LECTURE 15: LINEAR ARRAYS – PART III

(*Broadside N-element linear arrays with uniform spacing and non-uniform amplitude: Binomial and Dolph–Tschebyscheff arrays. Directivity and design.*)

1. Advantages of Linear Arrays with Nonuniform Amplitude Distribution

The most often met BSAs, classified according to the type of their excitation amplitudes, are:

a) the uniform BSA – relatively high directivity, but the side-lobe levels are high;
b) Dolph–Tschebyscheff (or Chebyshev)[1] BSA – for a given directivity with a fixed number of array elements, achieves the lowest side-lobe level;
c) binomial BSA – does not have good directivity (for a given number of elements) but has low side-lobes (if $d = \lambda / 2$, no side lobes at all).

2. Array Factor of Linear Arrays with Nonuniform Amplitude Distribution

Let us consider a linear array with an even number (2*M*) of elements, located symmetrically along the *z*-axis, with excitation, which is also symmetrical with respect to $z = 0$. For a *broadside array* $(\beta = 0)$,

$$\begin{aligned} AF^e = a_1 e^{j\frac{1}{2}kd\cos\theta} + a_2 e^{j\frac{3}{2}kd\cos\theta} + \cdots + a_M e^{j\frac{2M-1}{2}kd\cos\theta} + \\ + a_1 e^{-j\frac{1}{2}kd\cos\theta} + a_2 e^{-j\frac{3}{2}kd\cos\theta} + \cdots + a_M e^{-j\frac{2M-1}{2}kd\cos\theta}, \end{aligned} \tag{15.1}$$

$$\Rightarrow AF^e = 2\sum_{n=1}^{M} a_n \cos\left[\left(\frac{2n-1}{2}\right) kd\cos\theta\right]. \tag{15.2}$$

If the linear array consists of an odd number (2*M*+1) of elements, located symmetrically along the *z*-axis, the array factor is

$$\begin{aligned} AF^o = 2a_1 + a_2 e^{jkd\cos\theta} + a_3 e^{j2kd\cos\theta} + \ldots + a_{M+1} e^{jMkd\cos\theta} + \\ + a_2 e^{-jkd\cos\theta} + a_3 e^{-j2kd\cos\theta} + \ldots + a_{M+1} e^{-jMkd\cos\theta}, \end{aligned} \tag{15.3}$$

$$\Rightarrow AF^o = 2\sum_{n=1}^{M+1} a_n \cos\left[(n-1) kd\cos\theta\right]. \tag{15.4}$$

[1] Russian spelling is Чебышёв.

