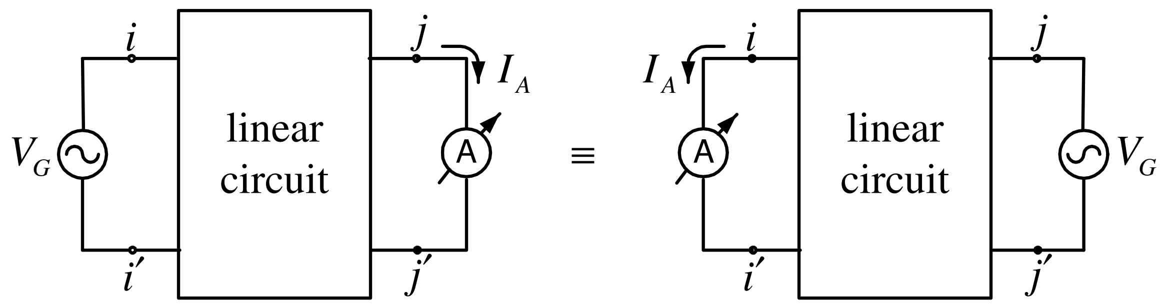
LECTURE 10: Reciprocity. Cylindrical Antennas – Analytical Models

(*Reciprocity theorem. Implications of reciprocity in antenna measurements. Self-impedance of a dipole using the induced **emf** method.*)

1. Reciprocity Theorem for Antennas

1.1. Reciprocity theorem in circuit theory

If a voltage (current) generator is placed between any pair of nodes of a linear circuit, and a current (voltage) response is measured between any other pair of nodes, the interchange of the generator's and the measurement's locations would lead to the same measurement results.



$$\frac{V_i}{I_j} = \frac{V_j}{I_i} \quad \text{or} \quad Z_{ji} = Z_{ij},\, Y_{ji} = Y_{ij}. \tag{10.1}$$

1.2. Reciprocity theorem in EM field theory (Lorentz' reciprocity theorem)

Consider a volume $V_{[S]}$ bounded by the surface S, where two pairs of sources exist: $(\mathbf{J}_1, \mathbf{M}_1)$ and $(\mathbf{J}_2, \mathbf{M}_2)$. The medium is linear. We denote the field associated with the $(\mathbf{J}_1, \mathbf{M}_1)$ sources as $(\mathbf{E}_1, \mathbf{H}_1)$, and the field generated by $(\mathbf{J}_2, \mathbf{M}_2)$ as $(\mathbf{E}_2, \mathbf{H}_2)$.

$$\left| \begin{aligned} &\nabla \times \mathbf{E}_1 = -j\omega\mu \mathbf{H}_1 - \mathbf{M}_1 \qquad / \cdot \mathbf{H}_2 \\ &\nabla \times \mathbf{H}_1 = j\omega\varepsilon \mathbf{E}_1 + \mathbf{J}_1 \qquad / \cdot \mathbf{E}_2 \end{aligned} \right. \tag{10.2}$$

$$\left| \begin{aligned} &\nabla \times \mathbf{E}_2 = -j\omega\mu \mathbf{H}_2 - \mathbf{M}_2 \qquad / \cdot \mathbf{H}_1 \\ &\nabla \times \mathbf{H}_2 = j\omega\varepsilon \mathbf{E}_2 + \mathbf{J}_2 \qquad / \cdot \mathbf{E}_1 \end{aligned} \right. \tag{10.3}$$

The vector identity

$$\mathbf{H}_2 \cdot \nabla\times\mathbf{E}_1 - \mathbf{E}_1 \cdot \nabla\times\mathbf{H}_2 - \mathbf{H}_1 \cdot \nabla\times\mathbf{E}_2 + \mathbf{E}_2 \cdot \nabla\times\mathbf{H}_1 = \nabla\cdot(\mathbf{E}_1\times\mathbf{H}_2 - \mathbf{E}_2\times\mathbf{H}_1)$$

is used along with (10.2) and (10.3) to obtain

$$\nabla\cdot(\mathbf{E}_1\times\mathbf{H}_2 - \mathbf{E}_2\times\mathbf{H}_1) = -\mathbf{E}_1\cdot\mathbf{J}_2 + \mathbf{H}_1\cdot\mathbf{M}_2 + \mathbf{E}_2\cdot\mathbf{J}_1 - \mathbf{H}_2\cdot\mathbf{M}_1. \tag{10.4}$$

Equation (10.4) is written in its integral form as

$$\oiint_S (\mathbf{E}_1\times\mathbf{H}_2 - \mathbf{E}_2\times\mathbf{H}_1)\cdot d\mathbf{s} = \iiint_{V[S]} (-\mathbf{E}_1\cdot\mathbf{J}_2 + \mathbf{H}_1\cdot\mathbf{M}_2 + \mathbf{E}_2\cdot\mathbf{J}_1 - \mathbf{H}_2\cdot\mathbf{M}_1)dv. \tag{10.5}$$

Equations (10.4) and (10.5) represent the general ***Lorentz reciprocity theorem in differential and integral forms***, respectively.

One special case of the reciprocity theorem is of fundamental importance to antenna theory, namely, its application to unbounded (open) problems. In this case, the surface S is a sphere of infinite radius. Therefore, the fields integrated over it are far-zone fields. This means that the left-hand side of (10.5) vanishes:

$$\oiint_S \left(\frac{|\mathbf{E}_1||\mathbf{E}_2|}{\eta}\cos\gamma - \frac{|\mathbf{E}_1||\mathbf{E}_2|}{\eta}\cos\gamma \right) ds = 0. \tag{10.6}$$

Here, γ is the angle between the polarization vectors of both fields, $\mathbf{E}_1$ and $\mathbf{E}_2$. Note that in the far zone, the field vectors are orthogonal to the direction of propagation and, therefore are orthogonal to $d\mathbf{s}$. Thus, in the case of open problems, the reciprocity theorem reduces to

$$\iiint_{V[S]} (\mathbf{E}_1\cdot\mathbf{J}_2 - \mathbf{H}_1\cdot\mathbf{M}_2)dv = \iiint_{V[S]} (\mathbf{E}_2\cdot\mathbf{J}_1 - \mathbf{H}_2\cdot\mathbf{M}_1)dv. \tag{10.7}$$

Each of the integrals in (10.7) can be interpreted as ***coupling energy*** between the field produced by some sources and another set of sources generating another field. The quantity

$$\langle 1,2\rangle = \iiint_{V[S]} (\mathbf{E}_1\cdot\mathbf{J}_2 - \mathbf{H}_1\cdot\mathbf{M}_2)dv$$

is called the ***reaction*** of the field $(\mathbf{E}_1,\mathbf{H}_1)$ to the sources $(\mathbf{J}_2,\mathbf{M}_2)$. Similarly,

$$\langle 2,1\rangle = \iiint_{V[S]} (\mathbf{E}_2\cdot\mathbf{J}_1 - \mathbf{H}_2\cdot\mathbf{M}_1)dv$$

is the ***reaction*** of the field $(\mathbf{E}_2,\mathbf{H}_2)$ to the sources $(\mathbf{J}_1,\mathbf{M}_1)$. Thus, in a shorthand notation, the reciprocity equation (10.7) is $\langle 1,2\rangle = \langle 2,1\rangle$.

