LECTURE 3: Radiation from Infinitesimal (Elementary) Sources

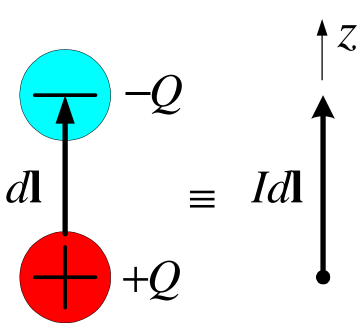
(*Radiation from an infinitesimal dipole. Duality of Maxwell's equations. Radiation from an infinitesimal loop. Radiation zones.*)

1. Radiation from Infinitesimal Dipole (Electric-current Element)

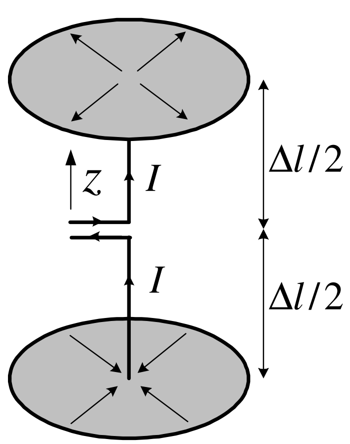
Definition: The infinitesimal dipole is a straight line segment of length Δl, which is much smaller than the radiation wavelength λ, $\Delta l \ll \lambda$ ($\Delta l < \lambda/50$), and which supports constant current distribution *I* along its length. The assumed positive direction of the current *I* determines the orientation of the line segment: $\Delta\mathbf{l} = \Delta l\hat{\mathbf{i}}$.

The infinitesimal dipole is mathematically described by a current element:

$$I d\mathbf{l} = -\frac{dQ}{dt} d\mathbf{l}.$$



A current element is best illustrated by a very short (compared to λ) piece of infinitesimally thin wire with constant current *I*. The ideal current element is difficult to realize in practice, but a good approximation of it is the short top-hat antenna. To realize a uniform current distribution along the wire, capacitive plates are added to provide enough charge storage at the end of the wire, so that the current is not zero there.



1.1. Magnetic vector potential due to a current element

The magnetic vector potential (VP) **A** due to a linear source is (see Lecture 2):

$$\mathbf{A}(P) = \int_L \mu I(Q) \frac{e^{-j\beta R_{PQ}}}{4\pi R_{PQ}} d\mathbf{l}_Q \tag{3.1}$$

$$\Rightarrow \mathbf{A}(P) = \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \int_{\Delta l} \frac{e^{-j\beta R_{PQ}}}{R_{PQ}} dl. \tag{3.2}$$

If we assume that the dipole's length Δl is much smaller than the distance from its center to the observation point *P*, then $R_{PQ} \approx r$ holds both in the exponential term and in the denominator. Therefore,

$$\mathbf{A} \approx \mu_0 I \Delta l \frac{e^{-j\beta r}}{4\pi r} \hat{\mathbf{z}}. \tag{3.3}$$

Equation (3.3) gives the vector potential due to an electric current element (infinitesimal dipole). This is an important result because ***the field radiated by any complex antenna in a linear medium is a superposition of the fields due to the current elements on the antenna***.

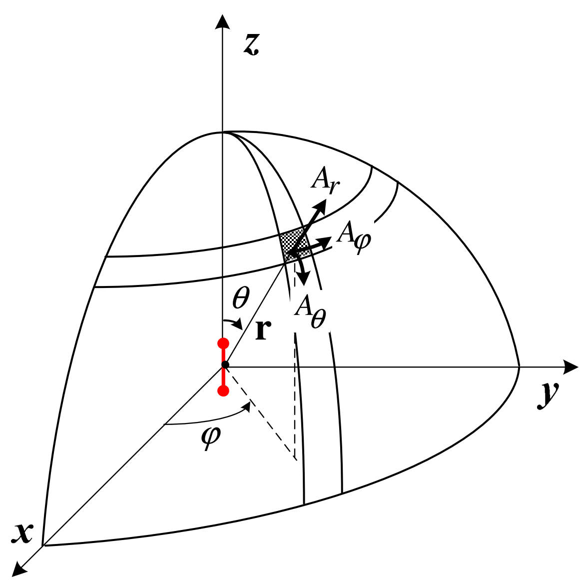
We represent **A** with its spherical components. In antenna theory, the preferred coordinate system is the spherical one. This is because the ***far field*** radiation is of interest where the field dependence on the distance r from the source is decoupled from its angular dependence. This angular dependence is described conveniently in terms of the two angles in the spherical coordinate system (SCS) φ and θ. Also, this field propagates radially (along $\hat{\mathbf{r}}$) when the source is located at the origin of the coordinate system.