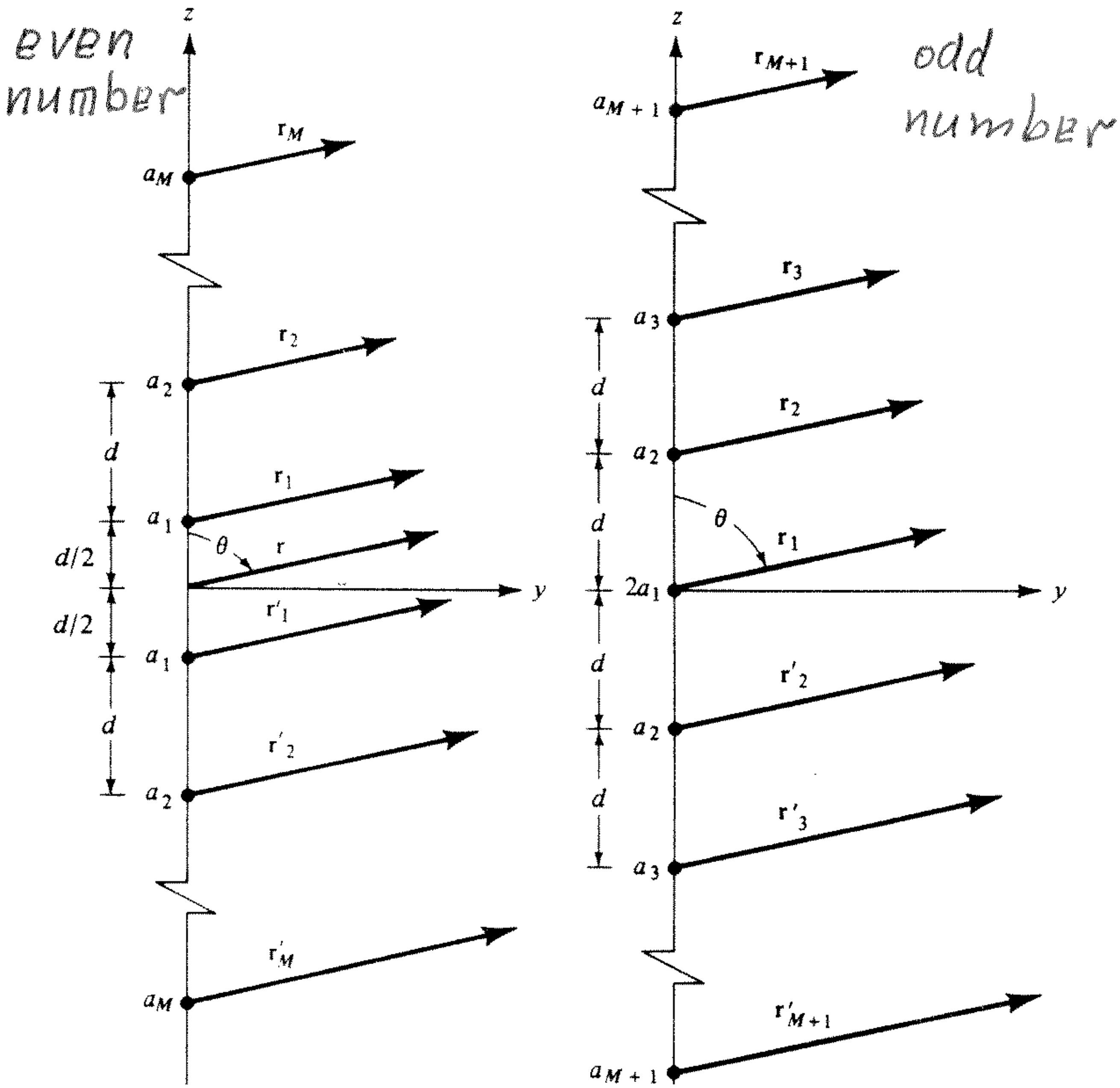


Fig. 6.17, p. 291, Balanis

The normalized AF derived from (15.2) and (15.4) can be written in the form

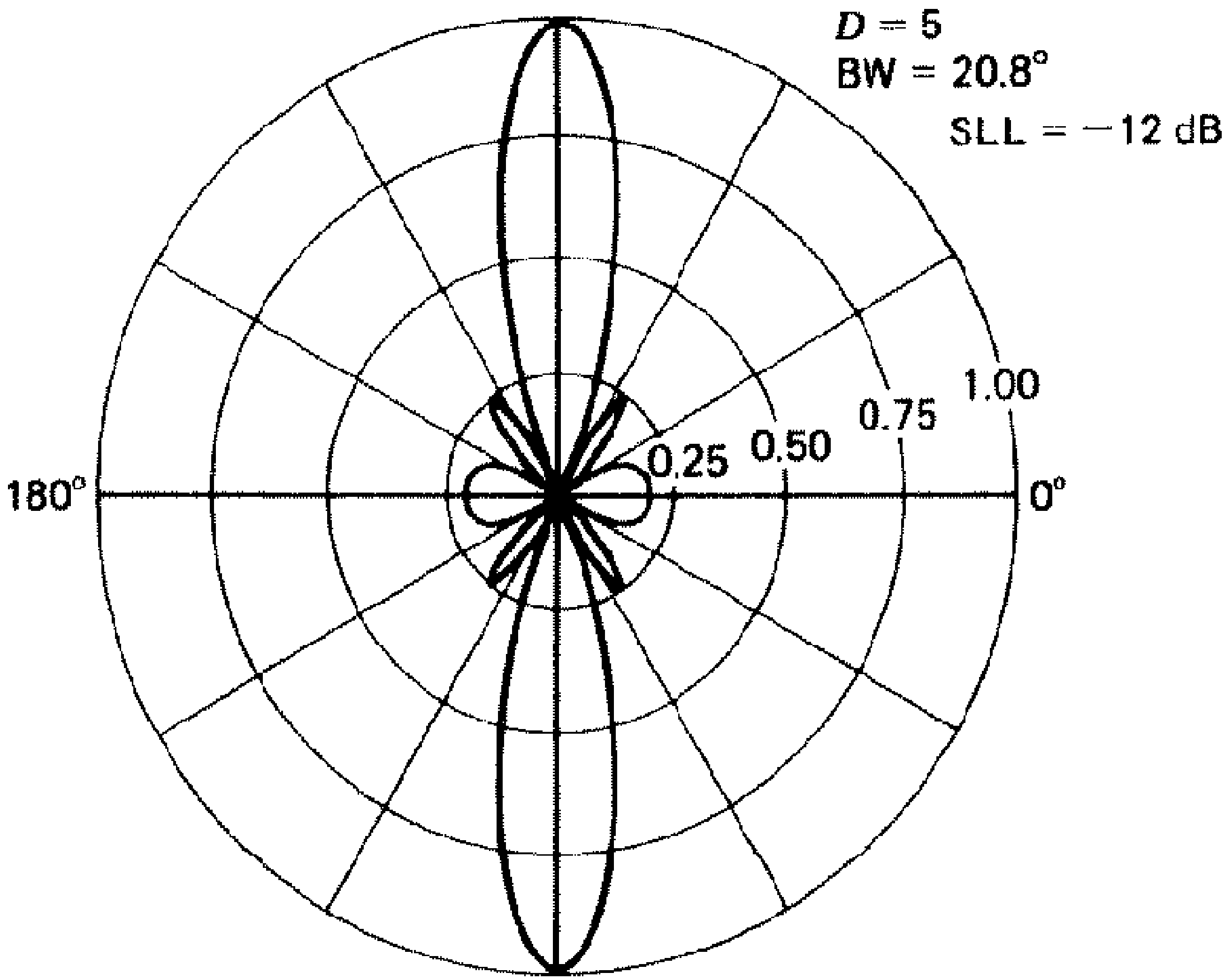
$$AF^e = \sum_{n=1}^{M} a_n \cos\left[(2n-1)u\right], \text{ for } N = 2M , \tag{15.5}$$

$$AF^o = \sum_{n=1}^{M+1} a_n \cos\left[2(n-1)u\right], \text{ for } N = 2M+1, \tag{15.6}$$

where $u = \left.\dfrac{\psi}{2}\right|_{\beta=0} = \dfrac{1}{2} kd\cos\theta = \dfrac{\pi d}{\lambda}\cos\theta$.

