LECTURE 13: LINEAR ARRAY THEORY - PART I

(*Linear arrays: the two-element array. N-element array with uniform amplitude and spacing. Broad-side array. End-fire array. Phased array*.)

1. Introduction

Usually the radiation patterns of single-element antennas are relatively wide, i.e., they have relatively low directivity (gain). In long distance communications, antennas with high directivity are often required. Such antennas are possible to construct by enlarging the dimensions of the radiating aperture (maximum size much larger than λ). This approach however may lead to the appearance of multiple side lobes. Besides, the antenna is usually large and difficult to fabricate.

Another way to increase the electrical size of an antenna is to construct it as an assembly of radiating elements in a proper electrical and geometrical configuration – ***antenna array***. Usually, the array elements are identical. This is not necessary but it is practical and simpler for design and fabrication. The individual elements may be of any type (wire dipoles, loops, apertures, etc.)

The total field of an array is a vector superposition of the fields radiated by the individual elements. To provide very directive pattern, it is necessary that the partial fields (generated by the individual elements) interfere constructively in the desired direction and interfere destructively in the remaining space.

There are five basic methods to control the overall antenna pattern:

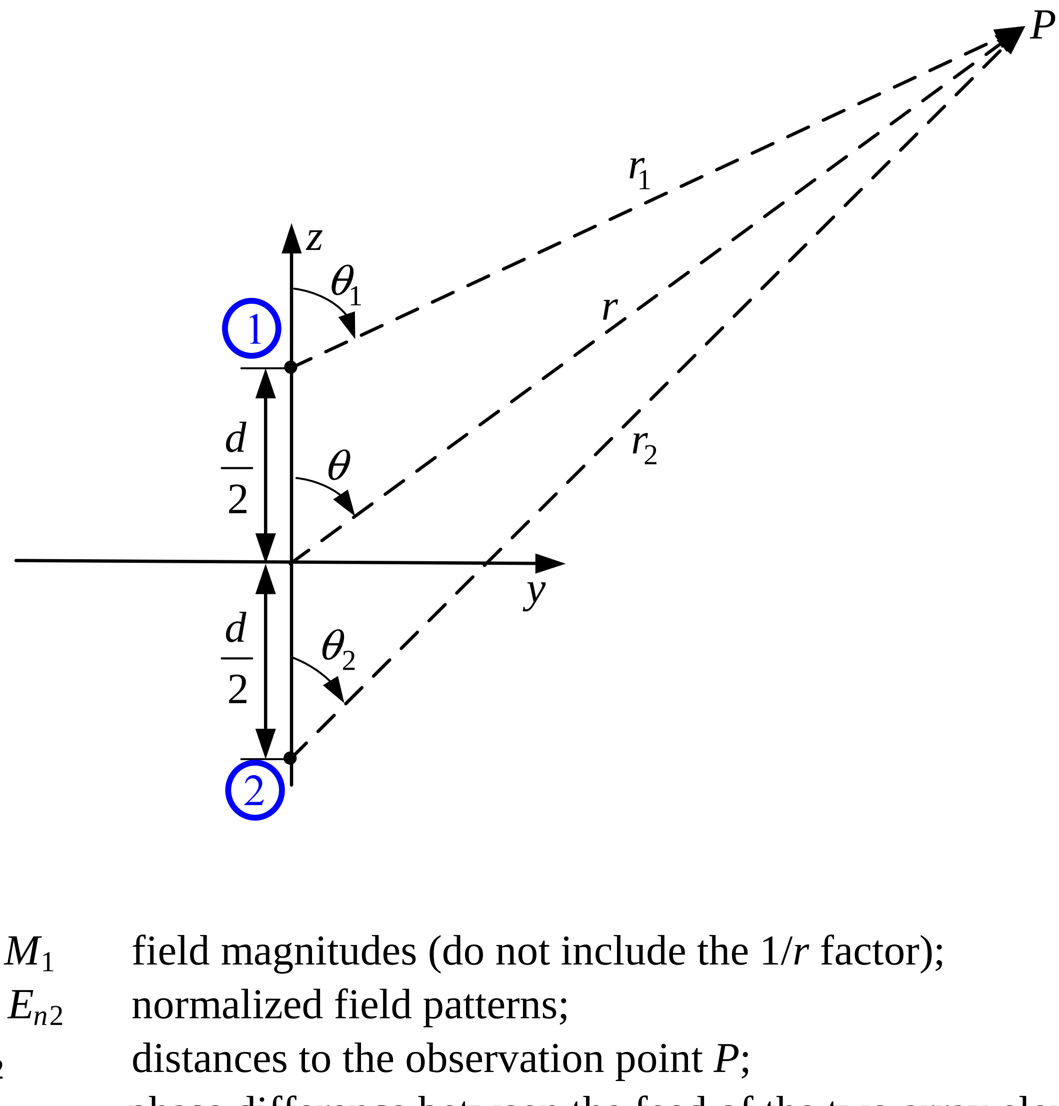
a) the geometrical configuration of the overall array (linear, circular, spherical, rectangular, etc.),
b) the relative placement of the elements,
c) the excitation amplitude of the individual elements,
d) the excitation phase of each element,
e) the individual pattern of each element.

2. Two-element Array

Let us represent the electric fields in the far zone of the array elements in the form

$$\mathbf{E}_1 = M_1 E_{n1}(\theta_1,\phi_1)\frac{e^{-j\left(kr_1-\frac{\beta}{2}\right)}}{r_1}\hat{\boldsymbol{\rho}}_1, \tag{13.1}$$

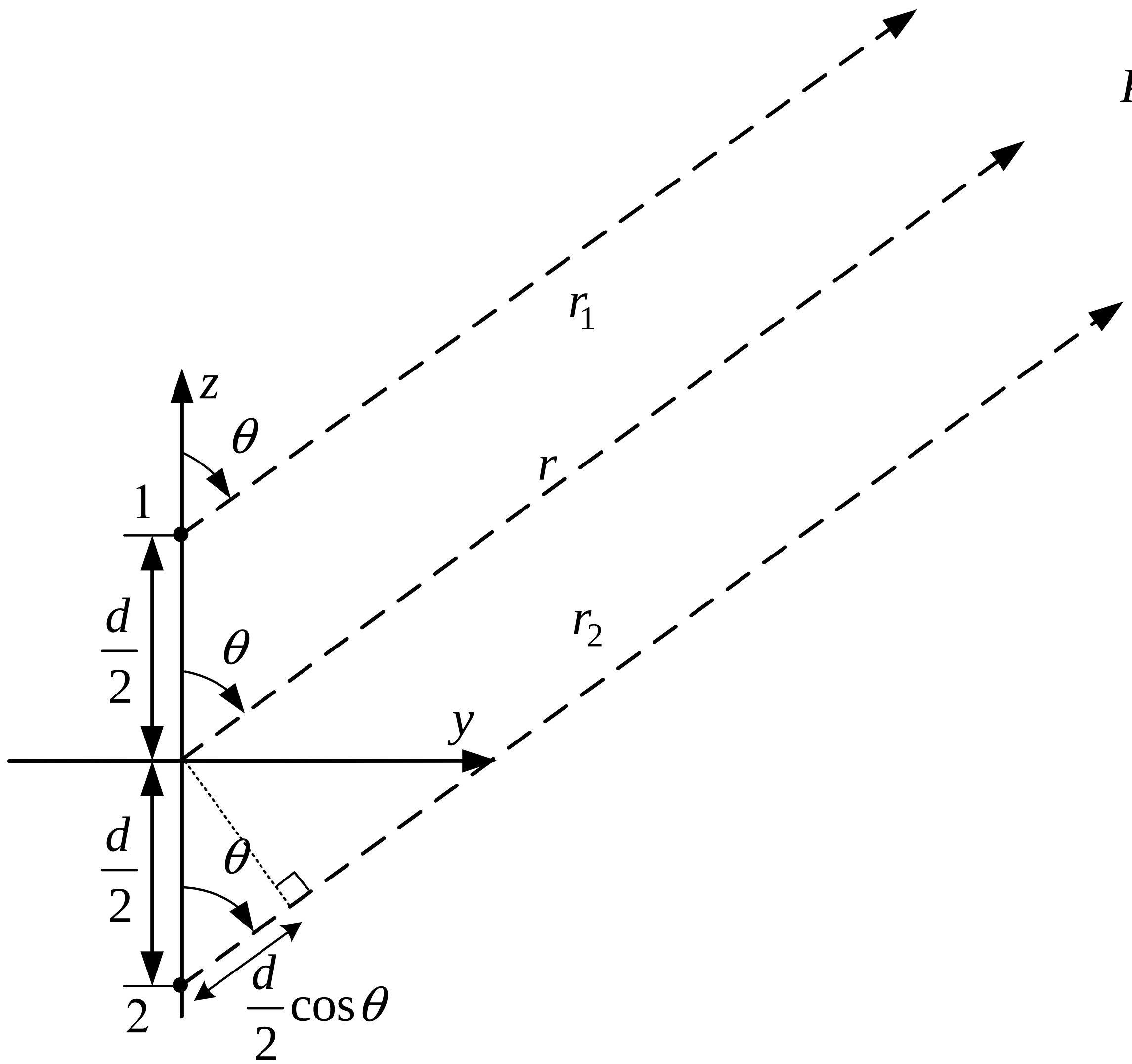
$$\mathbf{E}_2 = M_2 E_{n2}(\theta_2,\phi_2)\frac{e^{-j\left(kr_2+\frac{\beta}{2}\right)}}{r_2}\hat{\boldsymbol{\rho}}_2. \tag{13.2}$$



