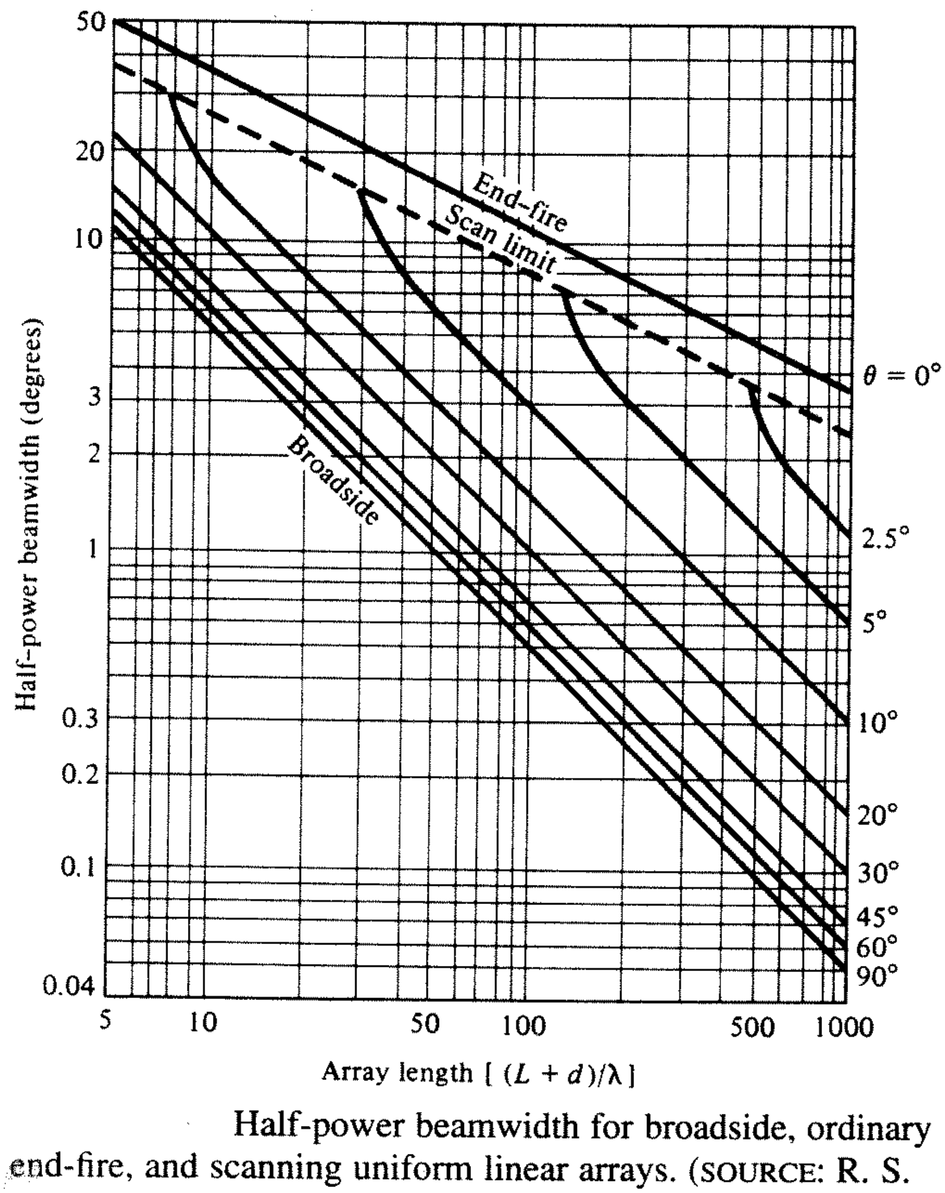
LECTURE 14: LINEAR ARRAY THEORY - PART II

(*Linear arrays: Hansen-Woodyard end-fire array, directivity of a linear array, linear array pattern characteristics – recapitulation; 3-D characteristics of an N-element linear array.*)

1. Hansen-Woodyard End-fire Array (HWEFA)

The end-fire arrays (EFA) have relatively large HPBW as compared to broadside arrays.