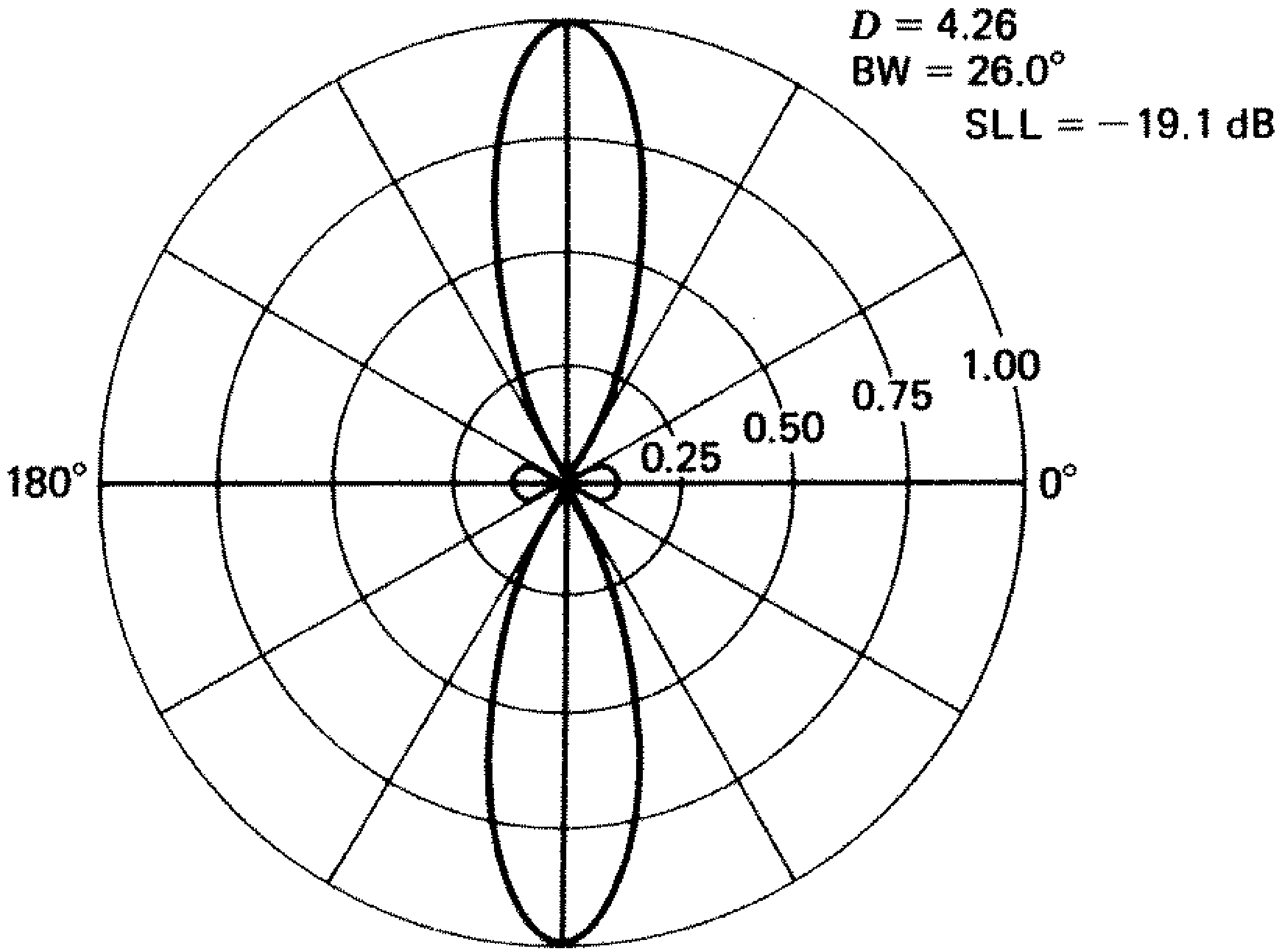
<u>Examples of AFs of arrays of nonuniform amplitude distribution</u>

a) **uniform** amplitude distribution ($N = 5,\ d = \lambda/2$, max. at $\theta_0 = 90°$)



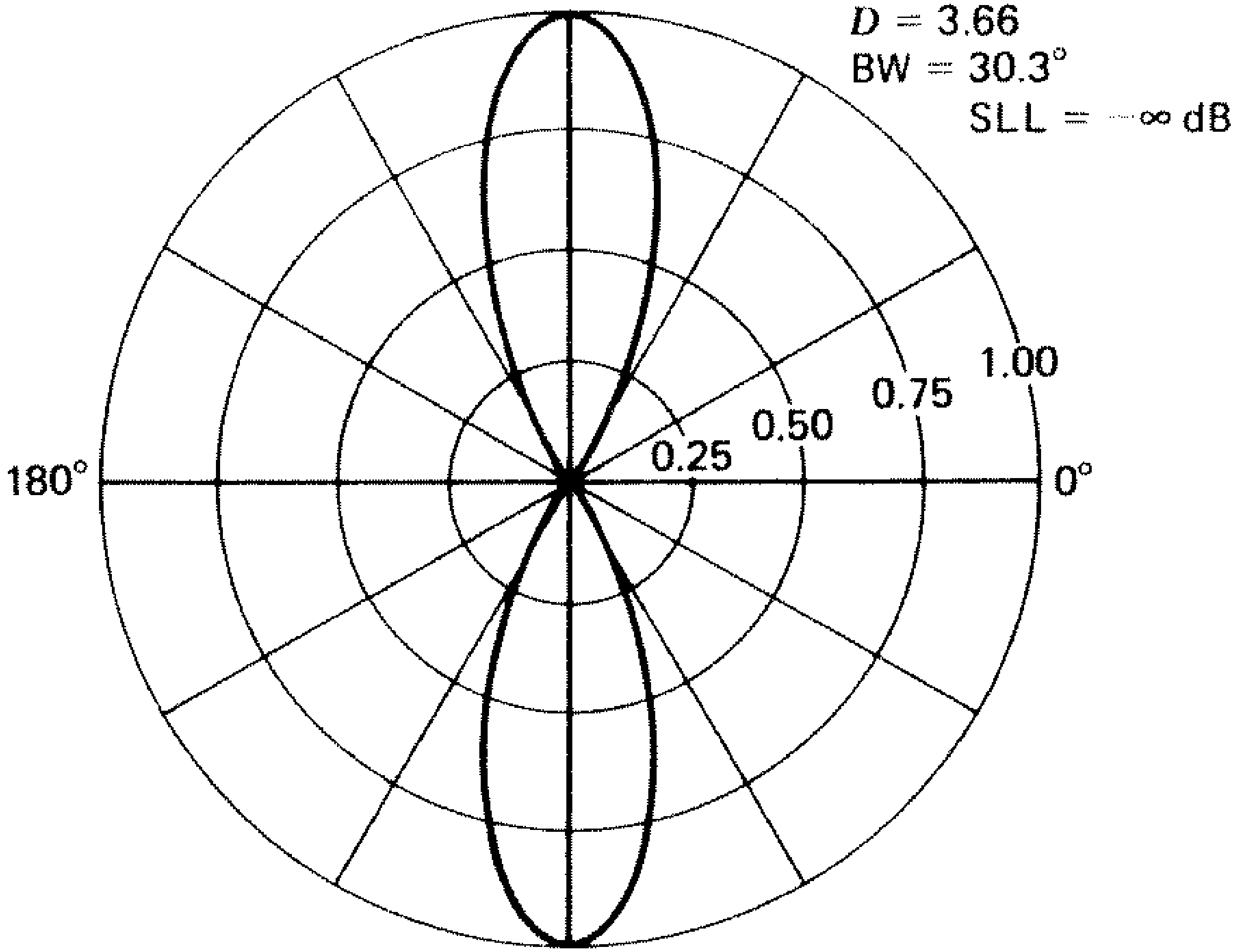
pp. 148-149, Stutzman

b) **triangular** (1:2:3:2:1) amplitude distribution (N = 5, $d = \lambda / 2$, max. at $\theta_0 = 90°$)