The Lorentz reciprocity theorem is the most general form of reciprocity in linear EM systems. Circuit reciprocity is a special case of lumped element sources and responses (local voltage or current measurements).

To illustrate the above statement, consider the following scenario. Assume that the sources in two measurements have identical amplitude and phase distributions in their respective volumes: $\mathbf{J}_1$ and $\mathbf{M}_1$ reside in V_1 whereas $\mathbf{J}_2$ and $\mathbf{M}_2$ reside in V_2. Note that the volumes V_1 and V_2 may or may not overlap. We can associate a local coordinate system with each source volume where the position is given by $\mathbf{x}_i = (r_i, \theta_i, \varphi_i)$, $i = 1,2$. If the sources have identical distributions in their respective volumes, i.e., $\mathbf{J}_1(\mathbf{x}_1) = \mathbf{J}_2(\mathbf{x}_2) = \mathbf{J}$, and $\mathbf{M}_1(\mathbf{x}_1) = \mathbf{M}_2(\mathbf{x}_2) = \mathbf{M}$, then, according to (10.7),

$$\iiint_{V_2} (\mathbf{E}_1 \cdot \mathbf{J} - \mathbf{H}_1 \cdot \mathbf{M}) dv_2 = \iiint_{V_1} (\mathbf{E}_2 \cdot \mathbf{J} - \mathbf{H}_2 \cdot \mathbf{M}) dv_1 . \tag{10.8}$$

It follows that $\mathbf{E}_1(\mathbf{x}_2) = \mathbf{E}_2(\mathbf{x}_1)$ and $\mathbf{H}_1(\mathbf{x}_2) = \mathbf{H}_2(\mathbf{x}_1)$. Here, $\mathbf{E}_1$ and $\mathbf{H}_1$ describe the observed field in V_2 (the volume where the sources $\mathbf{J}_2$ and $\mathbf{M}_2$ reside but are inactive), this field being due to the sources $\mathbf{J}_1$ and $\mathbf{M}_1$ (in V_1) which are active. Conversely, $\mathbf{E}_2$ and $\mathbf{H}_2$ describe the observed field in V_1 (the volume where $\mathbf{J}_1$ and $\mathbf{M}_1$ reside but are inactive), that field being due to the sources $\mathbf{J}_2$ and $\mathbf{M}_2$ (in V_2) which are active. These are two measurement scenarios which differ only in the interchanged locations of the source and the observation: in the former scenario, the observation is in V_2 whereas the source is in V_1; in the latter scenario, the observation is in V_1 whereas the source is in V_2. The field equality, $\mathbf{E}_1 = \mathbf{E}_2$ and $\mathbf{H}_1 = \mathbf{H}_2$, tells us that ***interchanging the locations of excitation and observation leaves the observed field unchanged***. This result is general in the sense that it holds in a heterogeneous medium. This is essentially the same principle that is postulated as reciprocity in circuit theory (see Section 1.1). Only that Lorentz' EM reciprocity considers volumes instead of nodes and branches, and field vectors instead of voltages and currents.

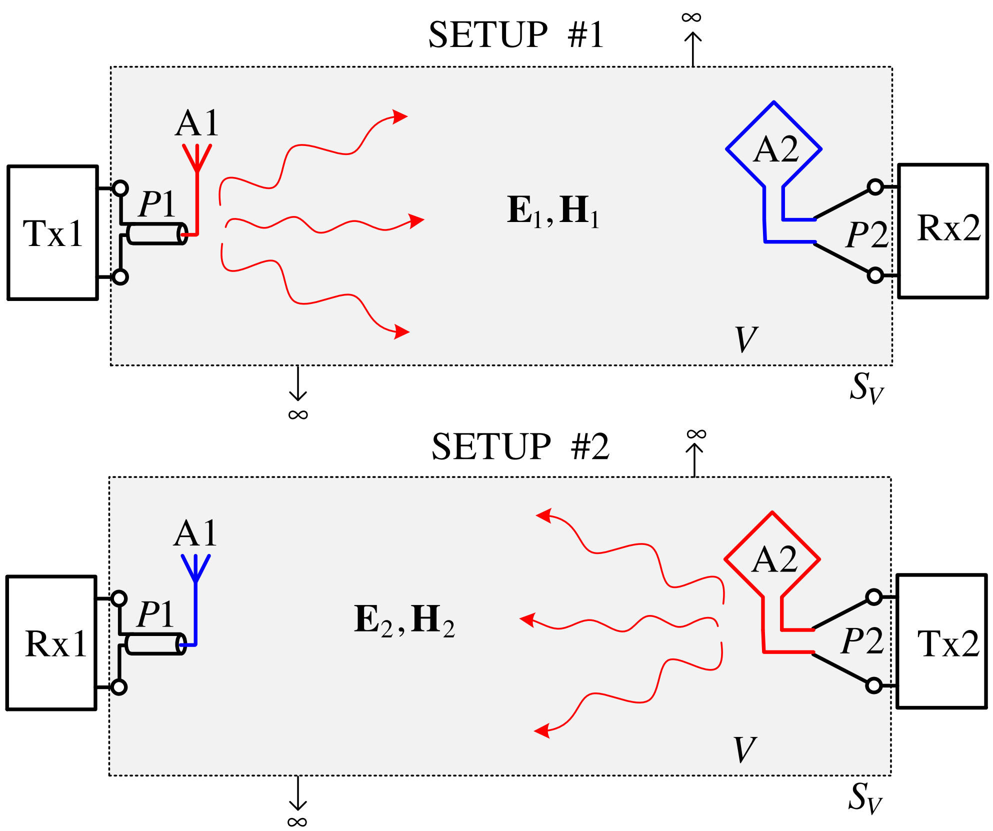
The reciprocity theorem can be postulated also as: *any network constructed of linear isotropic matter has a symmetrical impedance matrix*. This "network" can be two antennas and the space between them.

1.3. Implications of reciprocity for the received-to-transmitted power ratio

Using the reciprocity theorem, we next prove that ***the ratio of received to transmitted power P_r / P_t does not depend on whether antenna #1 transmits***

and antenna #2 receives or vice versa. We should reiterate that the reciprocity theorem holds only if the whole system (antennas + propagation environment) is isotropic and linear.