Half-power beamwidth for broadside, ordinary end-fire, and scanning uniform linear arrays. (SOURCE: R. S. Elliott, ''Beamwidth and Directivity of Large Scanning Arrays,'' First of Two Parts, *The Microwave Journal*, December 1963)

[Fig. 6-11, p. 270, Balanis]

To enhance the directivity of an end-fire array, Hansen and Woodyard proposed that the phase shift of an ordinary EFA

$$\beta = \pm kd \tag{14.1}$$

be increased as

$$\beta = -\left(kd + \frac{2.94}{N}\right) \text{ for a maximum at } \theta = 0°, \tag{14.2}$$

$$\beta = +\left(kd + \frac{2.94}{N}\right) \text{ for a maximum at } \theta = 180°. \tag{14.3}$$

Conditions (14.2)–(14.3) are known as the Hansen–Woodyard conditions for end-fire radiation. They follow from a procedure for maximizing the directivity, which we outline below.

The normalized pattern AF_n of a uniform linear array is

$$AF_n \approx \frac{\sin\left[\frac{N}{2}(kd\cos\theta + \beta)\right]}{\frac{N}{2}(kd\cos\theta + \beta)} \tag{14.4}$$

if $\psi = kd\cos\theta + \beta$ is sufficiently small (see previous lecture). We are looking for an optimal β, which results in maximum directivity. Let

$$\beta = -pd\,, \tag{14.5}$$

where d is the array spacing and p is the optimization parameter. Then,

$$AF_n = \frac{\sin\left[\frac{Nd}{2}(k\cos\theta - p)\right]}{\frac{Nd}{2}(k\cos\theta - p)}.$$

For brevity, use the notation $Nd/2 = q$. Then,

$$AF_n = \frac{\sin[q(k\cos\theta - p)]}{q(k\cos\theta - p)}, \tag{14.6}$$

$$\text{or } AF_n = \frac{\sin Z}{Z}, \text{ where } Z = q(k\cos\theta - p).$$

The radiation intensity becomes

$$U(\theta) = \left|AF_n\right|^2 = \frac{\sin^2 Z}{Z^2}, \tag{14.7}$$

$$U(\theta = 0) = \left\{ \frac{\sin\left[q(k-p)\right]}{q(k-p)} \right\}^2, \tag{14.8}$$

$$U_n(\theta) = \frac{U(\theta)}{U(\theta=0)} = \left(\frac{z}{\sin z} \cdot \frac{\sin Z}{Z} \right)^2, \tag{14.9}$$

where

$z = q(k-p)$,
$Z = q(k\cos\theta - p)$, and
$U_n(\theta)$ is normalized power pattern with respect to $\theta = 0°$.

The directivity at $\theta = 0°$ is

$$D_0 = \frac{4\pi U(\theta=0)}{P_{rad}} \tag{14.10}$$

where $P_{rad} = \oiint_\Omega U_n(\theta) d\Omega$. To maximize the directivity, $U_0 = P_{rad}/4\pi$ is minimized.

$$U_0 = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \left(\frac{z}{\sin z} \frac{\sin Z}{Z} \right)^2 \sin\theta d\theta d\theta, \tag{14.11}$$

$$U_0 = \frac{1}{2} \left(\frac{z}{\sin z} \right)^2 \int_0^{\pi} \left\{ \frac{\sin\left[q(k\cos\theta - p)\right]}{q(k\cos\theta - p)} \right\}^2 \sin\theta d\theta, \tag{14.12}$$

$$U_0 = \frac{1}{2kq} \left(\frac{z}{\sin z} \right)^2 \left[\frac{\pi}{2} + \frac{\cos 2z - 1}{2z} + \mathrm{Si}(2z) \right] = \frac{1}{2kq} g(z). \tag{14.13}$$

Here, $\mathrm{Si}(z) = \int_0^z (\sin t / t) dt$. The minimum of $g(z)$ occurs when

$$z = q(k-p) \approx -1.47, \tag{14.14}$$

$$\Rightarrow \frac{Nd}{2}(k-p) \approx -1.47\,.$$

$$\Rightarrow \frac{Ndk}{2} - \frac{Ndp}{2} \approx -1.47, \quad \text{where} \quad dp = -\beta$$

$$\Rightarrow \frac{N}{2}(dk+\beta) \approx -1.47$$

$$\beta \approx -\frac{2.94}{N} - kd = -\left(kd + \frac{2.94}{N}\right). \tag{14.15}$$

Equation (14.15) gives the Hansen-Woodyard condition for improved directivity along $\theta = 0°$. Similarly, for $\theta = 180°$,

$$\beta = +\left(kd + \frac{2.94}{N}\right). \tag{14.16}$$

Usually, conditions (14.15) and (14.16) are approximated by

$$\beta \approx \pm\left(kd + \frac{\pi}{N}\right), \tag{14.17}$$

which is easier to remember and gives almost identical results since the curve $g(z)$ at its minimum is fairly flat.

