pi R_{PQ}} \right) dv_Q . \tag{2.52}$$

Finally, we recall that not only *volume* sources are used to model current distributions. A useful approximation, especially for currents on a conductor surface, is the *surface* current density (or simply surface current):

$$\mathbf{J}_s(x, y) = \lim_{\delta \to 0} \int_{-\delta/2}^{\delta/2} \mathbf{J}(x, y, z) dz \text{, A/m.} \tag{2.53}$$

The magnetic vector potential **A** produced by distributed surface currents is then expressed as

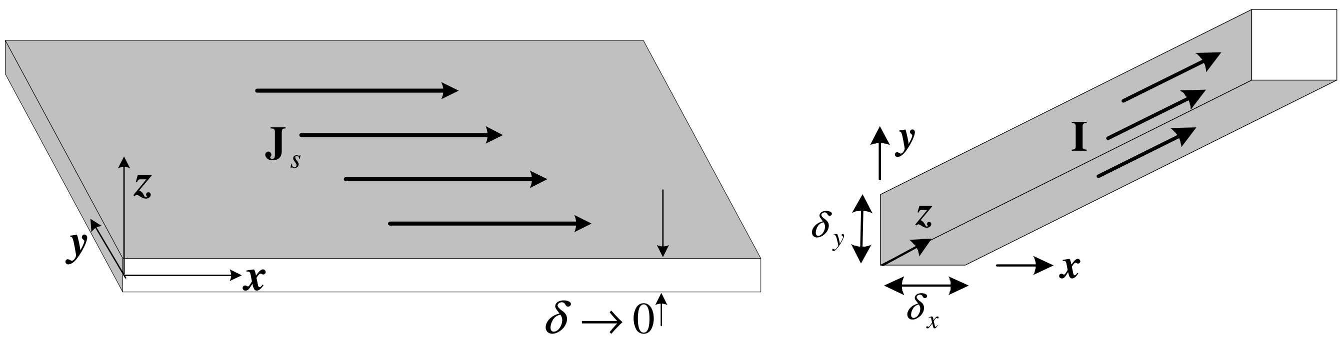
$$\mathbf{A}(P) = \iint_{S_Q} \mu \mathbf{J}_s(Q) \left(\frac{e^{-j\beta R_{PQ}}}{4\pi R_{PQ}} \right) ds_Q . \tag{2.54}$$

Currents on a very thin wire are usually approximated by a linear source, which is the current I flowing through the wire:

$$\mathbf{I}(z) = I(z)\mathbf{a}_l(z) = \lim_{\substack{\delta_x \to 0 \\ \delta_y \to 0}} \iint_{\delta_x \delta_y} \mathbf{J}(x, y, z) dx dy \text{, A.} \tag{2.55}$$

The vector potential of a line current is

$$\mathbf{A}(P) = \int_{L_Q} \mu I(Q) \mathbf{a}_l(Q) \left(\frac{e^{-j\beta R_{PQ}}}{4\pi R_{PQ}} \right) dl_Q . \tag{2.56}$$



(a) surface current on a sheet (b) linear current on a thin wire

7. Far Fields and Vector Potentials

7.1. Potentials

Antennas are sources of finite physical dimensions. The further away from the antenna the observation point is, the more the wave looks like a spherical wave (locally) and the more the antenna looks like a (directed) point source regardless of its actual shape. For such observation distances, we talk about *far field* and *far zone*. The exact meaning of these terms will be discussed later. For now, we will simply accept that the vector potentials behave locally like spherical waves, when the observation point is far from the source:

$$\mathbf{A} \approx \underbrace{\left[\hat{\mathbf{r}} A_r(\theta,\varphi) + \hat{\boldsymbol{\theta}} A_\theta(\theta,\varphi) + \hat{\boldsymbol{\varphi}} A_\varphi(\theta,\varphi) \right]}_{\substack{\text{dependence on observation angles only} \\ \text{(directionality of wave)}}} \cdot \underbrace{\frac{e^{-jkr}}{r}}_{\substack{\text{dependence on distance only} \\ \text{(spherical-wave dependence)}}}, \quad r \to \infty . \tag{2.57}$$