In this case, we view the two-antenna system as a two-port microwave network; see the figure below. Port 1 (P1) connects to antenna 1 (A1) while port 2 (P2) is at the terminals of antenna 2 (A2). Depending on whether an antenna transmits or receives, its terminals are connected to a transmitter (Tx) or a receiver (Rx), respectively. We consider two measurement setups. In Setup #1, A1 transmits and A2 receives whereas in Setup #2 A1 receives and A2 transmits.



The volume V in both setups excludes the power sources in the respective transmitters and, therefore, it does not have impressed currents sources, i.e., $\mathbf{J}_1 = \mathbf{J}_2 = 0$ and $\mathbf{M}_1 = \mathbf{M}_2 = 0$. The reciprocity integral (10.5) becomes

$$\oint\!\!\!\oint_{S_V} (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot d\mathbf{s} = 0 \qquad (10.9)$$

where the subscripts refer to the measurement setups. Part of the surface S_V extends to infinity away from the antennas (top and bottom lines in the plots above) but it also crosses through P1 and P2. At infinity, the surface integration in (10.9) produces zero; however, at the cross-sections S_1 and S_2 of ports 1 and 2, respectively, the contributions are not zero. Then,

$$\iint_{S_1} (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot d\mathbf{s} + \iint_{S_2} (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot d\mathbf{s} = 0. \tag{10.10}$$

Let us now assume that the transmit power in both setups is 1 W. This is not going to affect the generality of the result. Let us denote the field vectors in the transmission lines of ports 1 and 2 corresponding to 1-W transferred power as $(\mathbf{e}_{\text{P1}}, \mathbf{h}_{\text{P1}})$ and $(\mathbf{e}_{\text{P2}}, \mathbf{h}_{\text{P2}})$, respectively.[1] We assume that these vectors correspond to power transfer *from* the antenna (out of V and toward the Tx or the Rx circuit). When the power is transferred toward the antenna (from the Tx or Rx circuit), due to the opposite direction of propagation, we have to change the sign of either the **e** or the **h** vector (but not both!) in the respective pair.

At P1, in Setup #1, the incident field is the 1-W field generated by Tx1, which is $(\mathbf{e}_{\text{P1}}, -\mathbf{h}_{\text{P1}})$. There could be a reflected field due to impedance mismatch at the A1 terminals, which can be expressed as $\Gamma_1(\mathbf{e}_{\text{P1}}, \mathbf{h}_{\text{P1}})$ where Γ_1 is the reflection coefficient at P1. At P1, in Setup #2, there is the field $(\mathbf{E}_2,\mathbf{H}_2)$ due to the radiation from A2.

Analogous field components can be identified at P2 in both setups: (i) $(\mathbf{e}_{\text{P2}}, -\mathbf{h}_{\text{P2}})$ is the field when in Setup #2 the Tx at A2 provides 1 W of power to the antenna, (ii) $(\mathbf{E}_1,\mathbf{H}_1)$ is the field received at P2 in Setup #1. Equation (10.10) now becomes

$$\begin{aligned}&\iint_{S_1} \left[(\mathbf{e}_{\text{P1}} \times \mathbf{H}_2 + \mathbf{E}_2 \times \mathbf{h}_{\text{P1}}) + \Gamma_1 (\mathbf{e}_{\text{P1}} \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{h}_{\text{P1}}) \right] \cdot d\mathbf{s} + \\ &\iint_{S_2} \left[(-\mathbf{E}_1 \times \mathbf{h}_{\text{P2}} - \mathbf{e}_{\text{P2}} \times \mathbf{H}_1) + \Gamma_2 (\mathbf{E}_1 \times \mathbf{h}_{\text{P2}} - \mathbf{e}_{\text{P2}} \times \mathbf{H}_1) \right] \cdot d\mathbf{s} = 0.\end{aligned} \tag{10.11}$$

Next, the received fields in both scenarios, $(\mathbf{E}_1,\mathbf{H}_1)$ at P2 and $(\mathbf{E}_2,\mathbf{H}_2)$ at P1, can be expressed in terms of $(\mathbf{e}_{\text{P2}}, \mathbf{h}_{\text{P2}})$ and $(\mathbf{e}_{\text{P1}}, \mathbf{h}_{\text{P1}})$, which represent 1-W received powers at the respective ports:

$$\begin{aligned}(\mathbf{E}_2, \mathbf{H}_2)\big|_{\text{P1}} &= R_{1,2} \underbrace{(\mathbf{e}_{\text{P1}}, \mathbf{h}_{\text{P1}})}_{\text{1W power}} \\ (\mathbf{E}_1, \mathbf{H}_1)\big|_{\text{P2}} &= R_{2,1} \underbrace{(\mathbf{e}_{\text{P2}}, \mathbf{h}_{\text{P2}})}_{\text{1W power}}.\end{aligned} \tag{10.12}$$

[1] It can be shown that a propagating mode in a transmission line or a waveguide can be represented by real-valued phasor vectors **e** and **h**, known as *modal* vectors.

Note that (10.12) implies that the respective received-to-transmitted power ratios in Setup #1 and Setup #2 are

$$P_{r1} / P_{t1} = R_{2,1}^2 \tag{10.13}$$

$$P_{r2} / P_{t2} = R_{1,2}^2 . \tag{10.14}$$

Substituting (10.12) into (10.11) leads to

$$\begin{aligned} &R_{1,2} \iint_{S_1} \left[(\underbrace{\mathbf{e}_{\mathrm{P}1} \times \mathbf{h}_{\mathrm{P}1} + \mathbf{e}_{\mathrm{P}1} \times \mathbf{h}_{\mathrm{P}1}}_{=4}) + \Gamma_1 (\underbrace{\mathbf{e}_{\mathrm{P}1} \times \mathbf{h}_{\mathrm{P}1} - \mathbf{e}_{\mathrm{P}1} \times \mathbf{h}_{\mathrm{P}1}}_{=0}) \right] \cdot d\mathbf{s} + \\ &R_{2,1} \iint_{S_2} \left[(\underbrace{-\mathbf{e}_{\mathrm{P}2} \times \mathbf{h}_{\mathrm{P}2} - \mathbf{e}_{\mathrm{P}2} \times \mathbf{h}_{\mathrm{P}2}}_{=-4}) + \Gamma_2 (\underbrace{\mathbf{e}_{\mathrm{P}2} \times \mathbf{h}_{\mathrm{P}2} - \mathbf{e}_{\mathrm{P}2} \times \mathbf{h}_{\mathrm{P}2}}_{=0}) \right] \cdot d\mathbf{s} = 0. \end{aligned} \tag{10.15}$$