Conditions (14.15)-(14.16), or (14.17), ensure maximum directivity in the end-fire direction. There is, however, a trade-off in the side-lobe level, which is higher than that of the ordinary EFA. Besides, conditions (14.15)-(14.16) have to be complemented by additional requirements, which would ensure low level of the radiation in the direction opposite to the main beam.

(a) Maximum at $\theta = 0°$ [reminder: $\psi = kd\cos\theta + \beta$]

$$\beta = -\left(kd + \frac{2.94}{N}\right)\Bigg|_{\theta=0°} \Rightarrow \left| \begin{aligned} &\psi_{\theta=0°} = -\frac{2.94}{N} \\ &\psi_{\theta=180°} = -2kd - \frac{2.94}{N}. \end{aligned} \right. \tag{14.18}$$

Since we want to have a minimum of the pattern in the $\theta = 180°$ direction, we must ensure that

$$|\psi|_{\theta=180^\circ} \approx \pi\,. \tag{14.19}$$

The condition in (14.19) ensures that the AF argument $\psi/2$ falls in the middle between two major maxima, where minor maxima are the smallest; see the plot in page 13 of Lecture 13. It is easier to remember the Hansen-Woodyard conditions for maximum directivity in the $\theta = 0^\circ$ direction as

$$|\psi|_{\theta=0^\circ} = \frac{2.94}{N} \approx \frac{\pi}{N}, \quad |\psi|_{\theta=180^\circ} \approx \pi\,. \tag{14.20}$$

(b) Maximum at $\theta = 180^\circ$

$$\beta = kd + \left.\frac{2.94}{N}\right|_{\theta=180^\circ} \Rightarrow \left|\begin{array}{l} \psi_{\theta=180^\circ} = \dfrac{2.94}{N} \\ \psi_{\theta=0^\circ} = 2kd + \dfrac{2.94}{N}. \end{array}\right. \tag{14.21}$$

In order to have a minimum of the pattern in the $\theta = 0^\circ$ direction, we must ensure that

$$|\psi|_{\theta=0^\circ} \approx \pi\,. \tag{14.22}$$

We can now summarize the Hansen-Woodyard conditions for maximum directivity in the $\theta = 180^\circ$ direction as