The transformation from rectangular to spherical vector components is:

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}. \tag{3.4}$$

Applying (3.4) to **A** in (3.3) produces

$$\begin{aligned} A_r &= A_z \cos\theta = \mu_0 I \Delta l \frac{e^{-j\beta r}}{4\pi r} \cos\theta \\ A_\theta &= -A_z \sin\theta = -\mu_0 I \Delta l \frac{e^{-j\beta r}}{4\pi r} \sin\theta \\ A_\varphi &= 0 \end{aligned} \tag{3.5}$$



Note that:

1) **A** does not depend on φ (due to the cylindrical symmetry of the dipole);
2) the dependence on *r*, $e^{-j\beta r} / r$, is separable from the dependence on θ.

1.2. Field vectors due to current element

Next we find the field vectors **H** and **E** from **A**.

a) $$\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A} \tag{3.6}$$

The curl operator is expressed in spherical coordinates to obtain

$$\mathbf{H} = \frac{1}{\mu r}\left[\frac{\partial}{\partial r}(r\cdot A_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\varphi}}. \tag{3.7}$$

Thus, the magnetic field **H** has only a φ-component, i.e.,

$$\left| \begin{aligned} &H_\varphi = j\beta\cdot(I\Delta l)\cdot\sin\theta\cdot\left(1+\frac{1}{j\beta r}\right)\frac{e^{-j\beta r}}{4\pi r}, \\ &H_\theta = H_r = 0\ . \end{aligned} \right. \tag{3.8}$$

b) $$\mathbf{E} = \frac{1}{j\omega\varepsilon}\nabla\times\mathbf{H} = -j\omega\mathbf{A} - \frac{j}{\omega\mu\varepsilon}\nabla\nabla\cdot\mathbf{A} \tag{3.9}$$

In spherical coordinates, the **E** field components are:

$$\left| \begin{aligned} &E_r = j2\eta\beta(I\Delta l)\cdot\cos\theta\cdot\left[\frac{1}{j\beta r} - \frac{1}{(\beta r)^2}\right]\frac{e^{-j\beta r}}{4\pi r} \\ &E_\theta = j\eta\beta(I\Delta l)\cdot\sin\theta\cdot\left[1+\frac{1}{j\beta r} - \frac{1}{(\beta r)^2}\right]\frac{e^{-j\beta r}}{4\pi r} \\ &E_\varphi = 0\ . \end{aligned} \right. \tag{3.10}$$

Notes: 1) Equations (3.8) and (3.10) show that the **E** and **H** field vectors due to the current element are given by quite complicated expressions unlike that for the VP **A** in (3.3). The use of the VP instead of the field vectors is often advantageous in antenna studies.

2) The field vectors contain terms, which depend on the distance from the source as $1/r$, $1/r^2$ and $1/r^3$; the higher-order terms can be neglected at large distances from the dipole.

3) The longitudinal $\hat{\mathbf{r}}$-component of the **E** field vector decreases fast as the field propagates away from the source (as $1/r^2$ and $1/r^3$): it is neglected in the far zone. The longitudinal **H** field component of the infinitesimal electric dipole is zero everywhere.

4) The nonzero transverse field components, E_θ and H_φ, are orthogonal to each other, and they have terms that depend on the distance as $1/r$. These terms relate through the intrinsic impedance η and they describe a TEM wave. They represent the so-called *far field* which satisfies the Sommerfeld vector radiation boundary conditions. The concept of far field will be re-visited later, when the radiation zones are defined.

1.3. Power density and overall radiated power of the infinitesimal dipole

The complex Poynting vector **P** describes the complex power-flux density. In the case of infinitesimal dipole, it is

$$\mathbf{P} = \frac{1}{2}\left(\mathbf{E}\times\mathbf{H}^*\right) = \frac{1}{2}\left(E_r\hat{\mathbf{r}} + E_\theta\hat{\boldsymbol{\theta}}\right)\times\left(H_\varphi^*\hat{\boldsymbol{\varphi}}\right) = \frac{1}{2}\left(E_\theta H_\varphi^*\hat{\mathbf{r}} - E_r H_\varphi^*\hat{\boldsymbol{\theta}}\right). \quad (3.11)$$

Substituting (3.8) and (3.10) into (3.11) yields

$$\begin{vmatrix} P_r = \dfrac{\eta}{8}\left|\dfrac{I\Delta l}{\lambda}\right|^2 \dfrac{\sin^2\theta}{r^2}\left[1 - j\dfrac{1}{(\beta r)^3}\right], \\ P_\theta = j\eta\beta\dfrac{|I\Delta l|^2\cos\theta\sin\theta}{16\pi^2 r^3}\left[1 + \dfrac{1}{(\beta r)^2}\right]. \end{vmatrix} \quad (3.12)$$

The overall power Π is calculated over a sphere, and, therefore, only the radial component P_r contributes:

$$\Pi = \oiint_S \mathbf{P}\cdot d\mathbf{s} = \oiint_S \left(P_r\hat{\mathbf{r}} + P_\theta\hat{\boldsymbol{\theta}}\right)\cdot\hat{\mathbf{r}}\, r^2\sin\theta d\theta d\varphi, \quad (3.13)$$

$$\Pi = \frac{\pi}{3}\eta\left|\frac{I\Delta l}{\lambda}\right|^2\left[1 - \frac{j}{(\beta r)^3}\right], \text{W}. \quad (3.14)$$