Since the fields $(\mathbf{e}_{\mathrm{P}n}, \mathbf{h}_{\mathrm{P}n})$, $n = 1, 2$, correspond to 1 W of transferred power, their respective integrals over the port cross-sections (integration over the Poynting vector) have the same value:

$$\frac{1}{2} \iint_{S_n} (\mathbf{e}_{\mathrm{P}n} \times \mathbf{h}_{\mathrm{P}n}) \cdot d\mathbf{s} = 1 \text{ W}, \; n = 1, 2 . \tag{10.16}$$

Note that here we have assumed that the fields $(\mathbf{e}_{\mathrm{P}n}, \mathbf{h}_{\mathrm{P}n})$, $n = 1, 2$, are "magnitude" (not RMS) phasors. It follows from (10.15) and (10.16) that

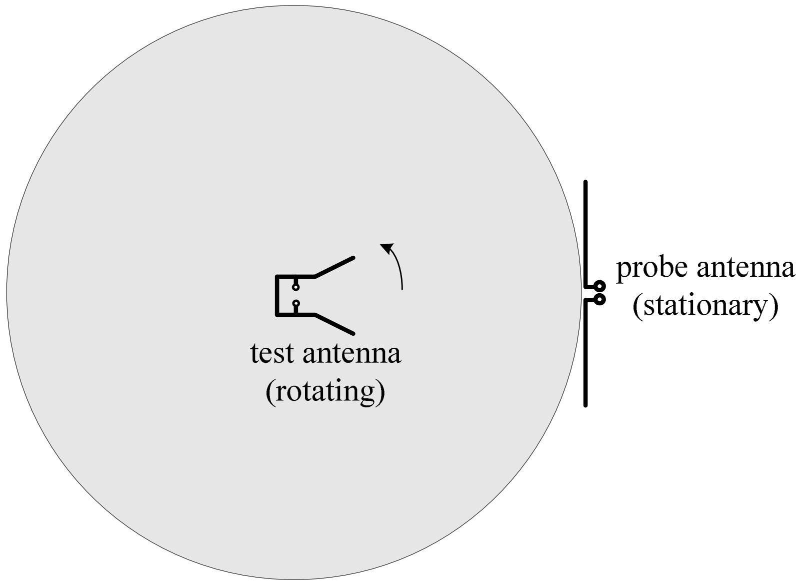
$$R_{1,2} = R_{2,1} . \tag{10.17}$$

This result together with (10.13) and (10.14) leads to the conclusion that the <u>received-to-transmitted power ratio in a two-antenna system does not depend on which antenna transmits and which receives</u>.

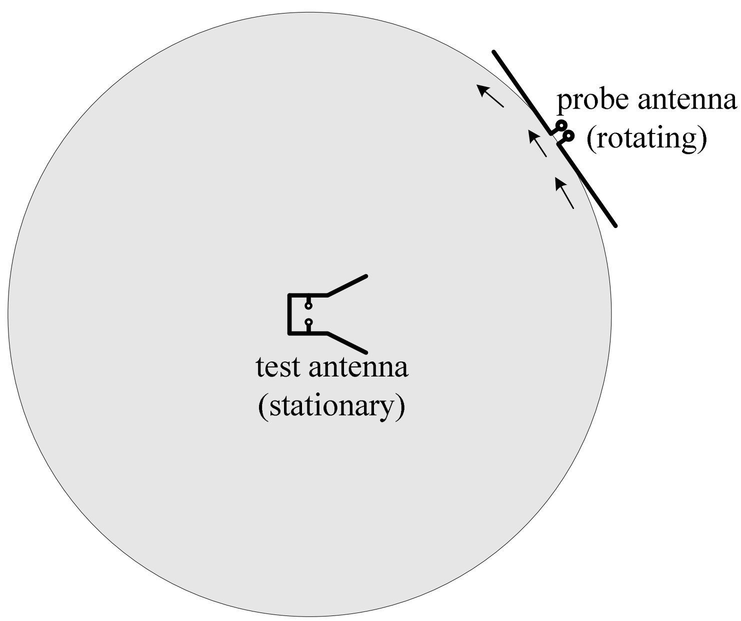
1.4. <u>Reciprocity of the radiation pattern</u>

The measured radiation pattern of an antenna is the same in receiving and in transmitting mode if the system is linear. Nonlinear devices such as diodes and transistors may make the system nonlinear, therefore, nonreciprocal.

In a two-antenna pattern measurement system, the pattern would not depend on whether the antenna under test (AUT) receives and the other antenna transmits, or *vice versa*. The pattern depends only on the mutual angular orientation of the two antennas (the distance between the two antennas must remain the same regardless of the angular orientation of the antennas). It also does not matter whether the AUT rotates and the other antenna is stationary, or *vice versa*.



(a)



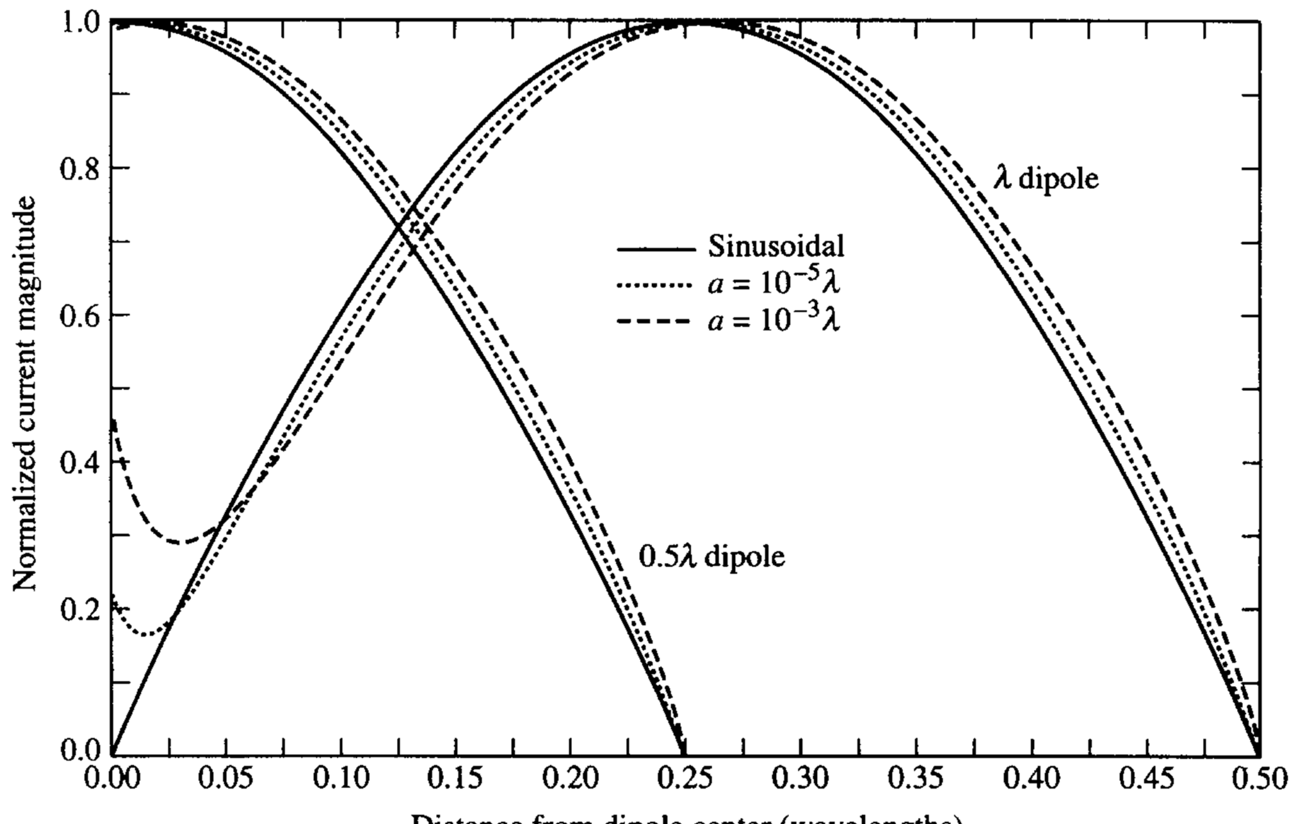
(b)