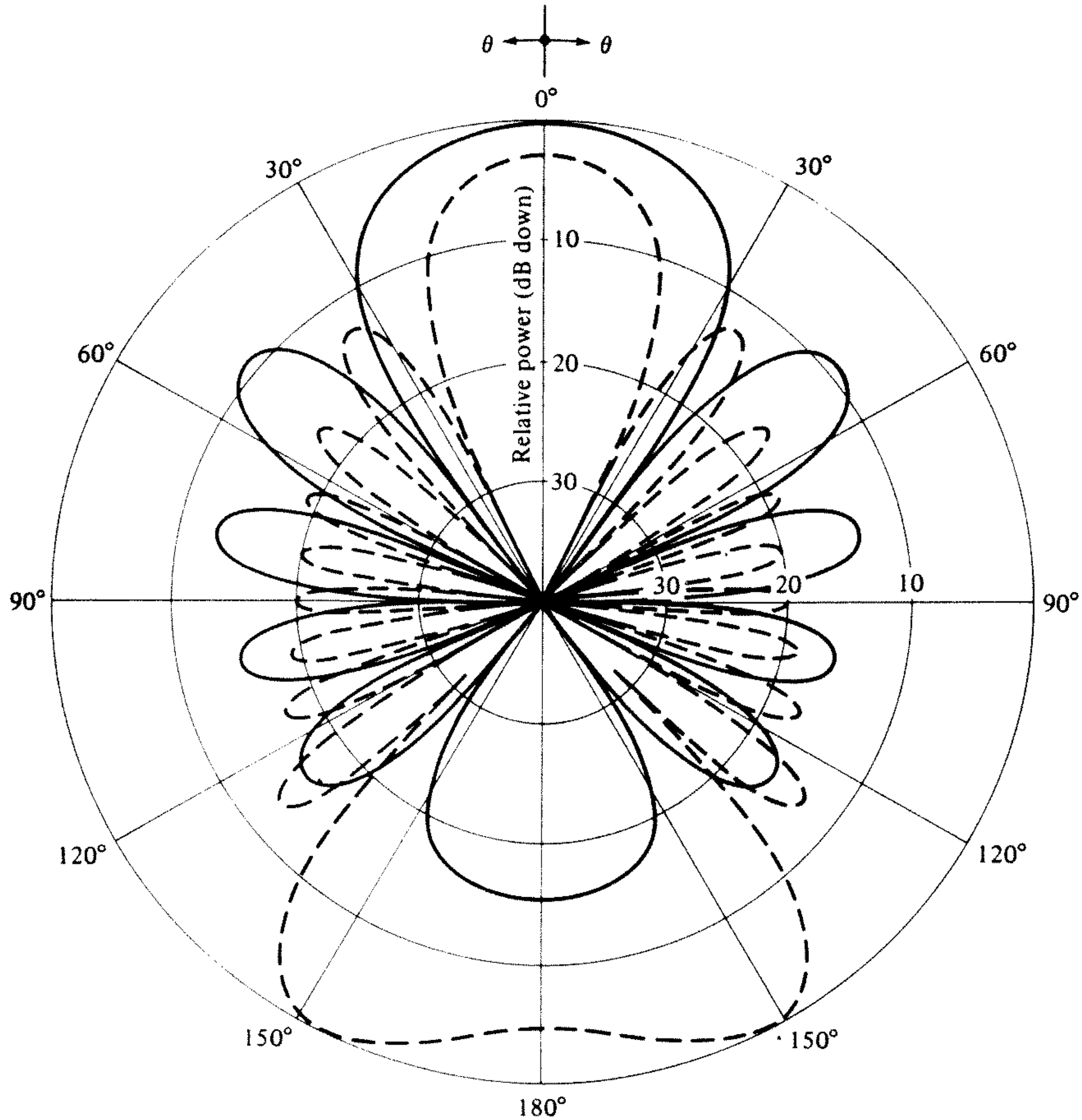
$$|\psi|_{\theta=180^\circ} = \frac{2.94}{N} \approx \frac{\pi}{N}, \quad |\psi|_{\theta=0^\circ} \approx \pi\,. \tag{14.23}$$

If (14.19) and (14.22) are not observed, the radiation in the opposite of the desired direction might even exceed the main beam level. It is easy to show (use the relation $2kd + \pi / N \approx \pi$) that the complementary requirement $|\psi| = \pi$ at the opposite direction can be met if the following relation is observed:

$$d \approx \left(\frac{N-1}{N}\right)\frac{\lambda}{4}. \tag{14.24}$$

If N is large, $d \approx \lambda / 4$. Thus, for a large uniform array, Hansen-Woodyard condition can yield improved directivity only if the spacing between the array elements is approximately $\lambda / 4$ or slightly less.

ARRAY FACTORS OF A 10-ELEMENT UNIFORM-AMPLITUDE HW EFA



Solid line: $d = \lambda / 4$

Dash line: $d = \lambda / 2$

$N = 10$

$$\beta = -\left(kd + \frac{\pi}{N}\right)$$

Fig. 6.12, p. 273, Balanis

2. Directivity of a Linear Array

2.1. Directivity of a BSA

Using the approximate expression for the AF, the normalized radiation intensity is obtained as

$$U(\theta) = |AF_n|^2 = \left[\frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{\frac{N}{2}kd\cos\theta}\right]^2 = \left(\frac{\sin Z}{Z}\right)^2 \tag{14.25}$$

$$D_0 = 4\pi\frac{U_0}{P_{rad}} = \frac{U_0}{U_{av}}, \tag{14.26}$$

where $U_{av} = P_{rad}/(4\pi)$. The radiation intensity in the direction of maximum radiation $\theta = \pi/2$ in terms of AF_n is unity:

$$U_0 = U_{\max} = U(\theta = \pi/2) = 1,$$

$$\Rightarrow D_0 = U_{av}^{-1}. \tag{14.27}$$

The radiation intensity averaged over all directions is calculated as

$$U_{av} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{\sin^2 Z}{Z^2}\sin\theta d\theta d\phi = \frac{1}{2}\int_0^{\pi}\left|\frac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{\frac{N}{2}kd\cos\theta}\right|^2 \sin\theta d\theta.$$