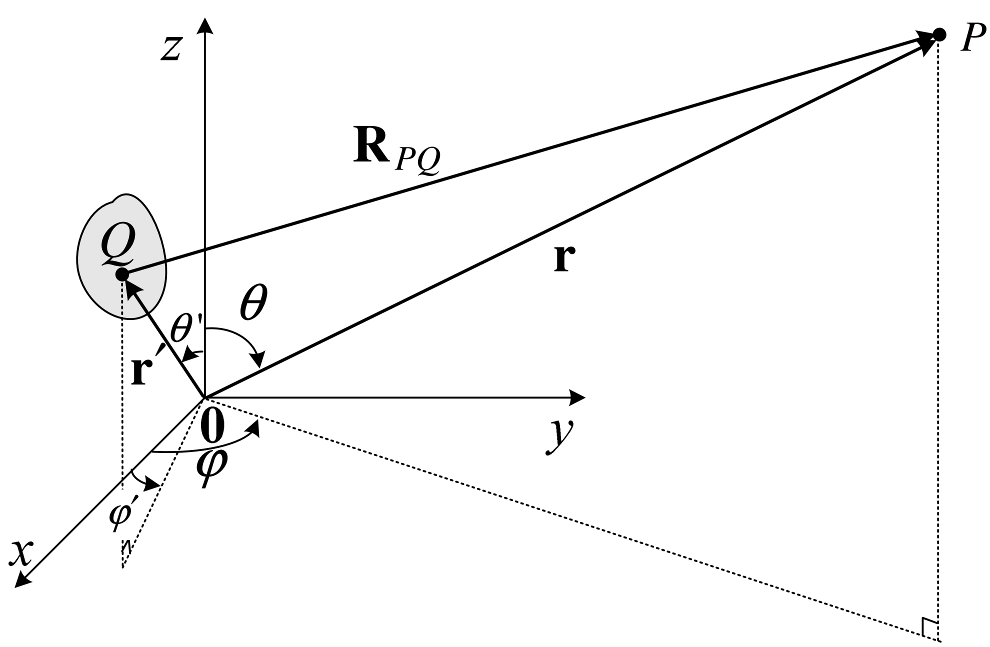
Here, $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}})$ are the unit vectors of the spherical coordinate system (SCS) centered on the antenna and $k = \omega\sqrt{\mu\varepsilon}$ is the wave number (or the phase constant). The phase-delay (retardation) term e^{-jkr} shows propagation along $\hat{\mathbf{r}}$ away from the antenna at the speed of light. The amplitude-decay term $1/r$ shows the potential's decrease in strength with distance.

Notice an important feature of the far-field potential: the dependence on the distance r is separable from the dependence on the observation angle (θ,φ), and it is the same for any antenna: e^{-jkr}/r.

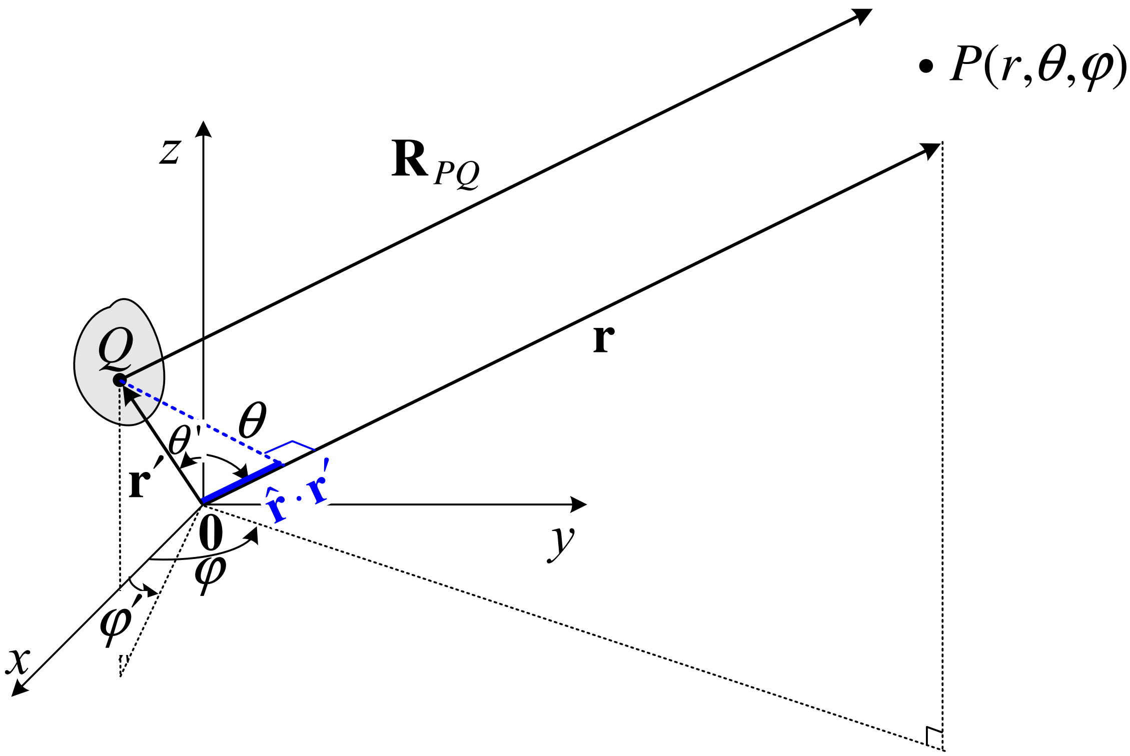
Formula (2.57) is a *far-field approximation* of the vector potential at distant points. We arrive at it starting from the integral in (2.51). When the observation point P is very far from the source, the distance R_{PQ} varies only slightly as Q sweeps the volume of the source. It is almost the same as the distance r from the origin (the antenna center) to P. The following first-order approximation (attributed to Kirchhoff) is made for the integrand:

$$\frac{e^{-jkR_{PQ}}}{R_{PQ}} \approx \frac{e^{-jk(r-\hat{\mathbf{r}}\cdot\mathbf{r}')}}{r}. \quad (2.58)$$

Here, $\mathbf{r}$ is the position vector of the observation point P and $r=|\mathbf{r}|$ is its length. Its direction is given by the unit vector $\hat{\mathbf{r}}$, so that $\mathbf{r}=\hat{\mathbf{r}}r$. The position vector of the integration point Q is $\mathbf{r}'$. Equation (2.58) is called the *far-field approximation*. The approximation in the phase term (in the exponent) is illustrated in the figures below. The first figure shows the real problem. The second one shows the (Kirchhoff) approximated problem, where, in effect, the vectors $\mathbf{R}_{PQ}$ and $\mathbf{r}$ are parallel.



(a) original problem



(b) far-field approximation of the original problem

We now apply the far-field approximation to the vector potential in (2.51):

$$\mathbf{A}(P) = \underbrace{\frac{e^{-jkr}}{4\pi r}}_{\text{dependence on distance from origin}} \cdot \underbrace{\iiint_{v_Q} \mu \mathbf{J}(Q) e^{jk\hat{\mathbf{r}}\cdot\mathbf{r}'} dv_Q}_{\text{dependence on source distribution and angular orientation}} . \quad (2.59)$$

The integrand in (2.59) no longer depends on the distance r between the origin (the antenna center) and the observation point. It depends only on the current distribution of the source, $\mathbf{J}(Q) \equiv \mathbf{J}(\mathbf{r}')$, and the angle between the position vector of the integration point $\mathbf{r}'$ and the unit position vector of the observation point $\hat{\mathbf{r}}$. This finally explains the general equation for the *far-field vector potential* in (2.57) and in particular the origin of its term in the square brackets, which is represented by the volume integral in (2.59).

7.2. Far-zone field

The far-field approximation of the vector potential leads to much simpler equations for the far-field vectors. Assume that there are only electrical currents $\mathbf{J}$. Then the field is fully described only by the magnetic vector potential $\mathbf{A}$. We have to substitute (2.57) into the equations of Table 2.1, where $\mathbf{F} = 0$:

$$\mathbf{E} = -j\omega\mathbf{A} - \frac{j}{\omega\mu\varepsilon}\nabla\nabla\cdot\mathbf{A}, \tag{2.60}$$

$$\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A} \ . \tag{2.61}$$

The differential operators $\nabla\times$ and $\nabla\nabla\cdot$ have to be expressed in spherical coordinates. All terms decreasing with the distance as $1/r^2$ and faster (near-field terms) are neglected. What remains is

$$\mathbf{E} = \underbrace{\frac{1}{r}\left\{-j\omega e^{-jkr}\left[\hat{\boldsymbol{\theta}}A_\theta(\theta,\varphi) + \hat{\boldsymbol{\varphi}}A_\varphi(\theta,\varphi)\right]\right\}}_{\text{far-field term}} + \underbrace{\frac{1}{r^2}\{\ \} + \cdots}_{\text{neglected}}, \quad r\to\infty, \tag{2.62}$$

$$\mathbf{H} = \underbrace{\frac{1}{r}\left\{j\frac{\omega}{\eta}e^{-jkr}\left[\hat{\boldsymbol{\theta}}A_\varphi(\theta,\varphi) - \hat{\boldsymbol{\varphi}}A_\theta(\theta,\varphi)\right]\right\}}_{\text{far-field term}} + \underbrace{\frac{1}{r^2}\{\ \} + \cdots}_{\text{neglected}}, \quad r\to\infty. \tag{2.63}$$