Scenario (a) is obviously more practical especially because the distance between the antennas must be sufficiently large to ensure a measurement in the far zone.

2. Self-impedance of a Dipole Using the Induced *EMF* Method

The induced ***emf*** (electro-motive force) method was developed by Carter[2] in 1932, when computers were not available and analytical (closed-form) solutions were much needed to calculate the self-impedance of wire antennas. The method was later extended to calculate mutual impedances of multiple wires (see, e.g., Elliot, *Antenna Theory and Design*). The ***emf*** method is restricted to straight parallel wires.

Measurements and full-wave simulations indicate that the current distribution on thin dipoles is nearly sinusoidal (except at the current minima). The induced ***emf*** method assumes this type of idealized distribution. It results in satisfactory accuracy for dipoles with length-to-diameter ratios larger than 100. The accuracy deteriorates closer to the feed point and is particularly poor for dipoles, the length of which approaches a wavelength.



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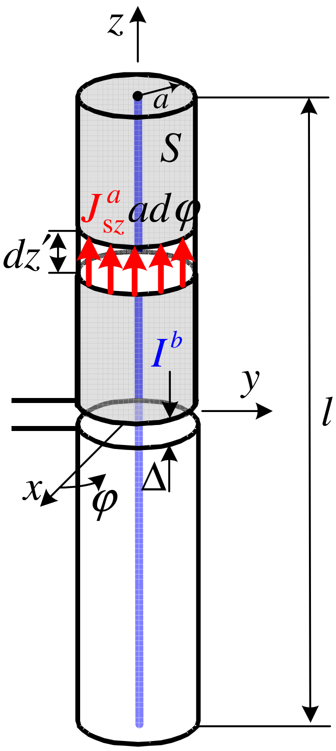
[2] P.S. Carter, "Circuit relations in radiating systems and applications to antenna problems," *Proc. IRE*, **20**, pp.1004-1041, June 1932.



Consider a tubular dipole the arms of which are made of perfect electric conductor (PEC). When excited by a voltage-gap source at its base, the dipole supports surface current along z, which radiates. This surface current density is $J_{sz}(z') = H_\varphi(z')$ as per the boundary conditions at the PEC surface where $E_z(z') = 0$.

Using the equivalence principle, we consider an equivalent problem where $J_{sz}^a(z') = J_{sz}(z')$ is a cylindrical current sheet that exists over a closed cylindrical surface S tightly enveloping the dipole. It radiates in open space generating the field $(\mathbf{E}^a, \mathbf{H}^a)$ such that

$$E_z^a(\rho \le a, z') = \begin{cases} V_{\text{in}}^a / \Delta, & -\Delta/2 \le z' \le \Delta/2, \\ 0, & \text{elsewhere.} \end{cases} \quad (10.18)$$

Here, Δ is the feed-gap length. Note that the **E**-field inside the tubular volume (except in the gap) is zero.

Next, consider a fictitious linear current source $I^b(z')$ along the axis of the cylinder (z axis) where I^b is nonzero only for $-l/2 < z' < l/2$. It also radiates in open space and its field is denoted as $(\mathbf{E}^b, \mathbf{H}^b)$. We require that $I^b(z')$ represents the actual current distribution on the metallic surface of the dipole, i.e., $I^b(z') = 2\pi a J_{sz}^a(z')$.

In the volume (air) bound by S, we apply the reciprocity formula (10.7):

$$\iiint_{V[S]} (\mathbf{E}^a \cdot \mathbf{J}^b - \mathbf{E}^b \cdot \mathbf{J}^a) dv = 0. \quad (10.19)$$

Bearing in mind the surface nature of the current source a, the linear nature of current source b, and equation (10.18), we write (10.19) as

$$\int_0^{2\pi} \int_{-l/2}^{l/2} E_z^b J_{sz}^a a dz' d\varphi = \int_{-\Delta/2}^{\Delta/2} E_z^a I^b dz'. \quad (10.20)$$

In (10.20), we assumed that the electric fields of the two sources have only z components.

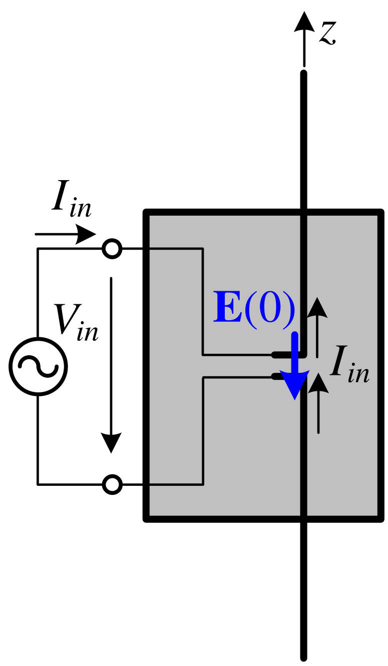
Assuming constant current distribution in the feed gap, we obtain

$$\int_0^{2\pi} \int_{-l/2}^{l/2} E_z^b J_{sz}^a a dz' d\varphi = -I_{\text{in}}^b V_{\text{in}}^a \quad (10.21)$$

where

$$V_{\text{in}}^a = -\int_{-\Delta/2}^{\Delta/2} E_z^a dz' \tag{10.22}$$

is the voltage at the terminals of the generator driving the current $J_{sz}^a(z')$. The minus sign in (10.22) reflects the fact that a positive V_{in}^a, which implies a "positive" current I_{in}, i.e., current flowing in the positive z direction, relates to a "negative" electric field at the dipole's base, i.e., E_z points in the negative z direction. This is illustrated below.