Here,

$M_1,\ M_1$	field magnitudes (do not include the $1/r$ factor);
$E_{n1},\ E_{n2}$	normalized field patterns;
$r_1,\ r_2$	distances to the observation point P;
β	phase difference between the feed of the two array elements;
$\hat{\boldsymbol{\rho}}_1,\ \hat{\boldsymbol{\rho}}_2$	polarization vectors of the far-zone **E** fields.

The far-field approximation of the two-element array problem:



Let us assume that:

1) the array elements are identical, i.e.,

$$E_{n1}(\theta,\phi) = E_{n2}(\theta,\phi) = E_n(\theta,\phi), \tag{13.3}$$

2) they are oriented in the same way in space (they have identical polarizations), i.e.,

$$\hat{\boldsymbol{\rho}}_1 = \hat{\boldsymbol{\rho}}_2 = \hat{\boldsymbol{\rho}}, \tag{13.4}$$

3) their excitation is of the same amplitude, i.e.,

$$M_1 = M_2 = M. \tag{13.5}$$

Then, the total field can be derived as

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2, \tag{13.6}$$

$$\mathbf{E} = \hat{\boldsymbol{\rho}} M E_n(\theta,\phi)\frac{1}{r}\left[e^{-jk\left(r-\frac{d}{2}\cos\theta\right)+j\frac{\beta}{2}} + e^{-jk\left(r+\frac{d}{2}\cos\theta\right)-j\frac{\beta}{2}} \right],$$

$$\mathbf{E} = \hat{\boldsymbol{\rho}} \frac{M}{r} e^{-jkr} E_n(\theta,\phi)\left[e^{j\left(\frac{kd}{2}\cos\theta+\frac{\beta}{2}\right)} + e^{-j\left(\frac{kd}{2}\cos\theta+\frac{\beta}{2}\right)} \right],$$

$$\boxed{\mathbf{E} = \underbrace{\hat{\boldsymbol{\rho}} M \frac{e^{-jkr}}{r} E_n(\theta,\phi)}_{EF} \times \underbrace{2\cos\left(\frac{kd\cos\theta+\beta}{2}\right)}_{AF}}. \tag{13.7}$$

The total field of the array is equal to the product of the field created by a single element located at the origin and the ***array factor*** (AF):

$$AF = 2\cos\left(\frac{kd\cos\theta+\beta}{2}\right). \tag{13.8}$$

Using the normalized field pattern of a single element, $E_n(\theta,\phi)$, and the normalized AF,

$$AF_n = \cos\left(\frac{kd\cos\theta+\beta}{2}\right), \tag{13.9}$$

the normalized field pattern of the array is expressed as their product:

$$f_n(\theta,\phi) = E_n(\theta,\phi) \times AF_n(\theta,\phi). \tag{13.10}$$

The concept expressed by (13.10) is the so-called ***pattern multiplication rule*** valid for arrays of identical elements. This rule holds for any array consisting of decoupled identical elements, where the excitation magnitudes, the phase shift between the elements and the displacement between them are not necessarily the same. The total pattern, therefore, can be controlled via the single–element pattern $E_n(\theta,\phi)$ or via the AF. The AF, in general, depends on the:

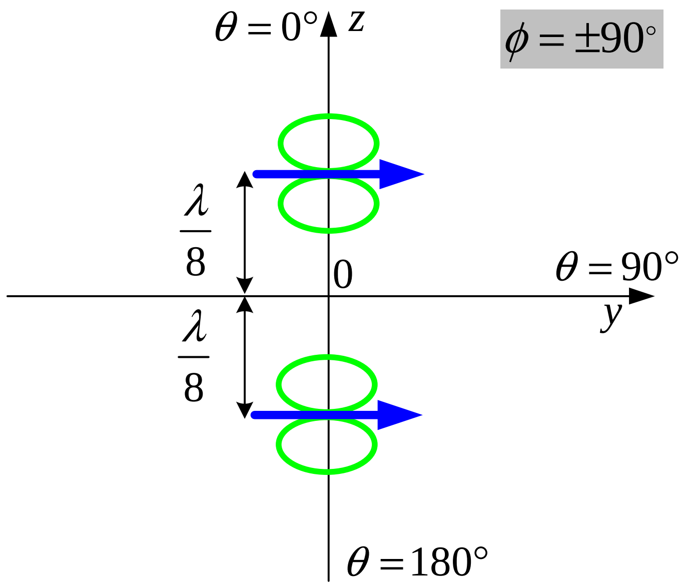
- number of elements,
- mutual placement,
- relative excitation magnitudes and phases.

Example 1: An array consists of two horizontal infinitesimal dipoles located at a distance $d=\lambda/4$ from each other. Find the nulls of the total field in the elevation plane $\phi=\pm 90^\circ$, if the excitation magnitudes are the same and the phase difference is:

a) $\beta=0$

b) $\beta=\pi/2$

c) $\beta=-\pi/2$



The element factor $E_n(\theta,\phi)=\sqrt{1-\sin^2\theta\sin^2\varphi}$ does not depend on β, and it produces in all three cases the same null. For $\phi=\pm 90^\circ$, $E_n(\theta,\phi)=|\cos\theta|$ and the null is at

$$\theta_1=\pi/2. \tag{13.11}$$

The *AF* depends on β and produces different results in the 3 cases:

a) $\beta=0$

$$AF_n=\cos\left(\frac{kd\cos\theta_n}{2}\right)=0 \quad \Rightarrow \cos\left(\frac{\pi}{4}\cos\theta_n\right)=0,$$

$$\Rightarrow \frac{\pi}{4}\cos\theta_n=(2n+1)\frac{\pi}{2} \quad \Rightarrow \cos\theta_n=(2n+1)\cdot 2, \;\; n=0,\pm1,\pm2,\ldots.$$

A solution with a real-valued angle does not exist. In this case, the total field pattern has only 1 null at $\theta = 90°$.