Here, $\eta = \sqrt{\mu/\varepsilon}$ is the intrinsic impedance of the medium. We write the far-field terms in equations (2.62) and (2.63) in a more compact way as

$$\left.\begin{aligned} &E_r \approx 0 \\ &E_\theta \approx -j\omega A_\theta \\ &E_\varphi \approx -j\omega A_\varphi \end{aligned}\right\} \Rightarrow \mathbf{E}^A \approx -j\omega\mathbf{A}_\perp, \ \text{ where } \mathbf{A}_\perp = \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\varphi}}A_\varphi, \tag{2.64}$$

$$\left.\begin{aligned} &H_r \approx 0 \\ &H_\theta \approx +j\frac{\omega}{\eta}A_\varphi = -\frac{E_\varphi}{\eta} \\ &H_\varphi \approx -j\frac{\omega}{\eta}A_\theta = +\frac{E_\theta}{\eta} \end{aligned}\right\} \Rightarrow \mathbf{H}^A \approx -j\frac{\omega}{\eta}\hat{\mathbf{r}}\times\mathbf{A} = \frac{1}{\eta}\hat{\mathbf{r}}\times\mathbf{E}^A. \tag{2.65}$$

In an analogous manner, we obtain the relations between the field vectors and the electric vector potential **F**, when only magnetic sources are present:

$$\left.\begin{aligned} &H_r \approx 0 \\ &H_\theta \approx -j\omega F_\theta \\ &H_\varphi \approx -j\omega F_\varphi \end{aligned}\right\} \Rightarrow \mathbf{H}^F \approx -j\omega\mathbf{F}_\perp, \ \text{ where } \mathbf{F}_\perp = \hat{\boldsymbol{\theta}}F_\theta + \hat{\boldsymbol{\varphi}}F_\varphi, \tag{2.66}$$

$$\left.\begin{array}{l} E_r \approx 0 \\ E_\theta \approx -j\omega\eta F_\varphi = \eta H_\varphi \\ E_\varphi \approx +j\omega\eta F_\theta = -\eta H_\theta \end{array}\right\} \Rightarrow \mathbf{E}^F \approx j\omega\eta \hat{\mathbf{r}} \times \mathbf{F} = -\eta \hat{\mathbf{r}} \times \mathbf{H}^F . \qquad (2.67)$$

In summary, the far field of any antenna has the following important features, which follow from equations (2.64) through (2.67):

- The far field has negligible radial components, $E_r \approx 0$ and $H_r \approx 0$. Since the radial direction is also the direction of propagation, the far field is a quasi-TEM (Transverse Electro-Magnetic) wave.
- The far-field **E** vector and **H** vector are mutually orthogonal, both of them being also orthogonal to the direction of propagation $\hat{\mathbf{r}}$.
- The magnitudes of the electric field and the magnetic field are related always as $|\mathbf{E}| = \eta |\mathbf{H}|$.

APPENDIX

Green's Function for the Helmholtz Equation

Suppose the following PDE must be solved:

$$L\Phi(\mathbf{x}) = f(\mathbf{x}) \qquad (2.68)$$

where $\mathbf{x}$ denotes the set of variables, e.g., $\mathbf{x} = (x, y, z)$ and L operates on $\mathbf{x}$. Suppose also that a Green's function exists such that it allows for the integral solution

$$\Phi(\mathbf{x}) = \iiint_{V'} G(\mathbf{x}, \mathbf{x}') \cdot f(\mathbf{x}') d\mathbf{x}' . \qquad (2.69)$$

Applying the operator L to both sides of (2.62), leads to

$$L\Phi(\mathbf{x}) = \iiint_{V'} \left[LG(\mathbf{x}, \mathbf{x}') \right] \cdot f(\mathbf{x}') d\mathbf{x}' = f(\mathbf{x}) . \qquad (2.70)$$

Note that L operates on the variable $\mathbf{x}$ while the integral in (2.70) is over $\mathbf{x}'$. This allows for the insertion of L inside the integral. From (2.70), we conclude that Green's function must satisfy the same PDE as Φ with a point source described by Dirac's delta function:

$$LG(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') . \qquad (2.71)$$

Here, $\delta(\mathbf{x} - \mathbf{x}')$ is Dirac's delta function in 3-D space, e.g., $\delta(\mathbf{x} - \mathbf{x}') = \delta(x - x')\delta(y - y')\delta(z - z')$. If Green's function of a problem is known and the source function $f(\mathbf{x})$ is known, the construction of an integral solution is possible via (2.69).