Further, due to the cylindrical symmetry, all quantities in the integral in (10.21) are independent of φ. Thus,

$$\int_{-l/2}^{l/2} E_z^b \left(2\pi a J_{sz}^a\right) dz' = -I_{\text{in}}^b V_{\text{in}}^a . \tag{10.23}$$

The quantity in the brackets in (10.23) is the total current I^a at position z'. Thus,

$$\int_{-l/2}^{l/2} E_z^b I^a dz' = -I_{\text{in}}^b V_{\text{in}}^a . \tag{10.24}$$

Here, I^a is the actual current distribution along the surface of the dipole's arms. Since we require the distribution of I^a and I^b along z' to be the same, we can now drop the superscripts:

$$\int_{-l/2}^{l/2} E_z I dz' = -I_{\text{in}} V_{\text{in}} . \tag{10.25}$$

As a reminder, $E_z \equiv E_z^b$ is the field at the fictitious cylindrical surface enveloping the dipole volume (air) due to $I^b(z') = I^a(z')$. The above result leads to the following self-impedance expression:

$$Z_{\text{in}}\Big|_{z'=0} = \frac{V_{\text{in}}}{I_{\text{in}}} = \frac{V_{\text{in}} \cdot I_{\text{in}}}{I_{\text{in}}^2} = -\frac{1}{I_{\text{in}}^2} \int_{-l/2}^{l/2} E_z(z') I(z') dz' . \tag{10.26}$$

In the classical ***emf*** method, we assume that the current has a sinusoidal distribution:

$$I(z') = \begin{cases} I_0 \sin\left[\beta\left(\dfrac{l}{2} - z'\right)\right], \; 0 \le z' \le l/2 \\ I_0 \sin\left[\beta\left(\dfrac{l}{2} + z'\right)\right], -l/2 \le z' \le 0. \end{cases} \tag{10.27}$$

So far, we have obtained only the far-field components of the field generated by the current in (10.27) (see Lecture 9). However, when the input resistance and reactance are needed, the near field must be known. In our case, we are interested in E_z, which is the field produced by the filamentary current $I(z')$ at the fictitious cylindrical surface enveloping the dipole volume (air). If we know it, we can calculate the integral in (10.26) since we already know $I(z')$ from (10.27).

We use cylindrical coordinates to describe the locations of the integration point (primed coordinates) and the observation point. The electric field can be expressed in terms of the VP $\mathbf{A}$ and the scalar potential ϕ (see Lecture 2):

$$\mathbf{E} = -\nabla\phi - j\omega\mathbf{A}, \tag{10.28}$$

$$\Rightarrow E_z = -\frac{\partial\phi}{\partial z} - j\omega A_z . \tag{10.29}$$

The VP $\mathbf{A}$ is z-polarized,

$$A_z = \frac{\mu}{4\pi} \int_{-l/2}^{l/2} I_z(z') \frac{e^{-j\beta R}}{R} dz'. \tag{10.30}$$

The scalar potential is

$$\phi = \frac{1}{4\pi\varepsilon} \int_{-l/2}^{l/2} q_l(z') \frac{e^{-j\beta R}}{R} dz'. \tag{10.31}$$

Here, q_l stands for linear charge density in C/m. Knowing that the current depends only on z', the continuity relation is written as

$$j\omega q_l = -\frac{\partial I_z}{\partial z'}. \tag{10.32}$$

$$\Rightarrow q_l(z') = \begin{cases} -j\dfrac{I_0}{c}\cos\left[\beta\left(\dfrac{l}{2} - z'\right)\right], & 0 \le z' \le l/2 \\ +j\dfrac{I_0}{c}\cos\left[\beta\left(\dfrac{l}{2} + z'\right)\right], & -l/2 \le z' \le 0 \end{cases} \tag{10.33}$$

where $c = \omega / \beta$ is the speed of light. Now, we express **A** and ϕ as

$$A_z = \frac{\mu}{4\pi} I_0 \left\{ \int_{-l/2}^{0} \sin\left[\beta\left(\frac{l}{2} + z'\right)\right] \frac{e^{-j\beta R}}{R} dz' + \int_{0}^{l/2} \sin\left[\beta\left(\frac{l}{2} - z'\right)\right] \frac{e^{-j\beta R}}{R} dz' \right\} \tag{10.34}$$

$$\phi = j\frac{\eta I_0}{4\pi} \left\{ -\int_{-l/2}^{0} \cos\left[\beta\left(\frac{l}{2} + z'\right)\right] \frac{e^{-j\beta R}}{R} dz' + \int_{0}^{l/2} \cos\left[\beta\left(\frac{l}{2} - z'\right)\right] \frac{e^{-j\beta R}}{R} dz' \right\}. \tag{10.35}$$

Here, $\eta = \sqrt{\mu / \varepsilon}$ is the intrinsic impedance of the medium.

The distance between integration and observation point is

$$R = \sqrt{\rho^2 + (z - z')^2}. \tag{10.36}$$

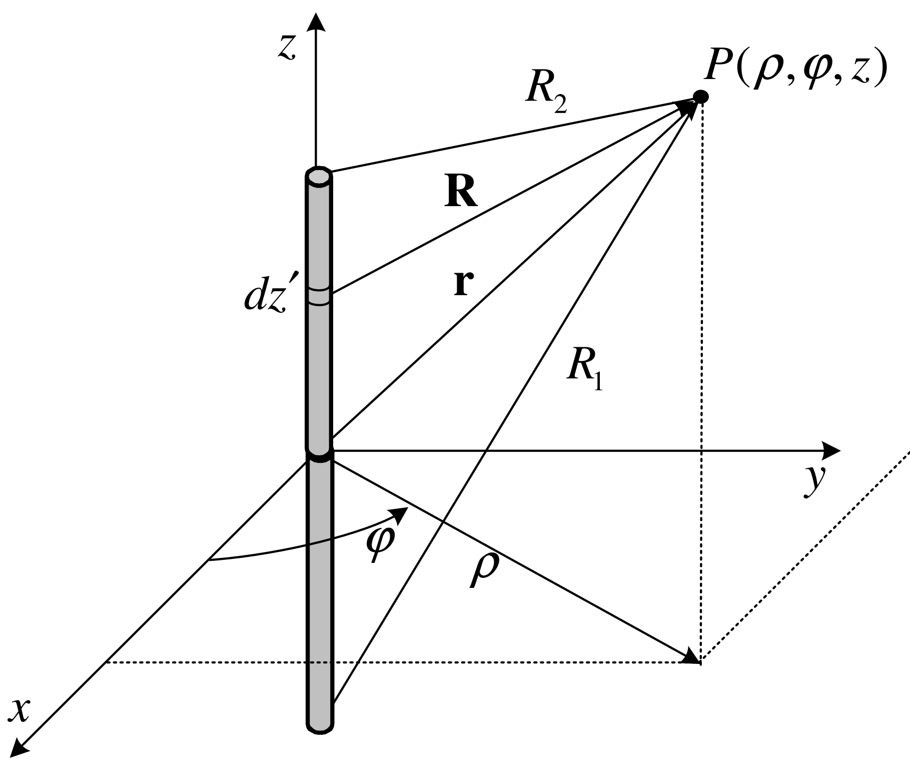
Equation (10.36) is substituted in (10.34) and (10.35). In addition, the resulting equations for A_z and ϕ are modified making use of Moivre's formulas:

$$\begin{aligned} \cos x &= 0.5\left(e^{jx} + e^{-jx}\right) \\ \sin x &= -j0.5\left(e^{jx} - e^{-jx}\right) \end{aligned}, \text{ where } x = \beta\left(\frac{l}{2} \pm z'\right). \tag{10.37}$$

Then, the equations for A_z and ϕ are substituted in (10.29) to derive the expression for E_z valid at any observation point P. This is a rather lengthy derivation, and we give the final result only:

$$E_z(P) = -j\frac{\eta I_0}{4\pi}\left[\frac{e^{-j\beta R_1}}{R_1} + \frac{e^{-j\beta R_2}}{R_2} - 2\cos\left(\frac{\beta l}{2}\right)\frac{e^{-j\beta r}}{r}\right]. \qquad (10.38)$$

Here, r is the distance from the observation point to the dipole's center, while R_1 and R_2 are the distances to the lower and upper vertices of the dipole, respectively (see figure below). The result in (10.38) is exact.



We need $E_z(z')$ at the dipole's surface where we employ the thin-wire approximation assuming that $a \ll l$:

$$r \approx z', \; R_1 = z' + l/2, \text{ and } R_2 = (l/2) - z', \; \rho \le a \;\; (a \ll l). \qquad (10.39)$$

The final goal of this development is to find the self-impedance (10.26) of the dipole. We substitute (10.39) in (10.38). The result for $E_z(z')$ is then substituted in (10.26), and the integration is performed. We give the final results for the real and imaginary parts of Z_{in}:

$$R_{\text{in}} = k\frac{\eta}{2\pi}\cdot\Big\{C + \ln(\beta l) - C_i(\beta l) + \frac{1}{2}\sin(\beta l)\left[S_i(2\beta l) - 2S_i(\beta l)\right] + \\ + \frac{1}{2}\cos(\beta l)\left[C + \ln\left(\frac{\beta l}{2}\right) + C_i(2\beta l) - 2C_i(\beta l)\right]\Big\} = k\frac{\eta}{2\pi}\cdot\Im\,, \tag{10.40}$$

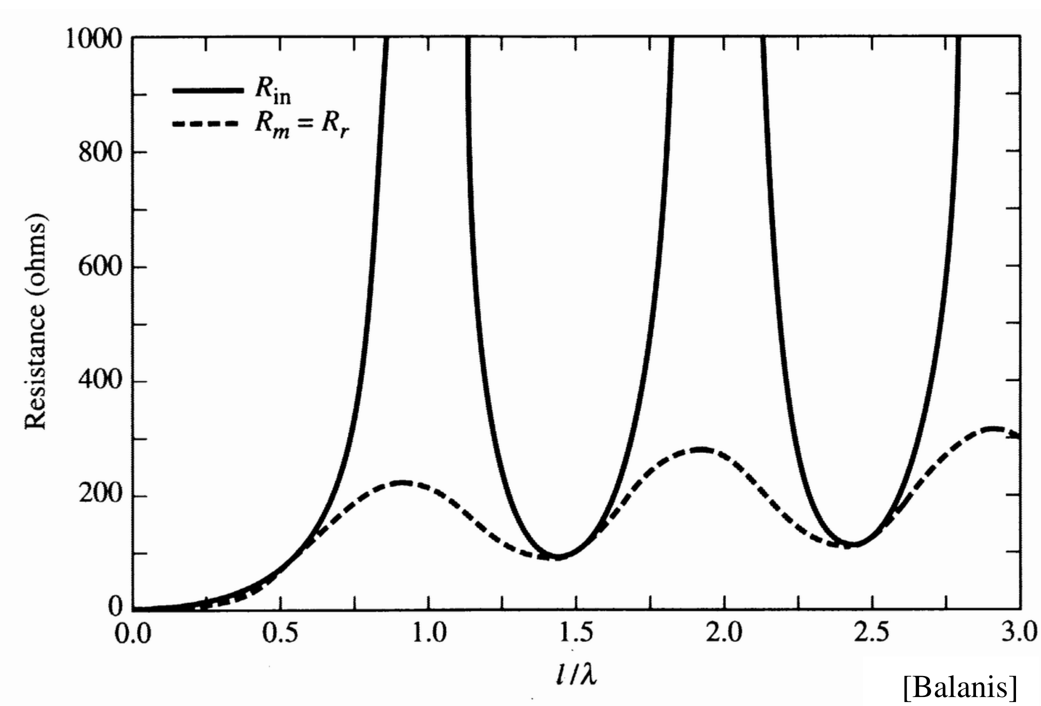
$$X_{\text{in}} = k\frac{\eta}{4\pi}\Big\{2S_i(\beta l) - \cos(\beta l)\left[S_i(2\beta l) - 2S_i(\beta l)\right] + \\ + \sin(\beta l)\left[C_i(2\beta l) - 2C_i(\beta l) + C_i(2\beta a^2/l)\right]\Big\}\,, \tag{10.41}$$

where $k = 1/\sin^2(\beta l/2)$ is the coefficient accounting for the difference between the maximum current magnitude along the dipole and the magnitude of the input current at the dipole's center [see Lecture 9, section 2]. Also, C is the Euler's constant, S_i is the sine integral and C_i is the cosine integral.