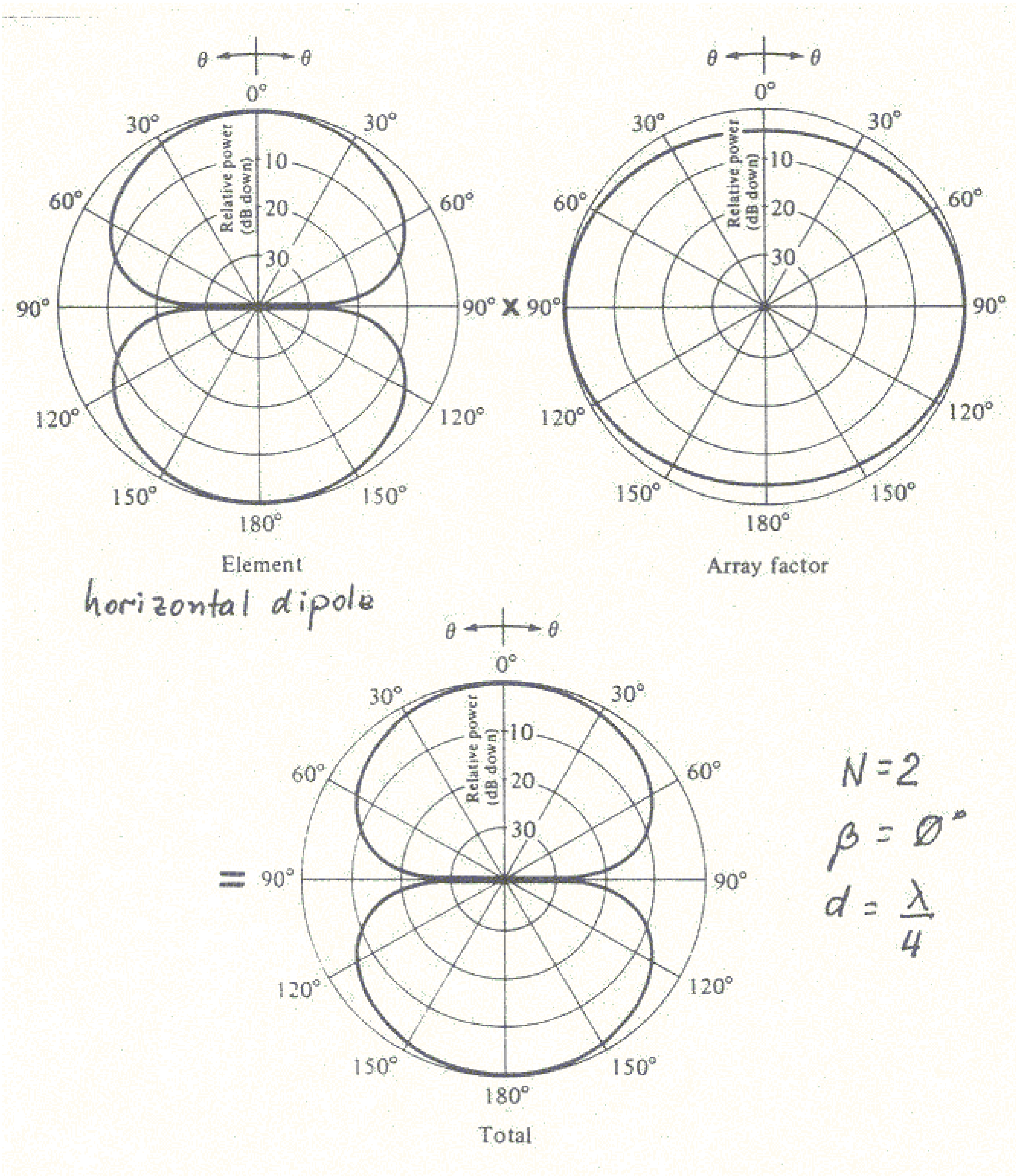


Fig. 6.3, p. 255, Balanis

b) $\beta = \pi / 2$

$$AF_n = \cos\left(\frac{\pi}{4}\cos\theta_n + \frac{\pi}{4}\right) = 0 \quad \Rightarrow \frac{\pi}{4}\left(\cos\theta_n + 1\right) = (2n+1)\frac{\pi}{2},$$

$$\Rightarrow \cos\theta_n + 1 = (2n+1)\cdot 2 \quad \Rightarrow \cos\theta_{(n=0)} = 1 \quad \Rightarrow \boxed{\theta_2 = 0}.$$

The solution for $n = 0$ is the only real-valued solution. Thus, the total field pattern has 2 nulls: at $\theta_1 = 90°$ and at $\theta_2 = 0°$:

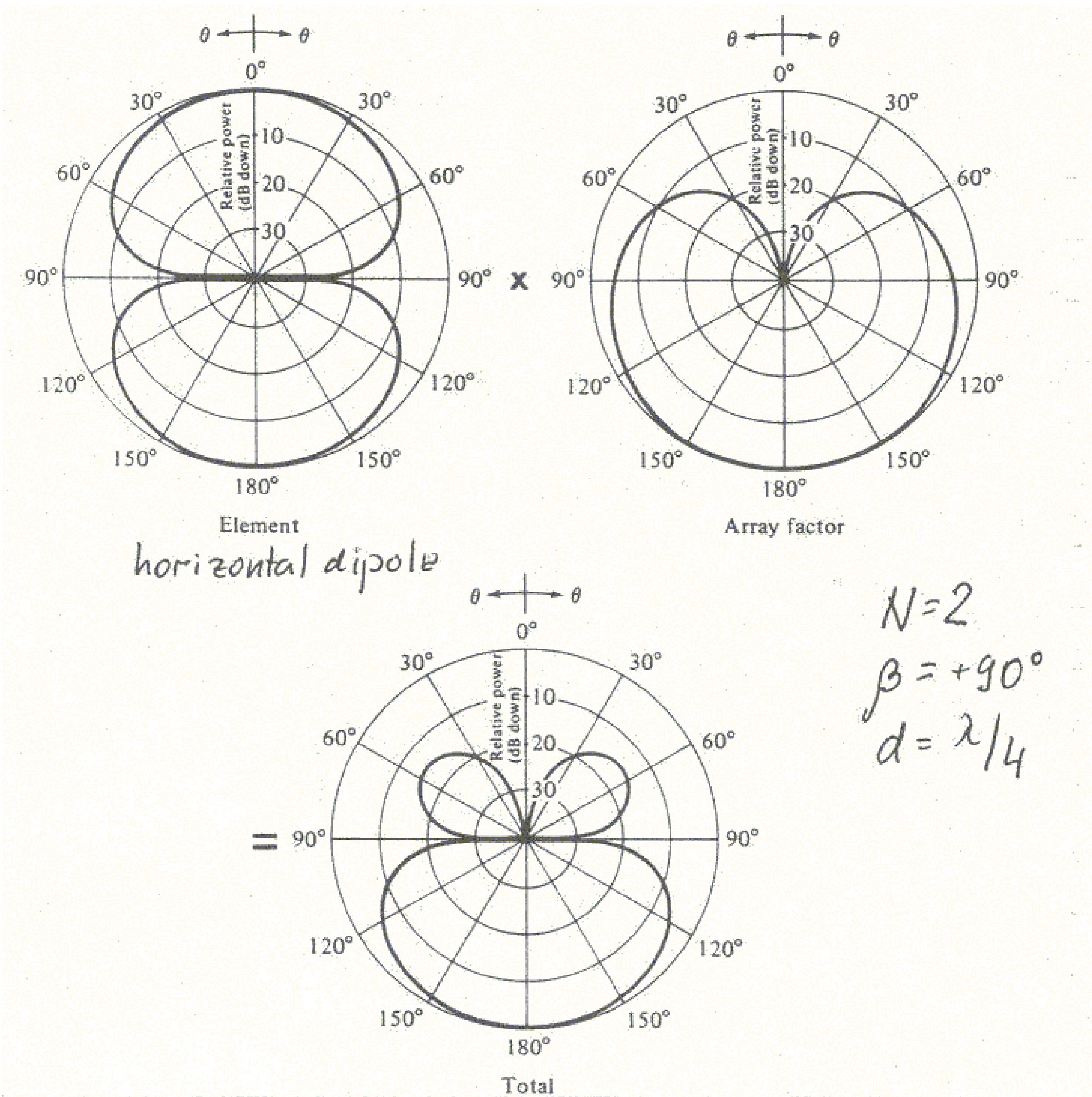


Fig. 6.4, p. 256, Balanis

c) $\beta = -\pi / 2$

$$AF_n = \cos\left(\frac{\pi}{4}\cos\theta_n - \frac{\pi}{4}\right) = 0 \quad \Rightarrow \frac{\pi}{4}(\cos\theta_n - 1) = (2n+1)\frac{\pi}{2},$$

$$\Rightarrow \cos\theta_n - 1 = (2n+1)\cdot 2 \quad \Rightarrow \cos\theta_{(n=-1)} = -1 \quad \Rightarrow \boxed{\theta_2 = \pi}.$$

The total field pattern has 2 nulls: at $\theta_1 = 90°$ and at $\theta_2 = 180°$.