The radiated power is equal to the real part of the complex power (the time-average of the total power flow, see Lecture 2). Therefore, the radiated power of an infinitesimal electric dipole is

$$\Pi_{\text{rad}} = \frac{\pi}{3}\eta\left(\frac{I\Delta l}{\lambda}\right)^2, \text{W}. \quad (3.15)$$

Here, we introduce the concept of radiation resistance R_r, which describes the power loss due to radiation in the equivalent circuit of the antenna:

$$\Pi_{\text{rad}} = \frac{1}{2} R_r I^2 \Rightarrow R_r = \frac{2\Pi_{\text{rad}}}{I^2} \tag{3.16}$$

$$\Rightarrow R_r^{id} = \frac{2\pi}{3}\eta\left(\frac{\Delta l}{\lambda}\right)^2,\ \Omega. \tag{3.17}$$

Note that (3.17) holds only for an infinitesimal dipole, i.e., when the current is assumed constant over the length Δl of the dipole.

2. Duality in Maxwell's Equations

Duality in electromagnetics means that the EM field is described by two sets of quantities, which correspond to each other in such a manner that substituting the quantities from one set with the respective quantities from the other set in any given equation produces a valid equation (the dual of the given one).

We deduce these dual sets by comparing the equations describing two dual fields: the field of electric sources and the field of magnetic sources. Note that duality exists even if there are no sources present in the region of interest. Tables 2.1 and 2.2 summarize the duality of the EM equations and quantities.

TABLE 2.1. DUALITY IN ELECTROMAGNETIC EQUATIONS

Electric sources $(\mathbf{J}\neq 0, \mathbf{M}=0)$	Magnetic sources $(\mathbf{J}=0, \mathbf{M}\neq 0)$
$\nabla\times\mathbf{E} = -j\omega\mu\mathbf{H}$	$\nabla\times\mathbf{H} = j\omega\varepsilon\mathbf{E}$
$\nabla\times\mathbf{H} = j\omega\varepsilon\mathbf{E}+\mathbf{J}$	$\nabla\times\mathbf{E} = -j\omega\mu\mathbf{H}-\mathbf{M}$
$\nabla\cdot\mathbf{D} = \rho$	$\nabla\cdot\mathbf{B} = \rho_m$
$\nabla\cdot\mathbf{B} = 0$	$\nabla\cdot\mathbf{D} = 0$
$\nabla\cdot\mathbf{J} = -j\omega\rho$	$\nabla\cdot\mathbf{M} = -j\omega\rho_m$
$\nabla^2\mathbf{A}+\beta^2\mathbf{A} = -\mu\mathbf{J}$	$\nabla^2\mathbf{F}+\beta^2\mathbf{F} = -\varepsilon\mathbf{M}$
$\mathbf{A} = \iiint_V \mu\mathbf{J}\frac{e^{-j\beta R}}{4\pi R}dv$	$\mathbf{F} = \iiint_V \varepsilon\mathbf{M}\frac{e^{-j\beta R}}{4\pi R}dv$
$\mathbf{H} = \mu^{-1}\nabla\times\mathbf{A}$	$\mathbf{E} = -\varepsilon^{-1}\nabla\times\mathbf{F}$
$\mathbf{E} = -j\omega\mathbf{A}+(j\omega\mu\varepsilon)^{-1}\nabla\nabla\cdot\mathbf{A}$	$\mathbf{H} = -j\omega\mathbf{F}+(j\omega\mu\varepsilon)^{-1}\nabla\nabla\cdot\mathbf{F}$

TABLE 2.2. DUAL QUANTITIES IN ELECTROMAGNETICS

given	**E**	**H**	**J**	**M**	**A**	**F**	ε	μ	η	$1/\eta$	β
dual	**H**	**–E**	**M**	**–J**	**F**	**–A**	μ	ε	$1/\eta$	η	β

3. Radiation from Infinitesimal Magnetic Dipole (Electric-current Loop)

3.1. <u>The vector potential and the field vectors of a magnetic dipole (magnetic current element)</u> $I_m\Delta l$

Using the duality theorem, the field of a magnetic dipole $I_m\Delta l$ is readily found by a simple substitution of the dual quantities in equations (3.5), (3.8) and (3.10) as per Table 2.2. We denote the magnetic current, which is the dual of the electric current *I*, by I_m (measured in *volts*).