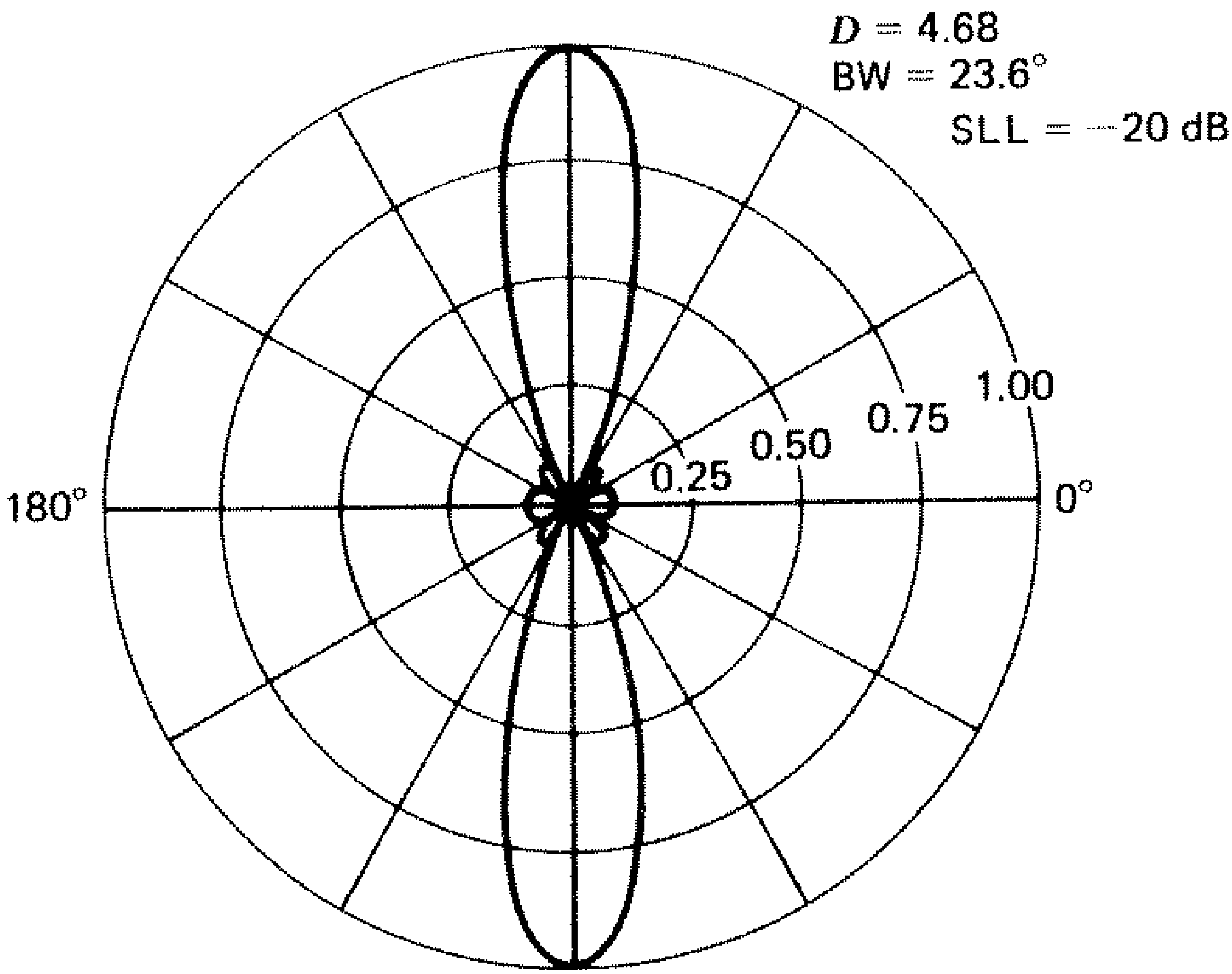
pp. 148-149, Stutzman

c) **binomial** (1:4:6:4:1) amplitude distribution ($N = 5$, $d = \lambda / 2$, max. at $\theta_0 = 90°$)