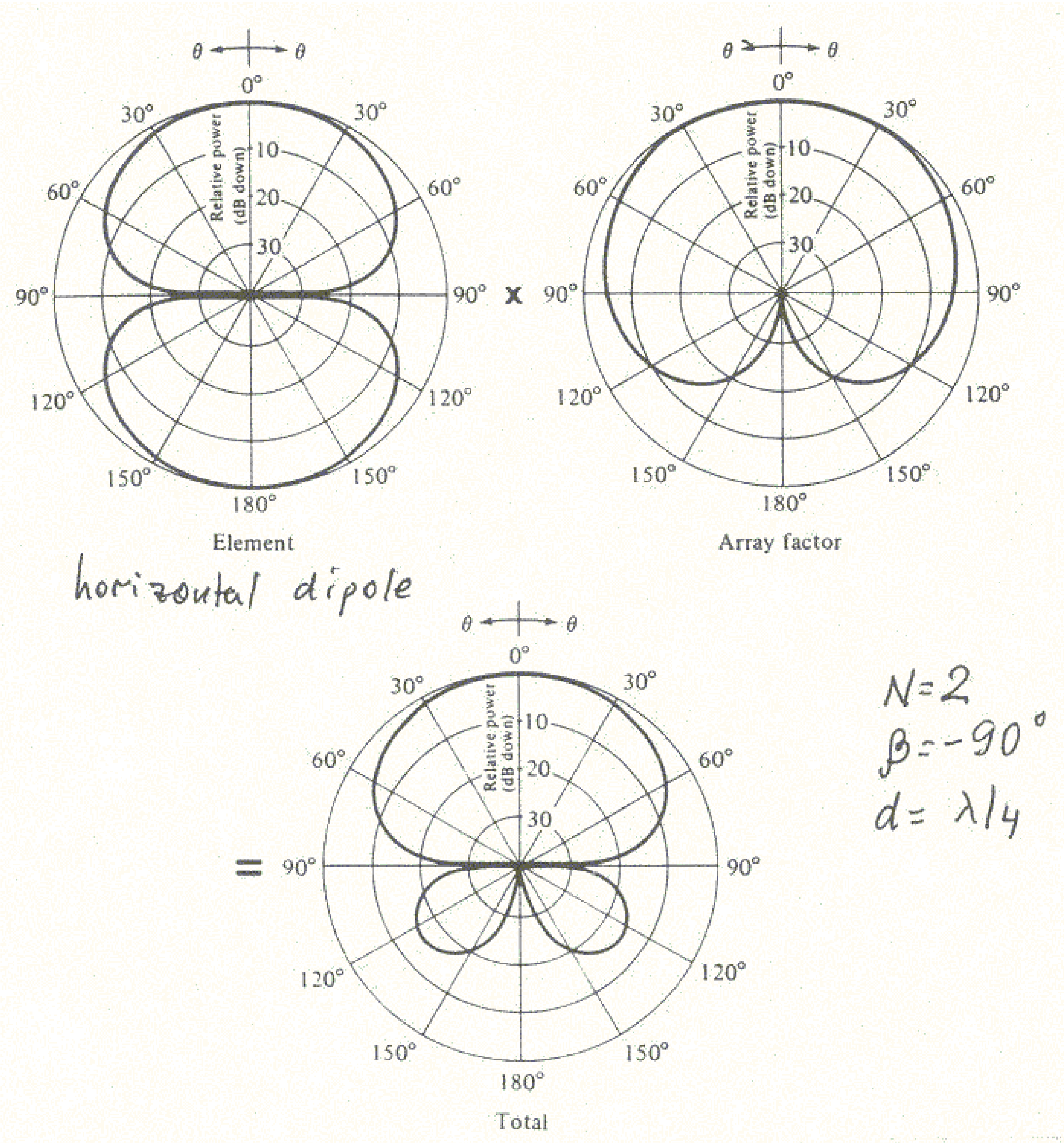


Fig. 6.4b, p. 257, Balanis

Example 2: Consider a 2-element array of identical (infinitesimal) dipoles oriented along the *y*-axis. Find the angles of observation where the nulls of the pattern occur in the plane $\phi = \pm 90^\circ$ as a function of the distance *d* between the dipoles and the phase difference β.

The normalized total field pattern is

$$f_n = |\cos\theta| \times \cos\left(\frac{kd\cos\theta + \beta}{2}\right). \tag{13.12}$$

In order to find the nulls, the equation

$$f_n = |\cos\theta| \cdot \cos\left(\frac{kd\cos\theta + \beta}{2}\right) = 0 \tag{13.13}$$

is solved.

The element factor $|\cos\theta|$ produces one null at

$$\theta_1 = \pi / 2. \tag{13.14}$$

The AF leads to the following solution:

$$\cos\left(\frac{kd\cos\theta + \beta}{2}\right) = 0 \;\; \Rightarrow \frac{kd\cos\theta + \beta}{2} = (2n+1)\frac{\pi}{2}, \; n = 0, \pm 1, \pm 2 \ldots$$

$$\underline{\underline{\theta_n = \arccos\left\{\frac{\lambda}{2\pi d}\left[-\beta \pm (2n+1)\pi\right]\right\}.}} \tag{13.15}$$

When there is no phase difference between the two element feeds $(\beta = 0)$, the separation *d* must satisfy

$$d \ge \frac{\lambda}{2}$$

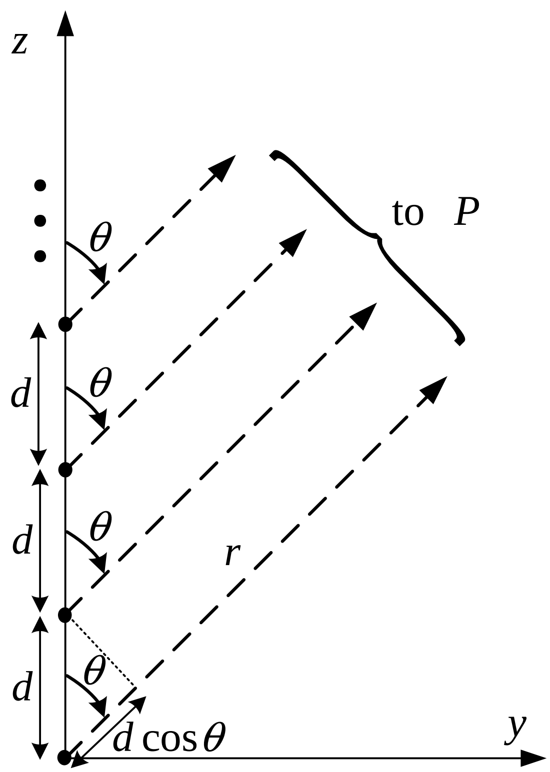
in order at least one null to occur due to (13.15).