(a) the electric vector potential

$$\begin{aligned} F_r &= F_z\cos\theta = \varepsilon_0\left(I_m\Delta l\right)\frac{e^{-j\beta r}}{4\pi r}\cos\theta \\ F_\theta &= -F_z\sin\theta = -\varepsilon_0\left(I_m\Delta l\right)\frac{e^{-j\beta r}}{4\pi r}\sin\theta \\ F_\varphi &= 0 \end{aligned} \tag{3.18}$$

(b) the electric field of the magnetic dipole

$$\begin{aligned} E_\varphi &= -j\beta\cdot(I_m\Delta l)\cdot\sin\theta\cdot\left(1+\frac{1}{j\beta r}\right)\frac{e^{-j\beta r}}{4\pi r} \\ E_\theta &= E_r = 0 \end{aligned} \tag{3.19}$$

(c) the magnetic field of the magnetic dipole

$$\begin{aligned} H_r &= \frac{2\left(I_m\Delta l\right)\cos\theta}{\eta}\left(\frac{1}{r}+\frac{1}{j\beta r^2}\right)\frac{e^{-j\beta r}}{4\pi r} \\ H_\theta &= \frac{j\beta\left(I_m\Delta l\right)\sin\theta}{\eta}\left(1+\frac{1}{j\beta r}-\frac{1}{\beta^2 r^2}\right)\frac{e^{-j\beta r}}{4\pi r} \\ H_\varphi &= 0 \end{aligned} \tag{3.20}$$

3.2. Equivalence between a magnetic dipole (magnetic current element) and an electric current loop

First, we prove the equivalence of the fields excited by particular configurations of electric and magnetic current densities. We write Maxwell's equations for the two cases:

(a) electric current density (EM field 1)

$$\left| \begin{array}{l} -\nabla \times \mathbf{E}_1 = j\omega\mu \mathbf{H}_1 \\ \nabla \times \mathbf{H}_1 = j\omega\varepsilon \mathbf{E}_1 + \mathbf{J} \end{array} \right. \tag{3.21}$$

$$\Rightarrow \nabla \times \nabla \times \mathbf{E}_1 - \omega^2 \mu\varepsilon \mathbf{E}_1 = -j\omega\mu \mathbf{J} \tag{3.22}$$

(b) magnetic current density (EM field 2)

$$\left| \begin{array}{l} -\nabla \times \mathbf{E}_2 = j\omega\mu \mathbf{H}_2 + \mathbf{M} \\ \nabla \times \mathbf{H}_2 = j\omega\varepsilon \mathbf{E}_2 \end{array} \right. \tag{3.23}$$

$$\Rightarrow \nabla \times \nabla \times \mathbf{E}_2 - \omega^2 \mu\varepsilon \mathbf{E}_2 = -\nabla \times \mathbf{M} \tag{3.24}$$

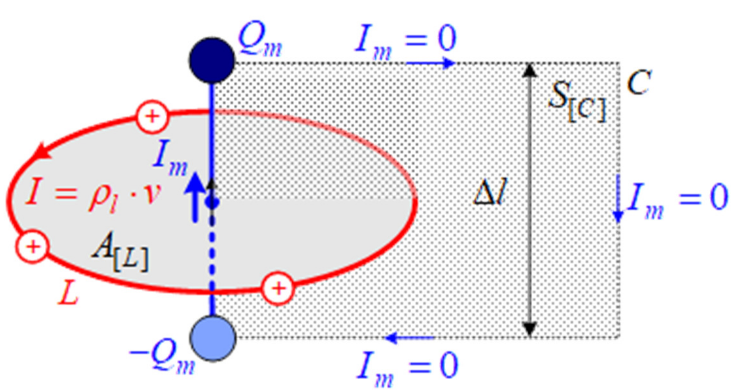
If the boundary conditions (BCs) for $\mathbf{E}_1$ in (3.22) are the same as the BCs for $\mathbf{E}_2$ in (3.24), and the excitations of both fields fulfill

$$j\omega\mu \mathbf{J} = \nabla \times \mathbf{M}, \tag{3.25}$$

then both fields are identical, i.e., $\mathbf{E}_1 \equiv \mathbf{E}_2$ and $\mathbf{H}_1 \equiv \mathbf{H}_2$.

Consider a loop [*L*] of electric current *I*. Equation (3.25) can be written in its integral form as

$$j\omega\mu \iint_{S[C]} \mathbf{J} \cdot d\mathbf{s} = \oint_C \mathbf{M} \cdot d\mathbf{l}. \tag{3.26}$$



The integral on the left side is the electric current I. $\mathbf{M}$ in a magnetic dipole is non-zero and constant only at the section Δl, which is normal to the loop's plane and passes through the loop's centre. Then,

$$j\omega\mu I = M\Delta l . \tag{3.27}$$

The magnetic current I_m corresponding to the loop [L] is obtained by multiplying the magnetic current density M by the area of the loop $A_{[L]}$, which yields

$$j\omega\mu I A_{[L]} = I_m \Delta l . \tag{3.28}$$

Thus, we show that *a small loop of electric current I and of area $A_{[L]}$ creates EM field equivalent to that of a small magnetic dipole (magnetic current element) $I_m\Delta l$, such that* (3.28) *holds*.

Here, it was assumed that the electric current is constant along the loop, which is true only for very small loops ($a < 0.1\lambda$, where a is the loop's radius and the loop has only 1 turn). If the loop is larger, the field expressions below are inaccurate and other solutions should be used. We will discuss the loop antennas in more detail in a dedicated lecture.