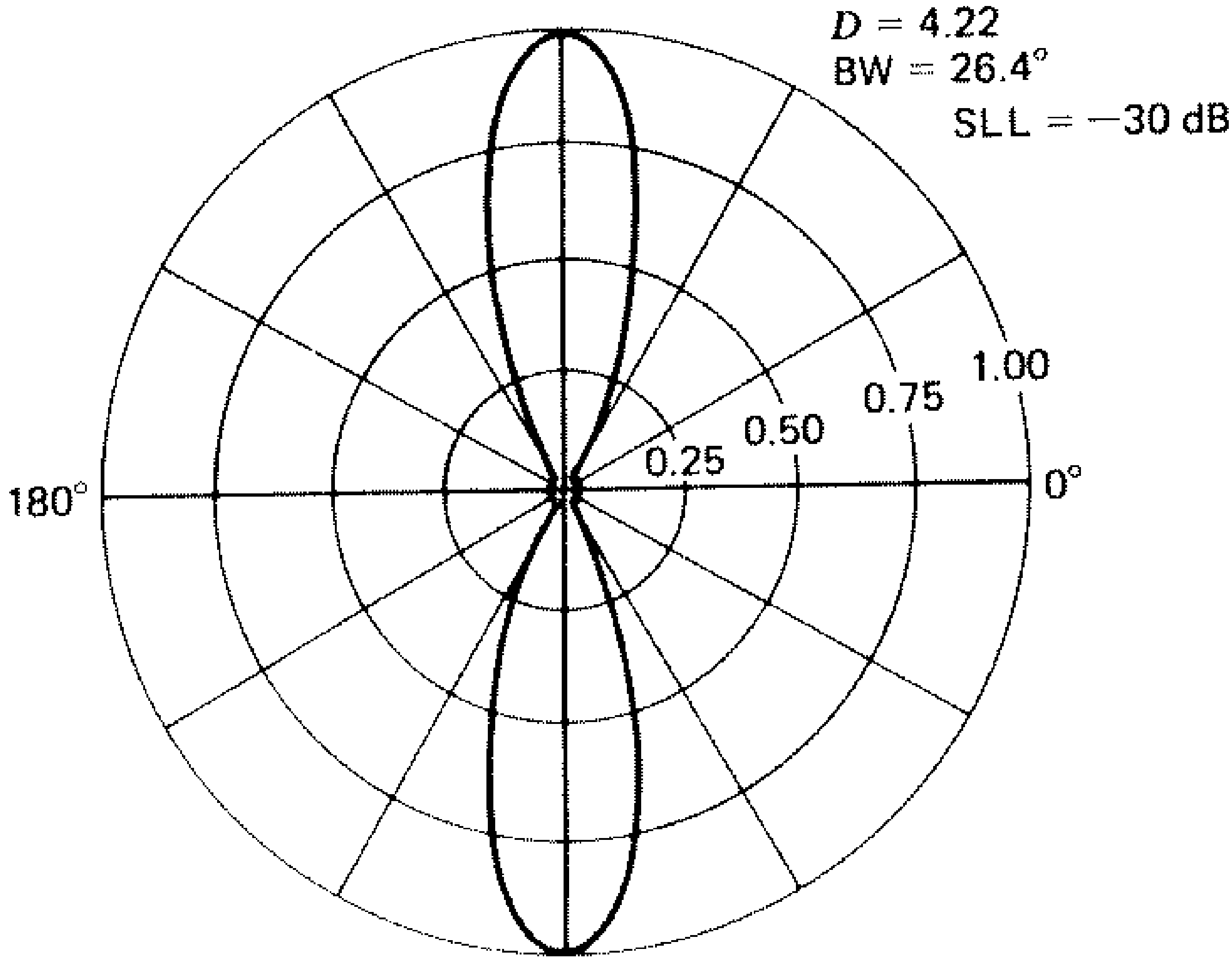
pp. 148-149, Stutzman

d) **Dolph-Tschebyschev** (1:1.61:1.94:1.61:1) amplitude distribution (N = 5, $d = \lambda / 2$, max. at $\theta_0 = 90^\circ$)



pp. 148-149, Stutzman

e) **Dolph-Tschebyschev** (1:2.41:3.14:2.41:1) amplitude distribution (N = 5, $d = \lambda / 2$, max. at $\theta_0 = 90°$)

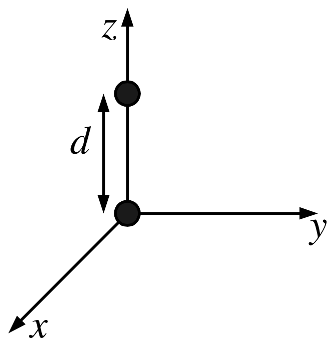


pp. 148-149, Stutzman

Notice that as the current amplitude is tapered more gradually toward the edges of the array, the side lobes tend to decrease, and the beamwidth tends to increase.

3. Binomial Broadside Array

The binomial BSA was investigated and proposed by J. S. Stone[2] to synthesize patterns without side lobes. First, consider a 2–element array (along the z-axis).



The elements of the array are identical and their excitations are the same. The array factor is of the form

$$AF = 1 + Z \text{, where } Z = e^{j\psi} = e^{j(kd\cos\theta+\beta)}. \tag{15.7}$$

If the spacing is $d \le \lambda/2$ and $\beta = 0$ (broad-side maximum), the array pattern $|AF|$ has no side lobes at all. This is proven as follows.

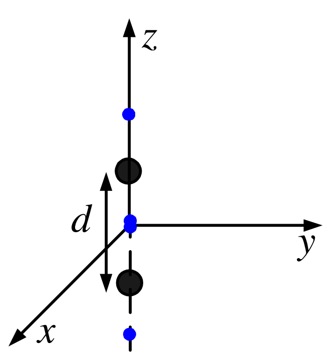
$$|AF|^2 = (1+\cos\psi)^2 + \sin^2\psi = 2(1+\cos\psi) = 4\cos^2(\psi/2) \tag{15.8}$$

where $\psi = kd\cos\theta$. The first null of the array factor is obtained from (15.8) as

$$\frac{1}{2}\cdot\frac{2\pi}{\lambda}\cdot d\cos\theta_{n1,2} = \pm\frac{\pi}{2} \quad \Rightarrow \theta_{n1,2} = \pm\arccos\left(\frac{\lambda}{2d}\right). \tag{15.9}$$

As long as $d < \lambda/2$, the first null does not exist. If $d = \lambda/2$, then $\theta_{n1,2} = 0$, $180°$. Thus, in the "visible" range of θ, all secondary lobes are eliminated.

Second, consider a 2–element array whose elements are identical and the same as the array given above. The distance between the two arrays is again d.



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[2] US Patents #1,643,323, #1,715,433.

This new array has an AF of the form

$$AF = (1+Z)(1+Z) = 1 + 2Z + Z^2. \tag{15.10}$$

Since $(1+Z)$ has no side lobes, $(1+Z)^2$ does not have side lobes either.

Continuing the process for an N-element array produces

$$AF = (1+Z)^{N-1}. \tag{15.11}$$

If $d \le \lambda / 2$, the above AF does not have side lobes regardless of the number of elements N. The excitation amplitude distribution can be obtained easily by the expansion of the binome in (15.11). Making use of Pascal's triangle,