Change variable:

$$Z = \frac{N}{2}kd\cos\theta \Rightarrow dZ = -\frac{N}{2}kd\sin\theta d\theta. \tag{14.28}$$

Then,

$$U_{av} = -\frac{1}{2N}\frac{2}{kd}\int_{\frac{Nkd}{2}}^{-\frac{Nkd}{2}}\left(\frac{\sin Z}{Z}\right)^2 dZ, \tag{14.29}$$

$$U_{av} = \frac{1}{Nkd} \int_{-\frac{Nkd}{2}}^{\frac{Nkd}{2}} \left(\frac{\sin Z}{Z} \right)^2 dZ . \tag{14.30}$$

The function $(Z^{-1} \sin Z)^2$ is a relatively fast decaying function as Z increases. That is why, for large arrays, where $Nkd/2$ is big enough (≥ 20), the integral (14.30) can be approximated by

$$U_{av} \approx \frac{1}{Nkd} \int_{-\infty}^{\infty} \left(\frac{\sin Z}{Z} \right)^2 dZ = \frac{\pi}{Nkd} , \tag{14.31}$$

$$D_0 = \frac{1}{U_{av}} \approx \frac{Nkd}{\pi} = 2N \left(\frac{d}{\lambda} \right). \tag{14.32}$$

Substituting the length of the array $L = (N-1)d$ in (14.32) yields

$$D_0 \approx 2 \underbrace{\left(1 + \frac{L}{d} \right)}_{N} \left(\frac{d}{\lambda} \right). \tag{14.33}$$

For a large array $(L \gg d)$,

$$D_0 \approx 2L/\lambda . \tag{14.34}$$

2.2. Directivity of ordinary EFA

Consider an EFA with maximum radiation at $\theta = 0°$, i.e., $\beta = -kd$.

$$U(\theta) = |AF_n|^2 = \left\{ \frac{\sin\left[\frac{N}{2} kd (\cos\theta - 1) \right]}{\left[\frac{N}{2} kd (\cos\theta - 1) \right]} \right\}^2 = \left(\frac{\sin Z}{Z} \right)^2 , \tag{14.35}$$

where $Z = \frac{N}{2} kd (\cos\theta - 1)$. The averaged (isotropic) radiation intensity is

$$U_{av} = \frac{P_{rad}}{4\pi} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \left(\frac{\sin Z}{Z} \right)^2 \sin\theta d\theta d\phi = \frac{1}{2} \int_0^{\pi} \left(\frac{\sin Z}{Z} \right)^2 \sin\theta d\theta .$$

Since

$$Z = \frac{N}{2} kd(\cos\theta - 1) \text{ and } dZ = -\frac{N}{2} kd \sin\theta d\theta , \tag{14.36}$$

it follows that

$$U_{av} = -\frac{1}{2} \frac{2}{Nkd} \int_0^{-Nkd/2} \left(\frac{\sin Z}{Z} \right)^2 dZ ,$$

$$U_{av} = \frac{1}{Nkd} \int_0^{Nkd/2} \left(\frac{\sin Z}{Z} \right)^2 dZ . \tag{14.37}$$

If (Nkd) is sufficiently large, the above integral can be approximated as

$$U_{av} \approx \frac{1}{Nkd} \int_0^{\infty} \left(\frac{\sin Z}{Z} \right)^2 dZ = \frac{1}{Nkd} \cdot \frac{\pi}{2} . \tag{14.38}$$

The directivity then becomes

$$\Rightarrow D_0 \approx \frac{1}{U_{av}} = \frac{2Nkd}{\pi} = 4N \left(\frac{d}{\lambda} \right) . \tag{14.39}$$

The comparison of (14.39) and (14.32) shows that the directivity of an EFA is approximately twice as large as the directivity of the BSA.