3. *N*-element Linear Array with Uniform Amplitude and Spacing

We assume that each succeeding element has a β progressive phase lead in the excitation relative to the preceding one. An array of identical elements with identical magnitudes and with a progressive phase is called a ***uniform array***. The *AF* of the uniform array can be obtained by considering the individual elements as point (isotropic) sources. Then, the total field pattern can be obtained by simply multiplying the *AF* by the normalized field pattern of the individual element (provided the elements are not coupled).

The *AF* of an *N*-element linear array of isotropic sources is

$$AF = 1 + e^{j(kd\cos\theta+\beta)} + e^{j2(kd\cos\theta+\beta)} + \ldots + e^{j(N-1)(kd\cos\theta+\beta)}. \qquad (13.16)$$



Phase terms of the partial fields:

$$1^{\text{st}} \rightarrow e^{-jkr}$$

$$2^{\text{nd}} \rightarrow e^{-jk(r-d\cos\theta)}$$

$$3^{\text{rd}} \rightarrow e^{-jk(r-2d\cos\theta)}$$

$$\ldots$$

$$N^{\text{th}} \rightarrow e^{-jk(r-(N-1)d\cos\theta)}$$

Equation (13.16) can be re-written as

$$AF = \sum_{n=1}^{N} e^{j(n-1)(kd\cos\theta+\beta)}, \tag{13.17}$$

$$AF = \sum_{n=1}^{N} e^{j(n-1)\psi}, \tag{13.18}$$

where $\psi = kd\cos\theta + \beta$.

From (13.18), it is obvious that the *AF*s of uniform linear arrays can be controlled by the relative phase β between the elements. The *AF* in (13.18) can be expressed in a closed form, which is more convenient for pattern analysis:

$$AF \cdot e^{j\psi} = \sum_{n=1}^{N} e^{jn\psi}, \tag{13.19}$$

$$AF \cdot e^{j\psi} - AF = e^{jN\psi} - 1,$$

$$AF = \frac{e^{jN\psi}-1}{e^{j\psi}-1} = \frac{e^{j\frac{N}{2}\psi}\left(e^{j\frac{N}{2}\psi} - e^{-j\frac{N}{2}\psi}\right)}{e^{j\frac{\psi}{2}}\left(e^{j\frac{\psi}{2}} - e^{-j\frac{\psi}{2}}\right)},$$

$$AF = e^{j\left(\frac{N-1}{2}\right)\psi} \cdot \frac{\sin(N\psi/2)}{\sin(\psi/2)}. \tag{13.20}$$

Here, N shows the location of the last element with respect to the reference point in steps of the length d. The phase factor, $\exp\left[j(N-1)\psi/2\right]$, is not important unless the array output signal is further combined with the output signal of another antenna. It represents the phase shift of the array's phase centre relative to the origin, and it would be equal to one if the origin were to coincide with the array centre. As we aim at obtaining the normalized AF, we will neglecting the phase factor, which gives

$$AF = \frac{\sin(N\psi/2)}{\sin(\psi/2)}. \tag{13.21}$$

For small values of ψ, (13.21) reduces to

$$AF = \frac{\sin(N\psi/2)}{\psi/2}. \tag{13.22}$$

To normalize (13.22) or (13.21), we need the maximum of the *AF*. We re-write (13.21) as

$$AF = N \cdot \frac{\sin(N\psi/2)}{N\sin(\psi/2)}. \tag{13.23}$$

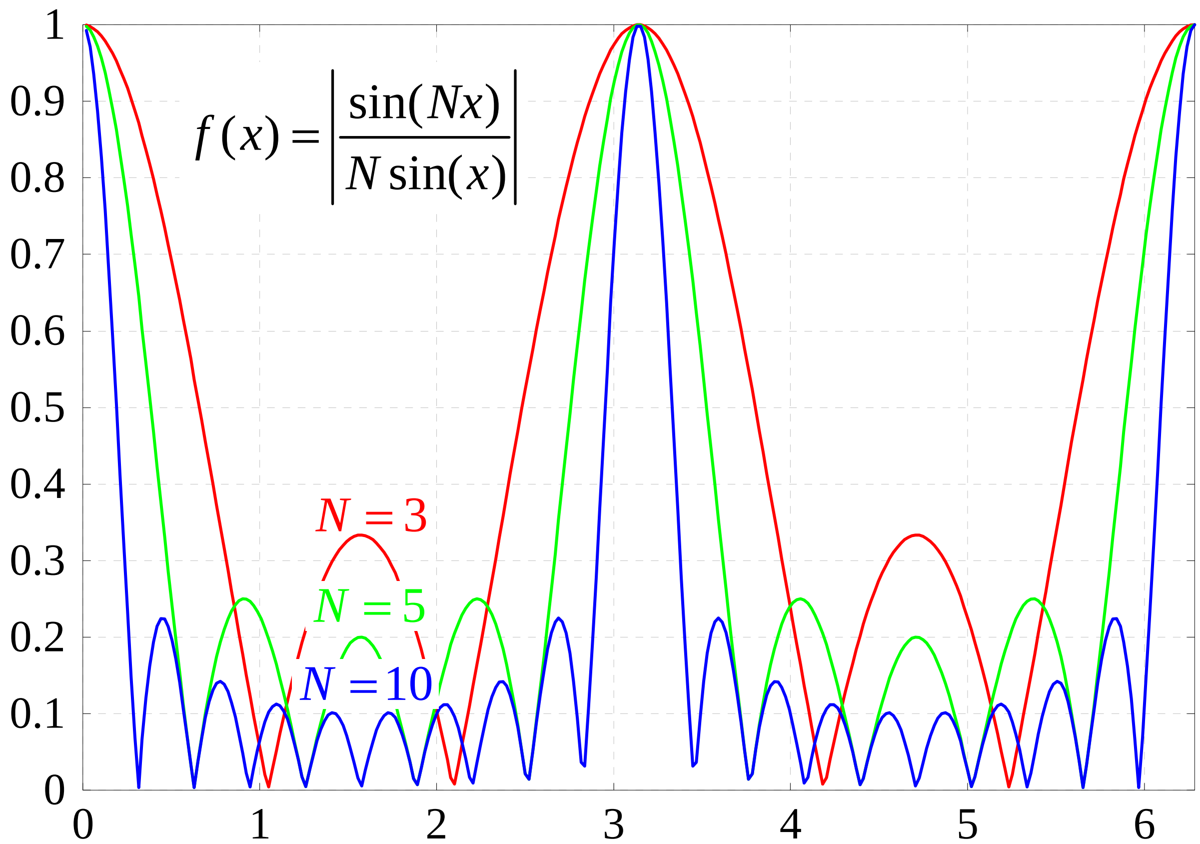
The function

$$f(x) = \frac{\sin(Nx)}{N\sin(x)}$$

has its maximum at $x = 0, \pi, \ldots$, and the value of this maximum is $f_{\max} = 1$. Therefore, $AF_{\max} = N$. The normalized *AF* is thus obtained as

$$\boxed{AF_n = \frac{\sin(N\psi/2)}{N\sin(\psi/2)}}. \tag{13.24}$$

The function $|f(x)|$ is plotted below.



For small ψ,

$$\boxed{AF_n = \frac{1}{N}\left[\frac{\sin(N\psi/2)}{\psi/2}\right]}. \quad (13.25)$$

<u>Nulls of the *AF*</u>

To find the nulls of the *AF*, equation (13.24) is set equal to zero:

$$\sin\left(\frac{N}{2}\psi\right) = 0 \quad \Rightarrow \frac{N}{2}\psi = \pm n\pi \quad \Rightarrow \frac{N}{2}(kd\cos\theta_n + \beta) = \pm n\pi, \quad (13.26)$$

$$\theta_n = \arccos\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2n}{N}\pi\right)\right], \; n = 1,2,3\ldots (n \neq 0, N, 2N, 3N\ldots). (13.27)$$

When $n = 0, N, 2N, 3N\ldots$, the *AF* attains its maximum values not nulls (see the case below). The values of n determine the order of the nulls. For a null to exist, *the argument of the arccosine must be between* –1 *and* +1.