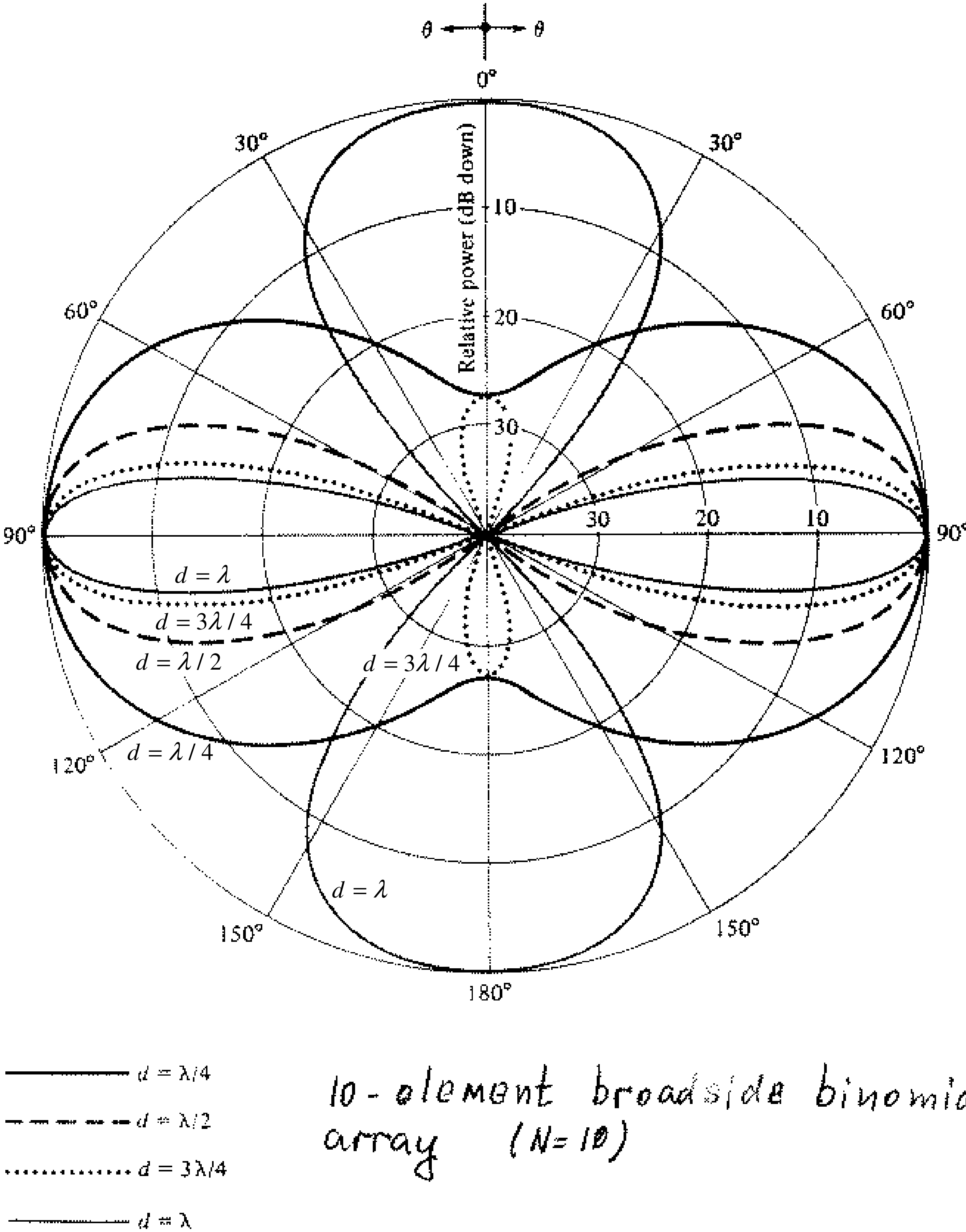
$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

the relative excitation amplitudes at each element of an (N+1)-element array can be determined. An array with a binomial distribution of the excitation amplitudes is called a ***binomial array***. The excitation distribution as given by the binomial expansion gives the ***relative*** values of the amplitudes. It is immediately seen that there is a fairly wide variation of the amplitude, which is a disadvantage of the BAs. The overall efficiency of such an array would be low. Besides, the BA has relatively wide beam. Its HPBW is the largest as compared to the uniform BSA or the DCA for a give number of elements.

An approximate closed-form expression for the HPBW of a BA with $d = \lambda / 2$ is

$$HPBW \approx \frac{1.06}{\sqrt{N-1}} = \frac{1.06}{\sqrt{2L/\lambda}} = \frac{1.75}{\sqrt{L/\lambda}}, \tag{15.12}$$

where $L = (N-1)d$ is the array's length. The AFs of 10-element broadside binomial arrays (N = 10) are given below.



10-element broadside binomial array (N=10)

Fig. 6.18, p.293, Balanis

The directivity of a broadside BA with spacing $d = \lambda / 2$ can be calculated as

$$D_0 = \frac{4\pi}{\bar{P}_{rad}} = \frac{2}{\int_0^{\pi} \left[\cos\left(\frac{\pi}{2} \cos\theta \right) \right]^{2(N-1)} d\theta}, \tag{15.13}$$

$$D_0 = \frac{(2N-2)\cdot(2N-4)\cdot\ldots\cdot 2}{(2N-3)\cdot(2N-5)\cdot\ldots\cdot 1}, \tag{15.14}$$

$$D_0 \approx 1.77\sqrt{N} = 1.77\sqrt{1 + 2L/\lambda}\,. \tag{15.15}$$

4. Dolph–Chebyshev Array (DCA)

Dolph proposed (in 1946) a method for designing arrays with any desired side-lobe level for a given HPBW. This method is based on the approximation of the pattern of the array by a Chebyshev polynomial of order *m*, high enough to meet the requirement for the side-lobe levels. A DCA with no side lobes (side-lobe level of −∞ dB) reduces to the binomial design.

4.1. Chebyshev polynomials

The Chebyshev polynomial of order *m* is defined by

$$T_m(z) = \begin{cases} (-1)^m \cosh(m \cdot \operatorname{arccosh} |z|), & z \le -1, \\ \cos\left(m \cdot \arccos(z)\right), & -1 \le z \le 1, \\ \cosh\left(m \cdot \operatorname{arccosh}(z)\right), & z \ge 1. \end{cases} \tag{15.16}$$

A Chebyshev polynomial $T_m(z)$ of any order *m* can be derived via a recursion formula, provided $T_{m-1}(z)$ and $T_{m-2}(z)$ are known:

$$T_m(z) = 2zT_{m-1}(z) - T_{m-2}(z). \tag{15.17}$$

Explicitly, from (15.16) we see that

$$m = 0, \;\; T_0(z) = 1$$

$$m = 1, \;\; T_1(z) = z\,.$$

Then, (15.17) produces:

$$m = 2,\ \ T_2(z) = 2z^2 - 1$$

$$m = 3,\ \ T_3(z) = 4z^3 - 3z$$

$$m = 4,\ \ T_4(z) = 8z^4 - 8z^2 + 1$$

$$m = 5,\ \ T_5(z) = 16z^5 - 20z^3 + 5z,\ \text{ etc.}$$

If $|z| \le 1$, then the Chebyshev polynomials are related to the cosine functions through $z = \cos x$ so that $T_m(x) = \cos(mx)$; see (15.16). Thus, the Chebyshev polynomials are handy in expanding the function cos(mx) as a polynomial of cos(x) of order m. For example, for $m = 2$,

$$\cos 2x = 2\cos^2 x - 1. \tag{15.18}$$

Similar expansion holds for the hyperbolic cosine function *cosh*. In general, the Chebyshev argument z can be related to the cosine argument x by

$$\begin{array}{l} z = \cos x \quad \leftrightarrow \quad x = \arccos z, |z| \le 1 \\ z = \cosh x \quad \leftrightarrow \quad x = \operatorname{arccosh} z, |z| > 1\,. \end{array} \tag{15.19}$$