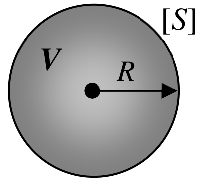
Consider Green's function for the Helmholtz equation in open space. It must satisfy

$$\nabla^2 G + \beta^2 G = \delta(x)\delta(y)\delta(z) \qquad (2.72)$$

together with the scalar radiation condition

$$\lim_{r\to\infty} r \cdot \left(\frac{\partial G}{\partial r} + j\beta G \right) = 0 \qquad (2.73)$$

if the source is centered at the origin of the coordinate system, i.e., $x' = y' = z' = 0$. Integrate (2.72) within a sphere with its center at (0,0,0) and a radius R:



$$\iiint_V \nabla^2 G dv + \iiint_V \beta^2 G dv = 1 \tag{2.74}$$

The function G is due to a point source and thus has a spherical symmetry, i.e., it depends on r only. The Laplacian ∇^2 in spherical coordinates is reduced to derivatives with respect to r only:

$$\frac{d^2G}{dr^2} + \frac{2}{r}\frac{dG}{dr} + \beta^2 G = \delta(x)\delta(y)\delta(z)\,. \tag{2.75}$$

Everywhere except at the point (x, y, z), G must satisfy the homogeneous equation

$$\frac{d^2G}{dr^2} + \frac{2}{r}\frac{dG}{dr} + \beta^2 G = 0 \tag{2.76}$$

the solution of which is well known for outgoing waves:

$$G(r) = C\frac{e^{-jkr}}{r}\,. \tag{2.77}$$

Here, C is a constant to be determined. Consider the 2nd integral from (2.74):

$$I_2 = \iiint_V \beta^2 G dv\,. \tag{2.78}$$

$$\Rightarrow I_2 = \iiint_v \beta^2 C\frac{e^{-j\beta r}}{r}dv = \int_0^R\int_0^{2\pi}\int_0^{\pi} \beta^2 C\frac{e^{-j\beta r}}{r} r^2 \sin\theta d\theta d\varphi dr \tag{2.79}$$

$$\Rightarrow I_2(R) = j4\pi\beta C\left(R\cdot e^{-j\beta R} + \frac{e^{-j\beta R}}{j\beta} - \frac{1}{j\beta}\right). \tag{2.80}$$

To evaluate the integral at (0,0,0), we let $R \to 0$, i.e., we let the sphere collapse into a point. We see that

$$\lim_{R\to 0} I_2(R) = 0\,. \tag{2.81}$$

Now, consider the 1st integral in (2.74),

$$I_1 = \iiint_V \nabla^2 G dv = \iiint_V \nabla\cdot(\nabla G)\, dv = \oiint_S \nabla G\cdot d\mathbf{s}\,. \tag{2.82}$$

Here, $d\mathbf{s} = R^2\sin\theta drd\theta d\varphi\cdot\hat{\mathbf{r}}$ is a surface element on S, and

$$\nabla G = \frac{\partial G}{\partial r}\hat{\mathbf{r}} = -C\left(jk\frac{e^{-jkr}}{r} + \frac{e^{-jkr}}{r^2}\right)\hat{\mathbf{r}}\,. \tag{2.83}$$

Substitute (2.83) in (2.82) and carry out the integration over the spherical surface:

$$I_1(R) = -C\left(jkR\cdot e^{-jkR} + e^{-jkR}\right)\int_0^{\pi}\int_0^{2\pi}\sin\theta d\varphi d\theta \tag{2.84}$$

$$\lim_{R\to 0} I_1(R) = -4\pi C\,. \tag{2.85}$$

Substituting (2.85) and (2.81) into (2.74) and bearing in mind the limit $\lim_{R\to 0}$, yields

$$C = -\frac{1}{4\pi}\,. \tag{2.86}$$

Finally,

$$G(r) = -\frac{e^{-jkr}}{4\pi r}\,. \tag{2.87}$$

It is not difficult to show that in the general case when the source is at a point $Q(x', y', z')$,

$$\nabla^2 G + \beta^2 G = \delta(x - x')\delta(y - y')\delta(z - z') \tag{2.88}$$

the Green function is

$$G(P,Q) = -\frac{e^{-jkR_{PQ}}}{4\pi R_{PQ}}, \tag{2.89}$$

where $R_{PQ} = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$ is the distance between the observation source points.