Another (equivalent) expression can be derived for D_0 of the EFA in terms of the array length $L = (N-1)d$:

$$D_0 = 4 \left(1 + \frac{L}{d} \right) \left(\frac{d}{\lambda} \right) . \tag{14.40}$$

For large arrays, the following approximation holds:

$$D_0 = 4L / \lambda \quad \text{if} \quad L \gg d . \tag{14.41}$$

2.3. Directivity of HW EFA

If the radiation has its maximum at $\theta = 0°$, then the minimum of U_{av} is obtained as in (14.13):

$$U_{av}^{\min} = \frac{1}{2k}\frac{2}{Nd}\left[\frac{Z_{\min}}{\sin Z_{\min}}\right]^2 \left[\frac{\pi}{2} + \frac{\cos(2Z_{\min})-1}{2Z_{\min}} + \mathrm{Si}(2Z_{\min})\right], \qquad (14.42)$$

where $Z_{\min} = -1.47 \approx -\pi/2$.

$$\Rightarrow U_{av}^{\min} = \frac{1}{Nkd}\left(\frac{\pi}{2}\right)^2 \left[\frac{\pi}{2} + \frac{2}{\pi} - 1.8515\right] = \frac{0.878}{Nkd}. \qquad (14.43)$$

The directivity is then

$$D_0 = \frac{1}{U_{av}^{\min}} = \frac{Nkd}{0.878} = 1.789\left[4N\left(\frac{d}{\lambda}\right)\right]. \qquad (14.44)$$

From (14.44), we can see that using the HW conditions leads to improvement of the directivity of the EFA with a factor of 1.789. Equation (14.44) can be expressed via the length L of the array as

$$D_0 = 1.789\left[4\left(1+\frac{L}{d}\right)\left(\frac{d}{\lambda}\right)\right] = 1.789\left[4\left(\frac{L}{\lambda}\right)\right]. \qquad (14.45)$$

<u>Example</u>: Given a linear uniform array of N isotropic elements ($N = 10$), find the directivity D_0 if:

a) $\beta = 0$ (BSA)

b) $\beta = -kd$ (ordinary EFA)

c) $\beta = -kd - \pi / N$ (Hansen-Woodyard EFA)

In all cases, $d = \lambda / 4$.

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a) BSA

$$D_0 \approx 2N\left(\frac{d}{\lambda}\right) = 5 \quad (6.999\ \text{dB})$$

b) Ordinary EFA

$$D_0 \approx 4N\left(\frac{d}{\lambda}\right) = 10 \quad (10\ \text{dB})$$

c) HW EFA

$$D_0 \approx 1.789\left[4N\left(\frac{d}{\lambda}\right)\right] = 17.89 \quad (12.53\ \text{dB})$$

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3. Pattern Characteristics of Linear Uniform Arrays – Recapitulation

A. Broad-side array

NULLS ($AF_n = 0$):

$$\theta_n = \arccos\left(\pm\frac{n}{N}\frac{\lambda}{d}\right), \text{ where } \ n = 1,2,3,4,\ldots \text{ and } n \neq N, 2N, 3N, \ldots$$

MAXIMA ($AF_n = 1$):

$$\theta_n = \arccos\left(\pm\frac{m\lambda}{d}\right), \text{ where } \ m = 0,1,2,3,\ldots$$

HALF-POWER POINTS:

$$\theta_h \simeq \arccos\left(\pm\frac{1.391\lambda}{\pi N d}\right), \text{ where } \frac{\pi \mathrm{d}}{\lambda} \ll 1$$

HALF-POWER BEAMWIDTH:

$$\Delta\theta_h = 2\left[\frac{\pi}{2} - \arccos\left(\frac{1.391\lambda}{\pi N d}\right)\right], \quad \frac{\pi d}{\lambda} \ll 1$$

MINOR LOBE MAXIMA:

$$\theta_s \simeq \arccos\left[\pm\frac{\lambda}{2d}\left(\frac{2s+1}{N}\right)\right], \text{ where } s = 1,2,3,\ldots \text{ and } \frac{\pi \mathrm{d}}{\lambda} \ll 1$$

FIRST-NULL BEAMWIDTH (FNBW):

$$\Delta\theta_n = 2\left[\frac{\pi}{2} - \arccos\left(\frac{\lambda}{Nd}\right)\right]$$

FIRST SIDE LOBE BEAMWIDH (FSLBW):

$$\Delta\theta_s = 2\left[\frac{\pi}{2} - \arccos\left(\frac{3\lambda}{2Nd}\right)\right], \quad \frac{\pi d}{\lambda} \ll 1$$

B. <u>Ordinary end-fire array</u>

NULLS ($AF_n = 0$):

$$\theta_n = \arccos\left(1 - \frac{n}{N}\frac{\lambda}{d}\right), \text{ where } n = 1,2,3,\ldots \text{ and } n \neq N, 2N, 3N, \ldots$$