For example, (15.18) can be written as:

$$\cos(2\arccos z) = 2\left[\cos(\arccos z)\right]^2 - 1 \Rightarrow \cos(2\arccos z) = 2z^2 - 1 = T_2(z). \tag{15.20}$$

<u>Properties of the Chebyshev polynomials of z</u>

1) All polynomials of any order m pass through the point (1,1).
2) Within the range $-1 \le z \le 1$, the polynomials have values within [–1,1].
3) All nulls occur within $-1 \le z \le 1$.
4) The maxima and minima in the $z \in [-1,1]$ range have values +1 and –1, respectively.
5) The higher the order of the polynomial, the steeper the slope for $|z| > 1$.

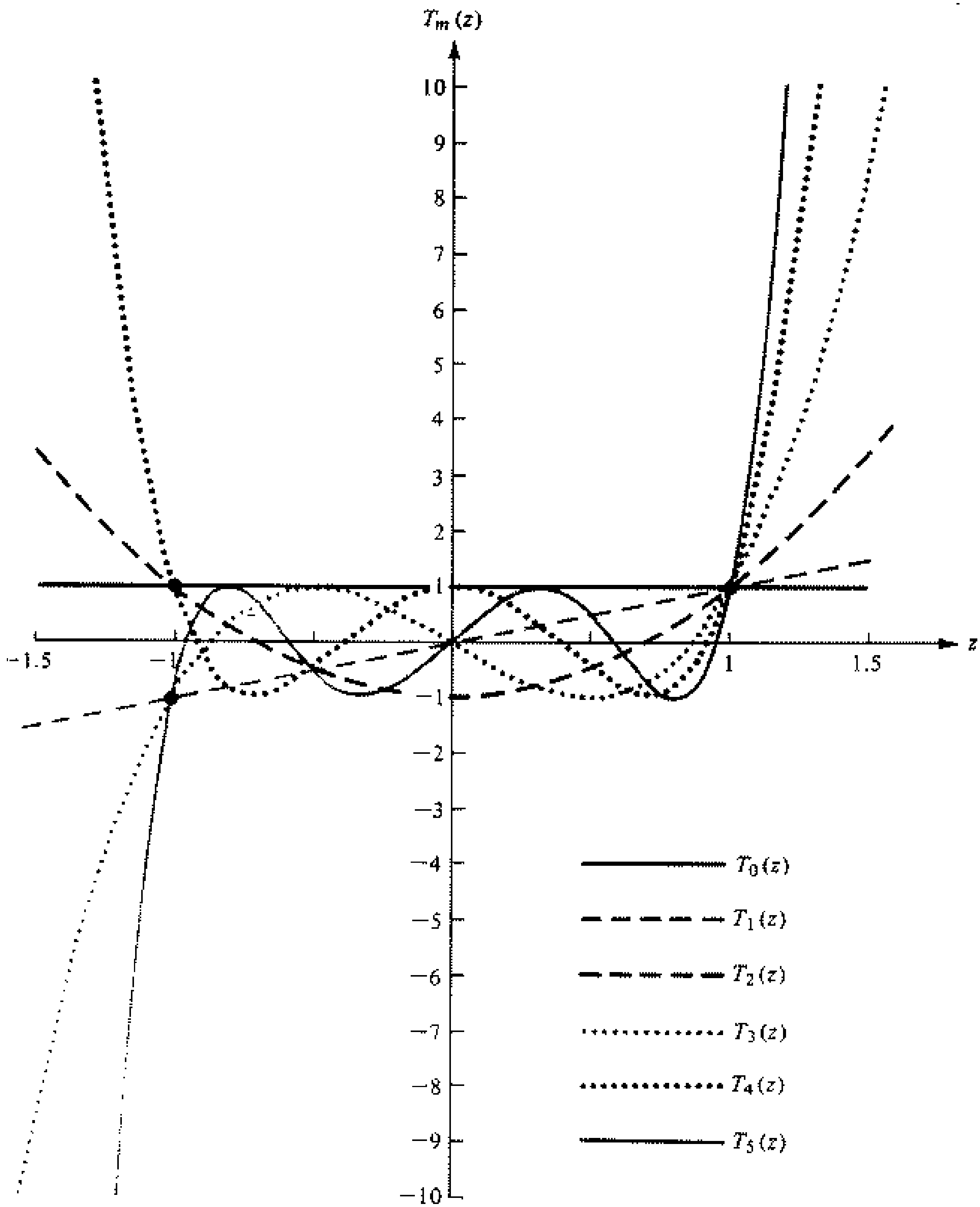


Fig. 6.19, pp. 296, Balanis

4.2. Chebyshev array design

The main goal is to approximate the desired AF with a Chebyshev polynomial such that

- the side-lobe level meets the requirements, and
- the main beam width is as small as possible.