3.3. Field vectors of an infinitesimal loop antenna

The expressions below are derived by substituting (3.28) into (3.19)-(3.20):

$$E_\varphi = \eta\beta^2 (IA)\cdot\sin\theta\left(1+\frac{1}{j\beta r}\right)\frac{e^{-j\beta r}}{4\pi r}, \tag{3.29}$$

$$H_r = j2\beta (IA)\cdot\cos\theta\left(\frac{1}{r}+\frac{1}{j\beta r^2}\right)\frac{e^{-j\beta r}}{4\pi r}, \tag{3.30}$$

$$H_\theta = -\beta^2 (IA)\cdot\sin\theta\left(1+\frac{1}{j\beta r}-\frac{1}{\beta^2 r^2}\right)\frac{e^{-j\beta r}}{4\pi r}, \tag{3.31}$$

$$E_r = E_\theta = H_\varphi = 0 . \tag{3.32}$$

The far-field terms ($1/r$ dependence on the distance from the source) show the same behaviour as in the case of an infinitesimal dipole antenna: (1) the electric field E_φ is orthogonal to the magnetic field H_θ; (2) E_φ and H_θ relate through η ; (3) the longitudinal $\hat{\mathbf{r}}$ components have no far-field terms.

The dependence of the Poynting vector and the complex power on the distance r is the same as in the case of an infinitesimal electric dipole. The radiated power can be found to be

$$\Pi_{rad} = \eta\beta^4 (IA)^2 / 12\pi . \tag{3.33}$$

4. Radiation Zones – Introduction

The space surrounding the antenna is divided into three regions according to the dominant field behaviour. The boundaries between the regions are not distinct and the field behaviour changes gradually as these boundaries are crossed. In this course, we are mostly concerned with the far-field characteristics of the antennas.

Next, we illustrate the three radiation zones through the field of the small electric dipole.

4.1. Reactive near-field region

This is the region immediately surrounding the antenna, where the reactive field dominates and the angular field distribution is different at different distances from the antenna. For most antennas, it is assumed that this region is a sphere with the antenna at its centre, and with a radius

$$r_{\text{RNF}} \approx 0.62\sqrt{D^3/\lambda}\,, \tag{3.34}$$

where D is the largest dimension of the antenna, and λ is the wavelength of the radiation. The above expression will be derived in Section 5. It must be noted that this limit is most appropriate for wire and waveguide aperture antennas while it is not valid for electrically large reflector antennas.

At this point, we discuss the general field behaviour making use of our knowledge of the infinitesimal electric-dipole field. When (3.34) is true, r is sufficiently small so that $\beta r \ll 1$ (note that $D \ll \lambda$ for the infinitesimal dipole). Then, the most significant terms in the field expressions (3.8) and (3.10) are

$$\left| \begin{aligned} & H_\varphi \approx \frac{(I\Delta l)e^{-j\beta r}}{4\pi r^2}\sin\theta \\ & E_\theta \approx -j\eta\frac{(I\Delta l)e^{-j\beta r}}{4\pi\beta r^3}\sin\theta \\ & E_r \approx -j\eta\frac{(I\Delta l)e^{-j\beta r}}{2\pi\beta r^3}\cos\theta \\ & H_r = H_\theta = E_\varphi = 0 \end{aligned} \right. ,\ \beta r \ll 1. \tag{3.35}$$

This approximated field is purely reactive (**H** and **E** are in phase quadrature). Since $e^{-j\beta r} \approx 1$ we see that: (1) H_φ has the distribution of the magnetostatic field of a current filament $I\Delta l$ (remember Bio-Savart's law); (2) E_θ and E_r have the distribution of the electrostatic field of a dipole.

That the field is almost purely reactive in the near zone is obvious from the power equation (3.14). Its imaginary part is

$$\mathrm{Im}\{\Pi\} = -\frac{\pi}{3}\eta\left(\frac{I\Delta l}{\lambda}\right)^2 \frac{1}{(\beta r)^3}. \tag{3.36}$$

$\mathrm{Im}\{\Pi\}$ dominates over the radiated power,

$$\Pi_{\mathrm{rad}} = \mathrm{Re}\{\Pi\} = \frac{\pi}{3}\eta\left(\frac{I\Delta l}{\lambda}\right)^2, \tag{3.37}$$

when $r \to 0$ because $\beta r \ll 1$ and Π_{rad} does not depend on *r*.