Major maxima of the *AF*

They are studied in order to determine the maximum directivity, the *HPBW*s, the direction of maximum radiation. The maxima of (13.24) occur when (see the plot in page 13, note that $x = \psi / 2$)

$$\frac{\psi}{2} = \frac{1}{2}\left(kd\cos\theta_m + \beta\right) = \pm m\pi, \tag{13.28}$$

$$\theta_m = \arccos\left[\frac{\lambda}{2\pi d}\left(-\beta \pm 2m\pi\right)\right], \quad m = 0,1,2\ldots. \tag{13.29}$$

When (13.28) is true, $AF_n = 1$, i.e., these are not maxima of minor lobes. The index m shows the maximum's order. It is usually desirable to have a single major lobe, i.e. $m = 0$ only. This can be achieved by choosing d/λ sufficiently small. Then the argument of the *arccosine* function in (13.29) becomes greater than unity for $m = 1,2\ldots$ and equation (13.29) has a single real-valued solution:

$$\theta_0 = \arccos\left(-\frac{\beta\lambda}{2\pi d}\right). \tag{13.30}$$

The *HPBW* of a major lobe

The *HPBW* of a major lobe is calculated by setting the value of AF_n equal to $1/\sqrt{2}$. For the approximate AF_n in (13.25),

$$\frac{N}{2}\psi = \frac{N}{2}\left(kd\cos\theta_h + \beta\right) \approx \pm 1.391.$$

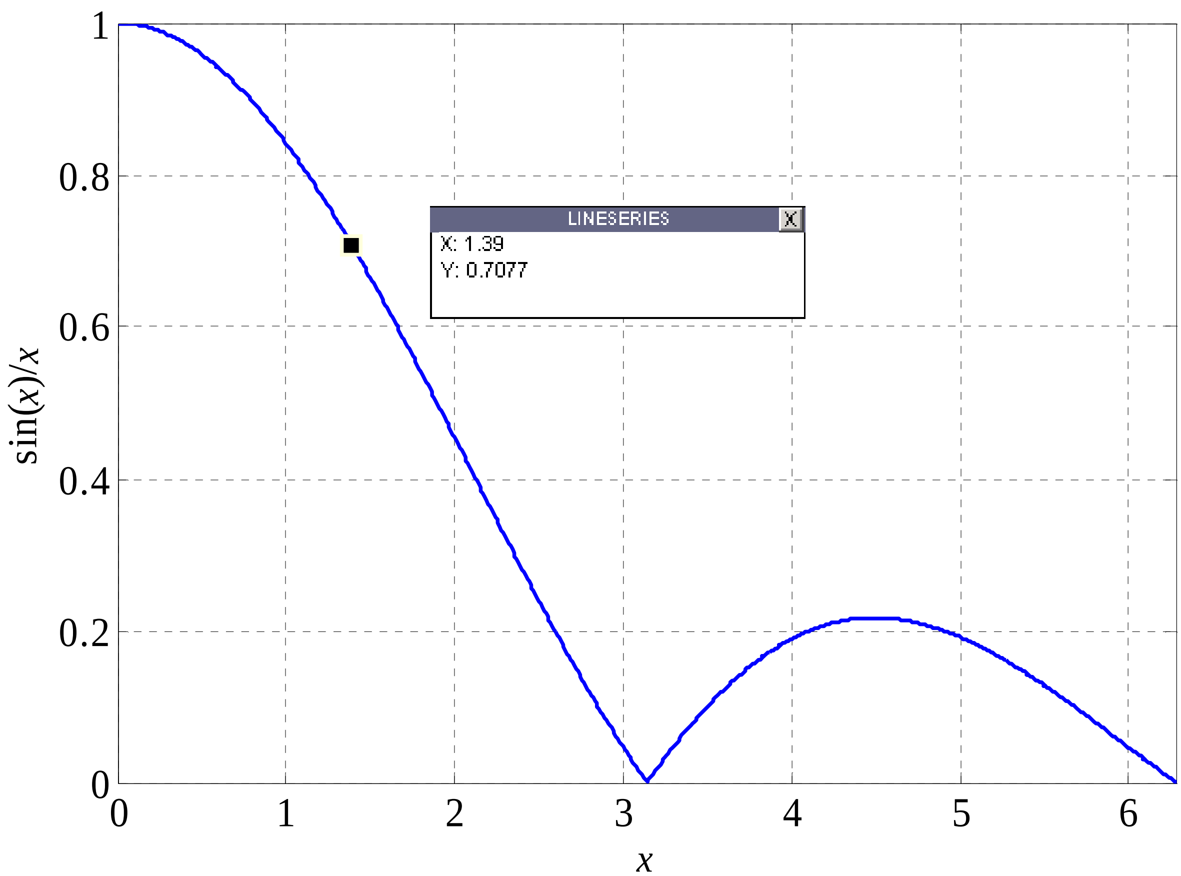
See plot of $(\sin x)/x$ below.

$$\Rightarrow \theta_h = \arccos\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2.782}{N}\right)\right]. \tag{13.31}$$

For a symmetrical pattern around θ_m (the angle at which maximum radiation occurs), the *HPBW* is calculated as

$$HPBW = 2\,|\,\theta_m - \theta_h\,|. \tag{13.32}$$

For a broadside array, for example, $\theta_m = \theta_0 = \pi/2$.



Maxima of minor lobes (secondary maxima)

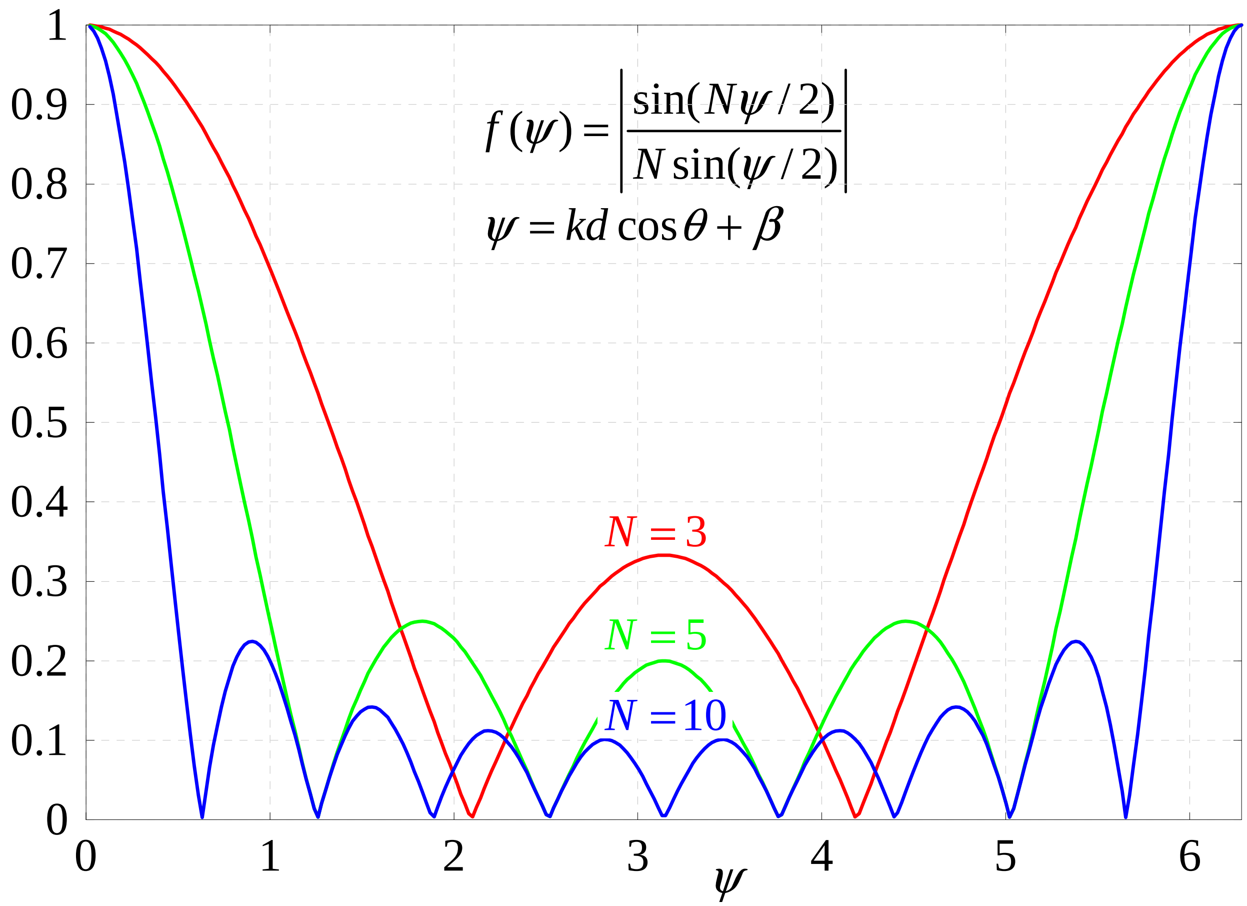
They are the maxima of AF_n, where $AF_n < 1$. These are illustrated in the plot below, which shows the array factors as a function of $\psi = kd\cos\theta + \beta$ for a uniform equally spaced linear array with N = 3, 5, 10.