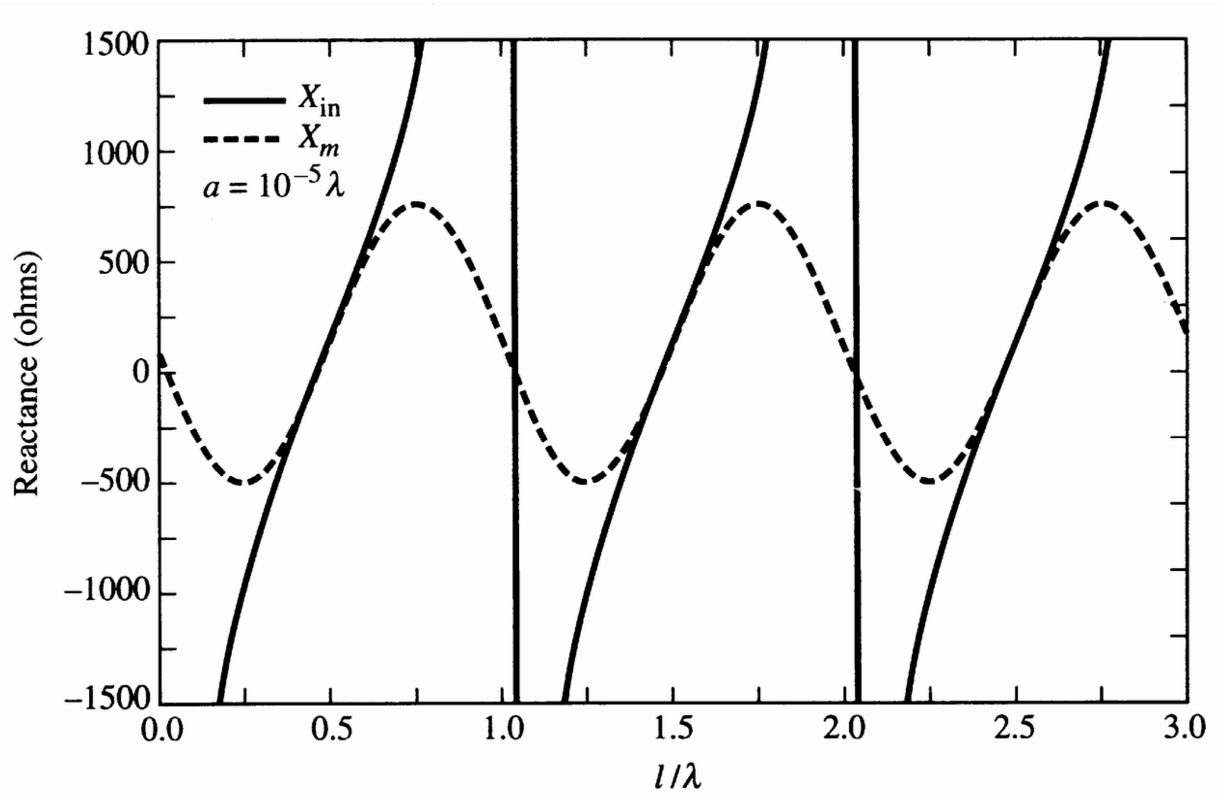
Equation (10.40) is identical with the expression found for the input resistance of an infinitesimally thin wire [see Lecture 9, Eqs. (9.37) and (9.38)]. Expression (10.41) for the dipole's reactance however is new. For a short dipole, the input reactance can be approximated by a simpler formula:

$$X_{in} \approx -120\left[\ln(l/a) - 1\right]/\tan(\beta l)\,. \tag{10.42}$$

The output of (10.40) and (10.41) *versus* l/λ is given below for $a = 10^{-5}\lambda$.



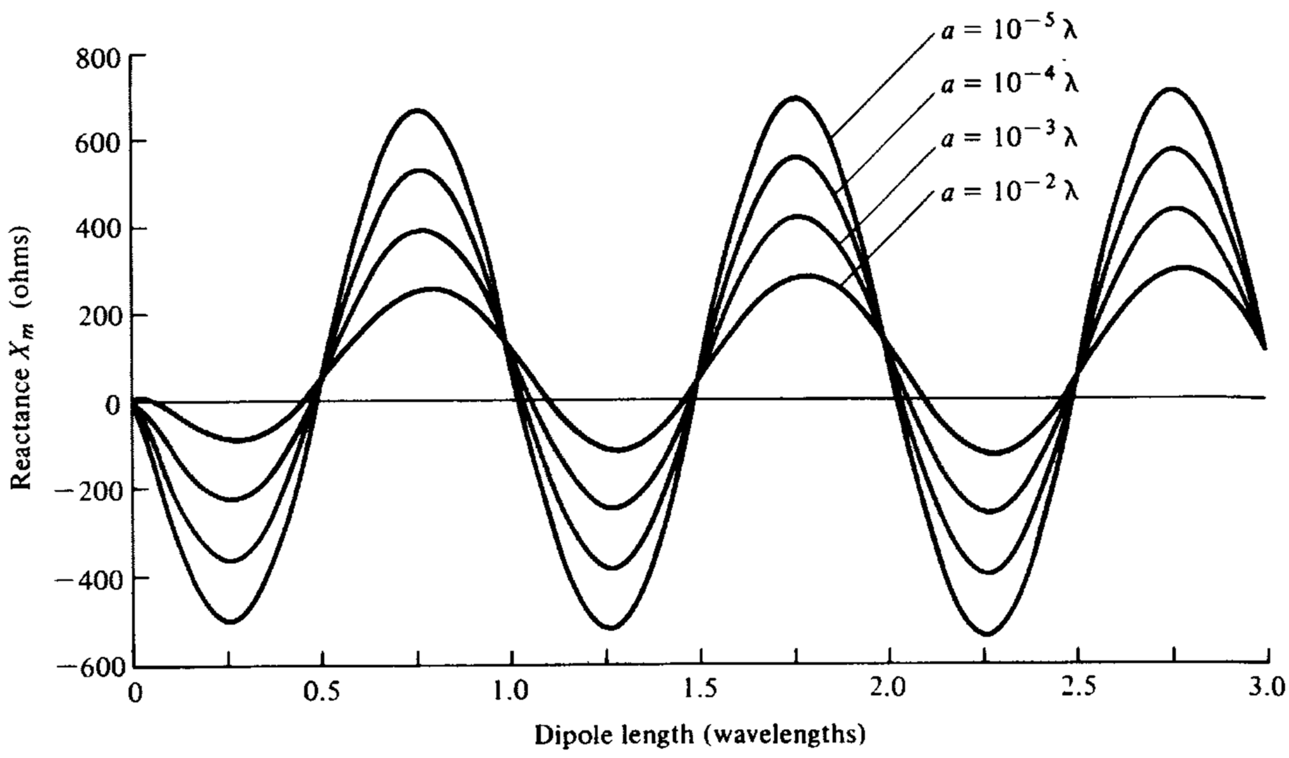
(a) Resistance



(b) Reactance

[Balanis]

INPUT REACTANCE OF A THIN DIPOLE (***EMF*** METHOD) FOR DIFFERENT RADII a



Note that:

- the reactance does not depend on the radius a, when the dipole length is a multiple of a half-wavelength ($l = n\lambda / 2$), as follows from (10.41);
- the resistance does not depend on a according to the assumptions made in the ***emf*** method (see equation (10.40)).
- In the plots above, R_m and X_m correspond to impedance values computed without the factor $k = 1/\sin^2(\beta l / 2)$.