The radial component of the near-field Poynting vector P_r has negative imaginary value and decreases as $1/r^5$:

$$P_r^{\mathrm{near}} = -j\frac{\eta}{8}\left(\frac{I\Delta l}{\lambda}\right)^2 \frac{\sin^2\theta}{\beta^3 r^5}. \tag{3.38}$$

The near-field P_θ component is also imaginary and has the same order of dependence on *r* but it is positive:

$$P_\theta^{\mathrm{near}} = j\eta\beta\frac{(I\Delta l)^2\cos\theta\sin\theta}{16\pi^2 r^3}\cdot\frac{1}{(\beta r)^2} \tag{3.39}$$

or

$$P_\theta^{\mathrm{near}} = j\frac{\eta}{8}\left(\frac{I\Delta l}{\lambda}\right)^2 \frac{\sin(2\theta)}{\beta^3 r^5}. \tag{3.40}$$

4.2. Radiating near-field (Fresnel) region

This is an intermediate region between the reactive near-field region and the far-field region, where the radiation field is more significant but the angular field distribution is still dependent on the distance from the antenna. In this region, $\beta r \geq 1$. For most antennas, it is assumed that the Fresnel region is enclosed between two spherical surfaces:

$$0.62\sqrt{\frac{D^3}{\lambda}} \leq r \leq \frac{2D^2}{\lambda}. \tag{3.41}$$

Here, *D* is the largest dimension of the antenna. This region is called the *Fresnel region* because its field expressions reduce to Fresnel integrals.

The fields of an infinitesimal dipole in the Fresnel region are obtained by neglecting the higher-order $(1/\beta r)^n$-terms, $n \geq 2$, in (3.8) and (3.10):

$$
\left|\begin{aligned}
&H_\varphi \approx \frac{j\beta(I\Delta l)\cdot e^{-j\beta r}}{4\pi r}\cdot\sin\theta\\
&E_r \approx \eta\frac{\beta(I\Delta l)\cdot e^{-j\beta r}}{2\pi\beta r^2}\cdot\cos\theta\\
&E_\theta \approx j\eta\frac{\beta(I\Delta l)\cdot e^{-j\beta r}}{4\pi r}\cdot\sin\theta\\
&H_\theta = H_r = E_\varphi = 0
\end{aligned}\right. ,\ \beta r \ge 1. \tag{3.42}
$$

The radial component E_r is not negligible yet but the transverse components E_θ and H_φ are dominant.

4.3. Far-field (Fraunhofer) region

Only the terms $\sim 1/r$ are considered when $\beta r \gg 1$. The angular field distribution does not depend on the distance from the source any more, i.e., the *far-field pattern* is already well established. The field is a transverse EM wave. For most antennas, the far-field region is defined as

$$
r \ge 2D^2/\lambda. \tag{3.43}
$$

The far-field of the infinitesimal dipole is obtained as

$$
\left|\begin{aligned}
&H_\varphi \approx \frac{j\beta\cdot(I\Delta l)\cdot e^{-j\beta r}}{4\pi r}\cdot\sin\theta\\
&E_\theta \approx j\eta\frac{\beta\cdot\left(I\Delta l\right)\cdot e^{-j\beta r}}{4\pi r}\cdot\sin\theta,\ \beta r \gg 1.\\
&E_r \approx 0\\
&H_\theta = H_r = E_\varphi = 0
\end{aligned}\right. \tag{3.44}
$$

The features of the far field are summarized below:

1) no radial components;
2) the angular field distribution is independent of *r*;
3) $\mathbf{E} \perp \mathbf{H}$;
4) $E_\theta = \eta H_\varphi$;
5) $\mathbf{P} = (\mathbf{E}\times\mathbf{H}^*)/2 = \hat{\mathbf{r}}0.5|E_\theta|^2/\eta = \hat{\mathbf{r}}0.5\eta|H_\varphi|^2$. (3.45)

5. Region Separation and Accuracy of the Approximations

In most practical cases, a closed form solution of the radiation integral (the VP integral) does not exist. For the evaluation of the far fields or the fields in the Fraunhofer region, standard approximations are applied, from which the boundaries of these regions are derived.

Consider the VP integral for a linear current source:

$$\mathbf{A} = \frac{\mu}{4\pi} \int_{L'} I(l') \frac{e^{-j\beta R}}{R} d\mathbf{l}', \tag{3.46}$$

where $R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$. The observation point is at $P(x, y, z)$ and the source point is at $Q(x', y', z')$, which belongs to the integration line L'.

So far, we have analyzed the infinitesimal dipole whose current is constant along L'. In practical antennas, the current distribution is not constant and the solution of (3.46) can be very complicated depending on the vector function $I(l')d\mathbf{l}'$. Besides, because of the infinitesimal size of this source, the distance R between the integration point and the observation point was considered constant and equal to the distance from the centre of the dipole, $R \approx r = (x^2 + y^2 + z^2)^{1/2}$. However, if $D_{\max}$ (the maximum dimension of the antenna) is larger and commensurate with the wavelength λ, the error, especially in the phase term βR, due to the above assumption for R would be unacceptable.

Let us divide the integral kernel $e^{-j\beta R} / R$ into two factors: (1) the amplitude-decay factor $(1/R)$, and (2) the phase-delay factor $e^{-j\beta R}$. The amplitude factor is not very sensitive to errors in R. In both, the Fresnel and the Fraunhofer regions, the approximation

$$1/R \approx 1/r \tag{3.47}$$

is acceptable, provided $r >> D_{\max}$.

The approximation $R \approx r$, however, is unacceptable in the phase term. ***To keep the phase term error low enough, the maximum error in*** (βR) ***must be kept below*** $\pi/8 = 22.5^\circ$.