MAXIMA ($AF_n = 1$):

$$\theta_n = \arccos\left(1 - \frac{m\lambda}{d}\right), \text{ where } m = 0,1,2,3,\ldots$$

HALF-POWER POINTS:

$$\theta_h = \arccos\left(1 - \frac{1.391\lambda}{\pi Nd}\right), \text{ where } \frac{\pi \mathrm{d}}{\lambda} \ll 1$$

HALF-POWER BEAMWIDTH:

$$\Delta\theta_h = 2\arccos\left(1 - \frac{1.391\lambda}{\pi Nd}\right), \quad \frac{\pi d}{\lambda} \ll 1$$

MINOR LOBE MAXIMA:

$$\theta_s = \arccos\left[1 - \frac{(2s+1)\lambda}{2Nd}\right], \text{ where } s = 1,2,3,\ldots \text{ and } \frac{\pi d}{\lambda} \ll 1$$

FIRST-NULL BEAMWIDTH:

$$\Delta\theta_n = 2\arccos\left(1 - \frac{\lambda}{Nd}\right)$$

FIRST SIDE LOBE BEAMWIDH:

$$\Delta\theta_s = 2\arccos\left(1 - \frac{3\lambda}{2Nd}\right), \quad \frac{\pi d}{\lambda} \ll 1$$

C. Hansen-Woodyard end-fire array

NULLS:

$$\theta_n = \arccos\left[1+(1-2n)\frac{\lambda}{2Nd}\right], \text{ where } n=1,2,\ldots \text{ and } n \neq N, 2N, \ldots$$

MINOR LOBE MAXIMA:

$$\theta_s = \arccos\left(1-\frac{s\lambda}{Nd}\right), \text{ where } s=1,2,3,\ldots \text{ and } \frac{\pi d}{\lambda} \ll 1$$

SECONDARY MAXIMA:

$$\theta_m = \arccos\left\{1+[1-(2m+1)]\frac{\lambda}{2Nd}\right\}, \text{ where } m=1,2,\ldots \text{ and } \frac{\pi d}{\lambda} \ll 1$$

HALF-POWER POINTS:

$$\theta_h = \arccos\left(1-0.1398\frac{\lambda}{Nd}\right), \text{ where } \frac{\pi d}{\lambda} \ll 1,\ N\text{-large}$$

HALF-POWER BEAMWIDTH:

$$\Delta\theta_h = 2\arccos\left(1-0.1398\frac{\lambda}{Nd}\right), \text{ where } \frac{\pi d}{\lambda} \ll 1,\ N\text{-Large}$$

FIRST-NULL BEAMWIDTH:

$$\Delta\theta_n = 2\arccos\left(1-\frac{\lambda}{2Nd}\right)$$

4. 3-D Characteristics of a Linear Array

In the previous considerations, it was always assumed that the linear-array elements are located along the z-axis, which is convenient to analyze in spherical coordinate system. If the array axis has an arbitrary orientation, the array factor can be expressed as

$$AF = \sum_{n=1}^{N} a_n e^{j(n-1)(kd\cos\gamma+\beta)} = \sum_{n=1}^{N} a_n e^{j(n-1)\psi}, \tag{14.46}$$

where a_n is the excitation amplitude and $\psi = kd\cos\gamma + \beta$.

The angle γ is subtended by the array axis and the position vector to the observation point. Thus, if the array axis is along the unit vector $\hat{\mathbf{a}}$,

$$\hat{\mathbf{a}} = \sin\theta_a \cos\phi_a \hat{\mathbf{x}} + \sin\theta_a \sin\phi_a \hat{\mathbf{y}} + \cos\theta_a \hat{\mathbf{z}} \tag{14.47}$$

and the position vector to the observation point is

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \tag{14.48}$$

the angle γ can be found as

$$\cos\gamma = \hat{\mathbf{a}}\cdot\hat{\mathbf{r}} = \sin\theta\cos\phi\sin\theta_a\cos\phi_a + \sin\theta\sin\phi\sin\theta_a\sin\phi_a + \cos\theta\cos\theta_a,$$

$$\Rightarrow \cos\gamma = \sin\theta\sin\theta_a\cos(\phi-\phi_a) + \cos\theta\cos\theta_a. \tag{14.49}$$

If $\hat{\mathbf{a}} = \hat{\mathbf{z}}$ $(\theta_a = 0°)$, then $\cos\gamma = \cos\theta,\ \ \gamma = \theta$.