The secondary maxima occur where the numerator attains a maximum and the AF is beyond its 1st null:

$$\sin\left(\frac{N}{2}\psi\right) = \pm 1 \quad \Rightarrow \frac{N}{2}\left(kd\cos\theta + \beta\right) = \pm(2s+1)\frac{\pi}{2}, \tag{13.33}$$

$$\Rightarrow \theta_s = \arccos\left\{\frac{\lambda}{2\pi d}\left(-\beta \pm \left(\frac{2s+1}{N}\right)\pi\right)\right\} \quad \text{or} \tag{13.34}$$

$$\theta_s = \frac{\pi}{2} - \arccos\left\{\frac{\lambda}{2\pi d}\left(-\beta \pm \left(\frac{2s+1}{N}\right)\pi\right)\right\}. \tag{13.35}$$



4. Broadside Array

A broadside array is an array, which has maximum radiation at $\theta = 90°$ (normal to the axis of the array). For optimal solution, both the element factor and the *AF*, should have their maxima at $\theta = 90°$.

From (13.28), it follows that the maximum of the *AF* occurs when

$$\psi = kd\cos\theta_m + \beta = 0. \tag{13.36}$$

Equation (13.36) is valid for the 0th order maximum, $m = 0$. If $\theta_m = \pi/2$, then

$$\boxed{\beta = 0}. \tag{13.37}$$

<u>The uniform linear array has its maximum radiation at $\theta = 90°$, if all array elements have their excitation with the same phase.</u>

To ensure that there are no maxima in other directions (grating lobes), the separation between the elements *d* must be smaller than the wavelength:

$$d < \lambda. \tag{13.38}$$

To illustrate the appearance of additional maxima, $AF_n = 1$, let us consider the case of $d = \xi\lambda$, where $\xi \geq 1$. Then,

$$\psi_{\beta=0} = kd\cos\theta = \frac{2\pi}{\lambda}\xi\lambda\cos\theta = 2\pi\xi\cos\theta. \tag{13.39}$$

The condition for $AF_n = 1$ from (13.28),

$$\psi_m = 2\pi \cdot m, \quad m = 0, \pm 1, \pm 2\ldots \tag{13.40}$$

is then fulfilled not only for $\theta_0 = \pi/2$ but also for

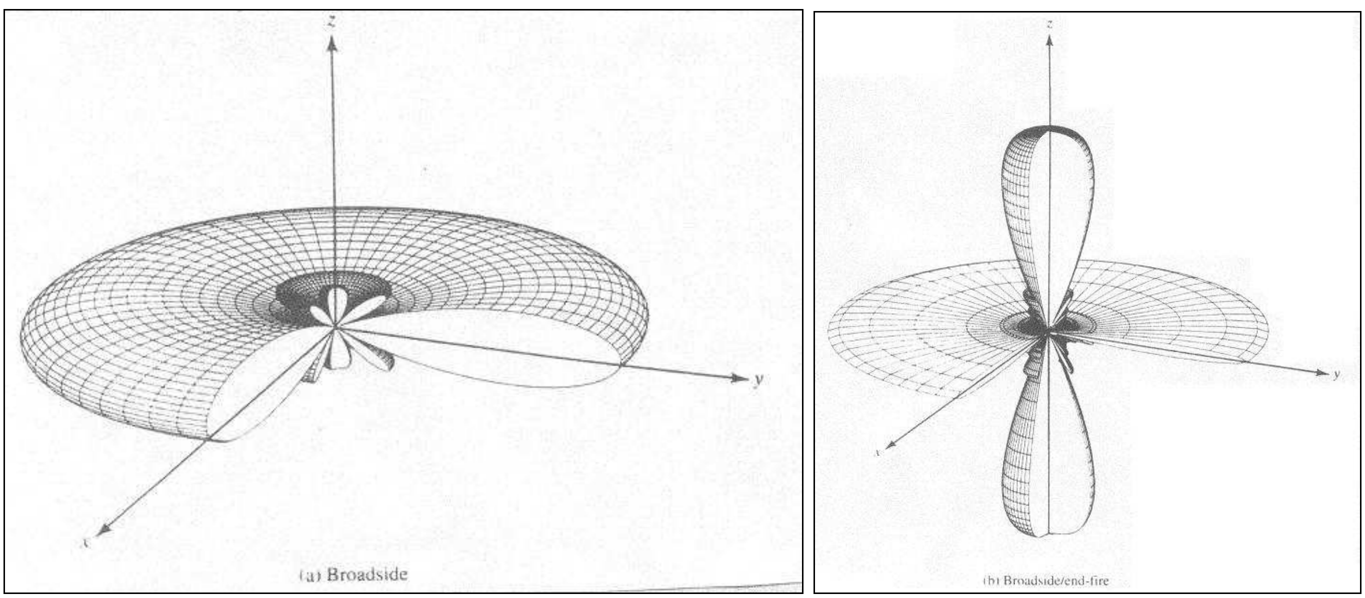
$$\theta_g = \arccos(m/\xi), \quad m = \pm 1, \pm 2\ldots. \tag{13.41}$$

As long as $m \leq \xi$, real-valued solutions for θ_g exist and grating lobes appear.

If, for example, $d = \lambda$ $(\xi = 1)$, equation (13.41) results in two additional major lobes at

$$\theta_g = \arccos(\pm 1) \Rightarrow \theta_{g_{1,2}} = 0°, 180°.$$

The resulting AF is illustrated in figure (b) below.