Neglect the antenna dimensions along the x- and y-axes (infinitesimally thin wire). Then,

$$x' = y' = 0 \Rightarrow R = \sqrt{x^2 + y^2 + (z-z')^2}, \tag{3.48}$$

$$\Rightarrow R = \sqrt{x^2 + y^2 + z^2 + (z'^2 - 2zz')} = \sqrt{r^2 + (z'^2 - 2rz' \cdot \cos\theta)}. \tag{3.49}$$

Using the binomial expansion,♣ R is expanded as

$$R=\left(r^2\right)^{1/2}+\frac{1}{2}\left(r^2\right)^{-1/2}\left(z'^2-2rz'\cos\theta\right)+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}\left(r^2\right)^{-3/2}\left(z'^2-2rz'\cos\theta\right)^2$$

$$+\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{6}\left(r^2\right)^{-5/2}\left(z'^2-2rz'\cos\theta\right)^3+\cdots$$

$$\Rightarrow R=r-z'\cos\theta+\frac{z'^2}{2r}-\frac{z'^2\cos^2\theta}{2r}+\frac{1}{2r^2}z'^3\cos\theta\sin^2\theta+O^3. \quad (3.50)$$

O^3 denotes terms of the order $(1/r^3)$ and higher. Neglecting these terms and simplifying further leads to the approximation

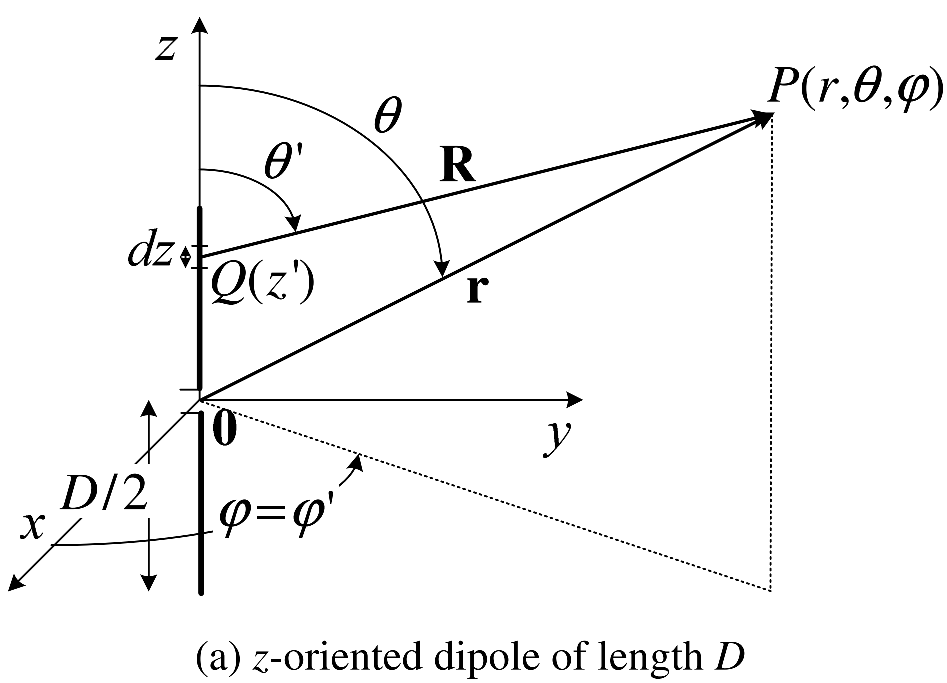
$$R\approx r-z'\cos\theta+\frac{1}{2r}z'^2\sin^2\theta+\frac{1}{2r^2}z'^3\cos\theta\sin^2\theta. \quad (3.51)$$

This expansion is used below to mathematically define the reactive near-field region, the radiating near-field region, and the far-field region.

(a) ***Far-field approximation***

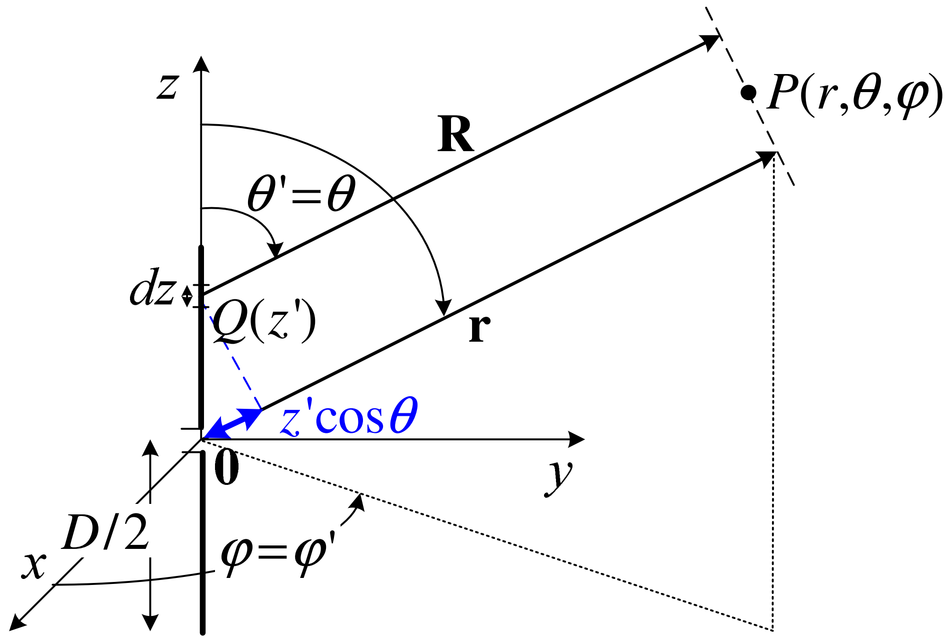
Only the first two terms in the expansion (3.51) are taken into account:

$$R\approx r-z'\cos\theta. \quad (3.52)$$



(a) z-oriented dipole of length D

♣ $\left(a+b\right)^n=a^n+na^{n-1}b+\frac{n(n-1)}{2!}a^{n-2}b^2+\frac{n(n-1)(n-2)}{3!}a^{n-3}b^3+\cdots$



(b) *z*-oriented dipole: far-field approximation

The most significant error term in *R* that was neglected in (3.52) is

$$e(r) = \frac{1}{2}\frac{(z')^2}{r}\sin^2\theta,$$

which has its maximum at $\theta = \pi/2$ and $z' = z'_{\max} = D/2$:

$$e_{\max}(r) = \frac{(z'_{\max})^2}{2r}. \tag{3.53}$$

The minimum *r*, at which the phase error (βR) is below $\pi/8$, is derived from:

$$\beta \cdot \frac{(z'_{\max})^2}{2r} \le \frac{\pi}{8}.$$

Thus, the smallest distance from the antenna centre *r*, at which the phase error is acceptable is

$$r_{\min}^{\text{far}} = 2D^2/\lambda. \tag{3.54}$$

This is the far-zone limit defined in (3.43).

As a word of caution, sometimes equation (3.54) produces too small values, which are in conflict with the assumptions made before. For example, in order the amplitude-factor approximation $1/R \approx 1/r$ to hold, the ratio of the maximum antenna dimension *D* and the distance *R* must fulfill $D/R \ll 1$. Otherwise, the first-order approximation based on the binomial expansion is too inaccurate.

Besides, in order to neglect all field components except the far-field ones, the condition $r \gg \lambda$ must hold, too. Therefore, in addition to (3.54), the calculated inner boundary of the far-field region should comply with two more conditions:

$$r \gg D \text{ and } r \gg \lambda. \tag{3.55}$$

Finally, we can generalize the far-zone limit as

$$r_{\min}^{\text{far}} = \max\left(2D^2/\lambda,\ \sim 2\lambda,\ \sim 2D\right). \tag{3.56}$$

(b) ***Radiating near-field (Fresnel region) approximation***

This region is adjacent to the Fraunhofer region, so its upper boundary is specified by

$$r \le r_{\min}^{\text{far}} = 2D^2/\lambda. \tag{3.57}$$

When the observation point belongs to this region, we must take one more term in the expansion of R as given by (3.51) to reduce sufficiently the phase error. The approximation this time is

$$R \approx r - z'\cos\theta + \frac{1}{2r} z'^2 \sin^2\theta. \tag{3.58}$$

The most significant error term is

$$e = \frac{1}{2}\frac{z'^3}{r^2}\cos\theta\sin^2\theta. \tag{3.59}$$

The angles θ_o must be found, at which e has its extrema:

$$\frac{\partial e}{\partial\theta} = \frac{z'^3}{2r^2}\sin\theta\left(-\sin^2\theta + 2\cos^2\theta\right) = 0. \tag{3.60}$$

The roots of (3.60) are

$$\begin{aligned} &\theta_o^{(1)} = 0 \to \min, \\ &\theta_o^{(2),(3)} = \arctan\left(\pm\sqrt{2}\right) \approx \pm 54.7^\circ \to \max\ . \end{aligned} \tag{3.61}$$

Following a procedure similar to case (a), we obtain:

$$\beta e_{\max}(r) = \frac{2\pi}{\lambda}\cdot\frac{1}{2}\frac{z'^3}{r^2}\cos\theta_o^{(2)}\sin^2\theta_o^{(2)} = \frac{\pi}{\lambda}\frac{z'^3}{r^2}\frac{2}{3\sqrt{3}},$$

$$\Rightarrow \beta e_{\max}(r) = \frac{\pi}{12\sqrt{3}}\frac{D^3}{\lambda r^2} \le \frac{\pi}{8}, \text{ note: } z'_{\max} = D/2$$

$$\Rightarrow r \ge \sqrt{\frac{2}{3\sqrt{3}}\frac{D^3}{\lambda}} \approx 0.62\sqrt{\frac{D^3}{\lambda}}. \tag{3.62}$$

Equation (3.62) states the lower boundary of the Fresnel region (for wire antennas) and is identical to the left-hand side of (3.41).