If $d = 2\lambda$ $(\xi = 2)$, equation (13.41) results in four additional major lobes at

$$\theta_g = \arccos(\pm 0.5, \pm 1) \Rightarrow \theta_{g_{1,2,3,4}} = 0°, 60°, 120°, 180°.$$

If $d = 1.25\lambda$ $(\xi = 1.25)$,

$$\theta_g = \arccos(\pm 0.8) \Rightarrow \theta_{g_{1,2}} \approx 37°, 143°.$$

5. Ordinary End-fire Array

An end-fire array is an array, which has its maximum radiation along the axis of the array $(\theta = 0°, 180°)$. It may be required that the array radiates only in one direction – either $\theta = 0°$ or $\theta = 180°$. For an *AF* maximum at $\theta = 0°$,

$$\psi = kd\cos\theta + \beta\Big|_{\theta=0°} = kd + \beta = 0, \tag{13.42}$$

$$\Rightarrow \boxed{\beta = -kd, \quad \text{for} \quad \theta_{\max} = 0°}. \tag{13.43}$$

For an *AF* maximum at $\theta = 180°$,

$$\psi = kd\cos\theta + \beta\Big|_{\theta=180°} = -kd + \beta = 0,$$

$$\Rightarrow \boxed{\beta = kd, \quad \text{for} \quad \theta_{\max} = 180°}. \tag{13.44}$$

If the element separation is multiple of a wavelength, $d = n\lambda$, then in addition to the end-fire maxima there also exist maxima in the broadside direction $(\theta = 90^\circ)$. As with the broadside array, in order to avoid grating lobes, the maximum spacing between the element should be less than λ:

$$d < \lambda.$$

(Show that an end-fire array with $d = \lambda/2$ has 2 maxima for $\beta = -kd$: at $\theta = 0^\circ$ and at $\theta = 180^\circ$.)

AF pattern of an *EFA*: $N = 10,\ d = \lambda / 4$

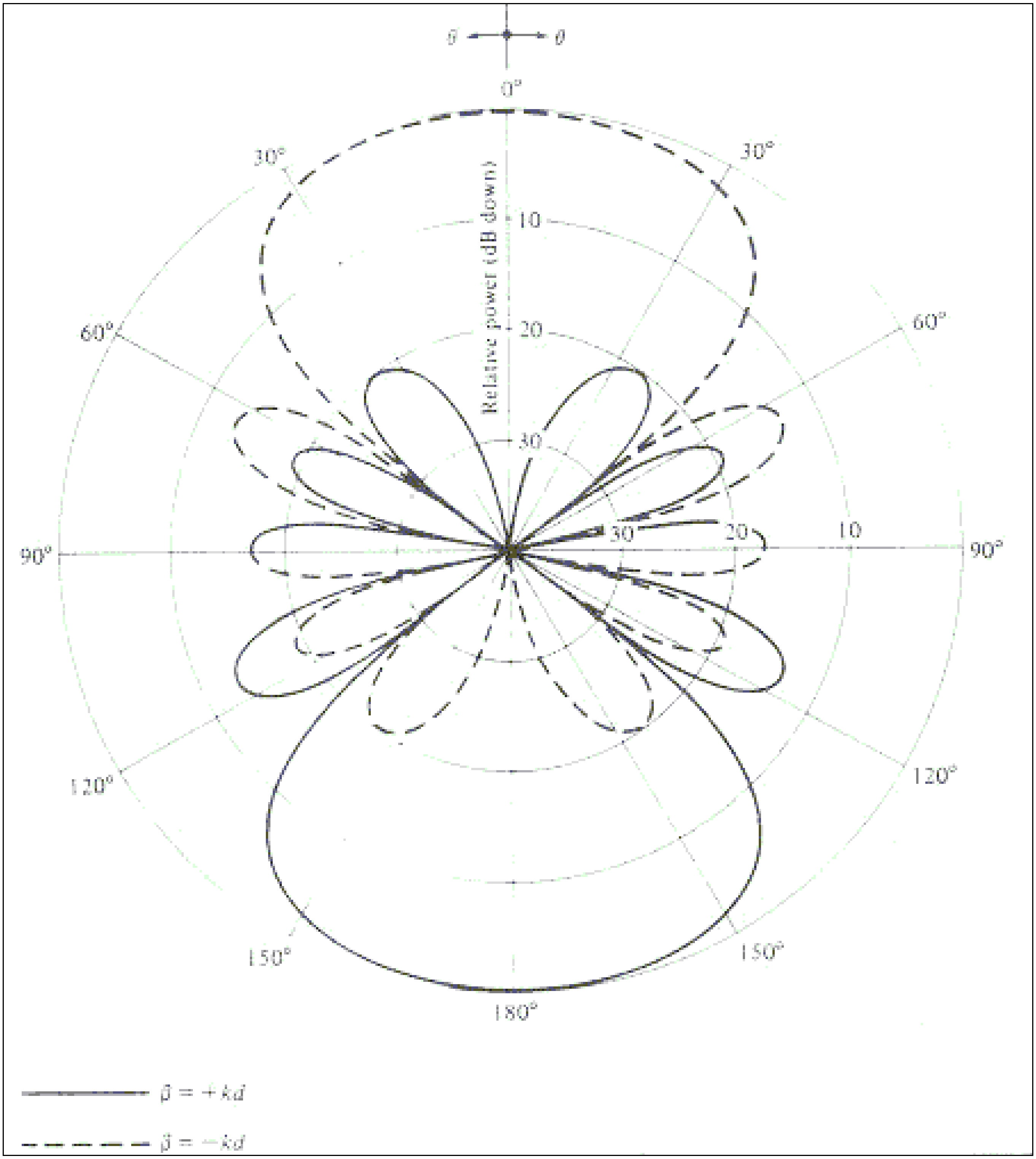


Fig. 6-11, p. 270, Balanis

6. Phased (Scanning) Arrays

It was already shown that the 0^{th} order maximum (m=0) of AF_n occurs when

$$\psi = kd\cos\theta_0 + \beta = 0. \tag{13.45}$$

This gives the relation between the direction of the main beam θ_0 and the phase difference β. The direction of the main beam can be controlled by the phase shift β. This is the basic principle of ***electronic scanning*** for phased arrays.

When the scanning is required to be continuous, the feeding system must be capable of continuously varying the progressive phase β between the elements. This is accomplished by ferrite or diode shifters (varactors).

Example: Derive the values of the progressive phase shift β as dependent on the direction of the main beam θ_0 for a uniform linear array with $d = \lambda / 4$.

From equation (13.45):

$$\beta = -kd\cos\theta_0 = -\frac{2\pi}{\lambda}\frac{\lambda}{4}\cos\theta_0 = -\frac{\pi}{2}\cos\theta_0$$

θ_0	β
0˚	-90˚
60˚	-45˚
120˚	+45˚
180˚	+90˚

The approximate *HPBW* of a scanning array is obtained using (13.31) with $\beta = -kd\cos\theta_0$:

$$\theta_{h\,1,2} = \arccos\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2.782}{N}\right)\right]. \tag{13.46}$$

The total beamwidth is

$$HPBW = \theta_{h1} - \theta_{h2}, \tag{13.47}$$

$$HPBW = \arccos\left[\frac{\lambda}{2\pi d}\left(kd\cos\theta_0 - \frac{2.782}{N}\right)\right] - \arccos\left[\frac{\lambda}{2\pi d}\left(kd\cos\theta_0 + \frac{2.782}{N}\right)\right] \tag{13.48}$$

Since $k = 2\pi/\lambda$,

$$\boxed{HPBW = \arccos\left[\cos\theta_0 - \frac{2.782}{Nkd}\right] - \arccos\left[\cos\theta_0 + \frac{2.782}{Nkd}\right].} \tag{13.49}$$

We can use the substitution $N = (L+d)/d$ to obtain

$$\boxed{\begin{aligned} HPBW = \arccos&\left[\cos\theta_0 - 0.443\left(\frac{\lambda}{L+d}\right)\right] - \\ &\arccos\left[\cos\theta_0 + 0.443\left(\frac{\lambda}{L+d}\right)\right]. \end{aligned}} \tag{13.50}$$

Here, L is the length of the array.

Equations (13.49) and (13.50) can be used to calculate the *HPBW* of a broadside array ($\theta_0 = 90° = \text{const}$). However, they are *not valid for end-fire arrays*, where

$$HPBW = 2\arccos\left(1 - \frac{2.782}{Nkd}\right). \tag{13.51}$$