An array of N elements has an AF approximated with a Chebyshev polynomial of order m, which is

$$m = N - 1. \tag{15.21}$$

In general, for a given side-lobe level, the higher the order m of the polynomial, the narrower the beamwidth. However, for $m > 10$, the difference is not substantial – see the slopes of $T_m(z)$ in the previous figure. The AFs of an N-element array in (15.5) or in (15.6) are shaped by a Chebyshev polynomial by requiring that

$$T_{N-1}(z) = \begin{cases} AF^e = \sum_{n=1}^{M} a_n \cos\left[(2n-1)u\right], \; M = N/2, & \text{even} \\ AF^o = \sum_{n=1}^{M+1} a_n \cos\left[2(n-1)u\right], M = (N-1)/2, & \text{odd} \end{cases} \tag{15.22}$$

where, as before, $u = (\pi d / \lambda)\cos\theta$. Let the side-lobe level be

$$R_0 = \frac{E_{\max}}{E_{sl}} = \frac{1}{AF_{sl}} \text{ (\textit{aka} voltage ratio).} \tag{15.23}$$

Then, we require that the maximum of T_{N-1} is fixed at an argument z_0, where

$$T_{N-1}^{\max}(z_0) = R_0, \, |z_0| > 1. \tag{15.24}$$

Equation (15.24) gives the maximum AF value, $AF(u) = AF^{\max}(u_0)$, and z_0 must satisfy the condition $|z_0| > 1$ so that $T_{N-1} > 1$. The maxima of $|T_{N-1}(z)|$ for $|z| \leq 1$ are equal to unity and they correspond to the side lobes of the AF. Thus, $AF(u)$ has side-lobe levels equal to $1/R_0$.

The AFs in (15.22) are sums of cosine functions of the form $\cos(mu)$, where $m = 2n - 1$ for an even-number array and $m = 2(n-1)$ for an odd-number array. Therefore, they can be expanded into polynomials of $\cos(u)$ of order m

using the Chebyshev recursion formula. On the other hand, $T_{N-1}(z)$ is a polynomial of z where z is limited to the range

$$-1 \le z \le z_0 > 1. \tag{15.25}$$

Since $-1 \le \cos u \le 1$, the relation between z and $\cos u$ must be normalized as

$$\cos u = z / z_0, \text{ where } u = (\pi d / \lambda)\cos\theta. \tag{15.26}$$

<u>Design of a DCA of *N* elements – general procedure:</u>

1) Expand the AF, as given by the right side of (15.22), by replacing each $\cos(mu)$ term with the power series of $\cos u$.
2) Determine z_0 such that $T_{N-1}(z_0) = R_0$ (voltage ratio).
3) Substitute $\cos u = z / z_0$ in the AF found in step 1.
4) Equate the AF found in Step 3 to $T_{N-1}(z)$ and determine the coefficients for each power of z.

<u>Example:</u> Design a DCA (broadside) of N=10 elements with a major-to-minor lobe ratio of $R_0 = 26$ dB. Find the excitation coefficients and the AF.

<u>Solution:</u>
The order of the Chebyshev polynomial is $m = N - 1 = 9$. The AF for an even-number array is:

$$AF_{2M} = \sum_{n=1}^{5} a_n \cos\left[(2n-1)u\right], \;\; u = \frac{\pi d}{\lambda}\cos\theta, \; M = 5.$$

<u>Step 1</u>: Write AF_{10} (see sum above) explicitly:

$$AF_{10} = a_1 \cos u + a_2 \cos 3u + a_3 \cos 5u + a_4 \cos 7u + a_5 \cos 9u.$$

Expand the $\cos(mu)$ terms as powers of $\cos u$:

$\cos 3u = 4\cos^3 u - 3\cos u$, ($a_2$ terms)

$\cos 5u = 16\cos^5 u - 20\cos^3 u + 5\cos u$, ($a_3$ terms)

$\cos 7u = 64\cos^7 u - 112\cos^5 u + 56\cos^3 u - 7\cos u$, ($a_4$ terms)

$\cos 9u = 256\cos^9 u - 576\cos^7 u + 432\cos^5 u - 120\cos^3 u + 9\cos u$. ($a_5$ terms)

Note that the above expansions can be readily obtained from the recursive Chebyshev relation (15.17) and the substitution $z=\cos u$. For example,

$$m=3,\;\; T_3(z)=4z^3-3z$$

translates into: $\cos(3u)=4\cos^3 u-3\cos u$.

<u>Step 2</u>: Determine z_0:

$$R_0=26\ \text{dB} \;\Rightarrow R_0=10^{26/20}\approx 20 \;\Rightarrow T_9(z_0)=20,$$
$$\cosh\left[9\operatorname{arccosh}(z_0)\right]=20,$$
$$9\operatorname{arccosh}(z_0)=\operatorname{arccosh}20=3.69,$$
$$\operatorname{arccosh}(z_0)=0.41,$$
$$z_0=\cosh 0.41 \quad\Rightarrow z_0=1.08515.$$

<u>Step 3</u>: Express the AF from Step 1 in terms of $\cos u=z/z_0$ and make equal to the Chebyshev polynomial:

$$\begin{aligned}
AF_{10}=&\frac{z}{z_0}\left(a_1-3a_2+5a_3-7a_4+9a_5\right)\\
&+\frac{z^3}{z_0^3}\left(4a_2-20a_3+56a_4-120a_5\right)\\
&+\frac{z^5}{z_0^5}\left(16a_3-112a_4+432a_5\right)\\
&+\frac{z^7}{z_0^7}\left(64a_4-576a_5\right)\\
&+\frac{z^9}{z_0^9}\left(256a_5\right)=\\
&=\underbrace{9z-120z^3+432z^5-576z^7+256z^9}_{T_9(z)}
\end{aligned}$$

<u>Step 4</u>: Find the coefficients by matching the power terms:

$$256a_5=256{z_0}^9 \Rightarrow a_5=2.0860$$
$$64a_4-576a_5=-576z_0^7 \Rightarrow a_4=2.8308$$
$$16a_3-112a_4+432a_5=432z_0^5 \Rightarrow a_3=4.1184$$

$$4a_2 - 20a_3 + 56a_4 - 120a_5 = -120z_0^7 \Rightarrow a_2 = 5.2073$$
$$a_1 - 3a_2 + 5a_3 - 7a_4 + 9a_5 = 9z_0^9 \Rightarrow a_1 = 5.8377$$

Normalize coefficients with respect to edge element (N=5):

$$a_5 = 1;\ a_4 = 1.357;\ a_3 = 1.974;\ a_2 = 2.496;\ a_1 = 2.789$$

$$\Rightarrow AF_{10} = 2.789\cos(u) + 2.496\cos(3u) + 1.974\cos(5u) + 1.357\cos(7u) + \cos(9u)$$

where $u = \dfrac{\pi d}{\lambda}\cos\theta$. Remember that the Chebyshev variable z relates to u as $z = z_0 \cos u$.

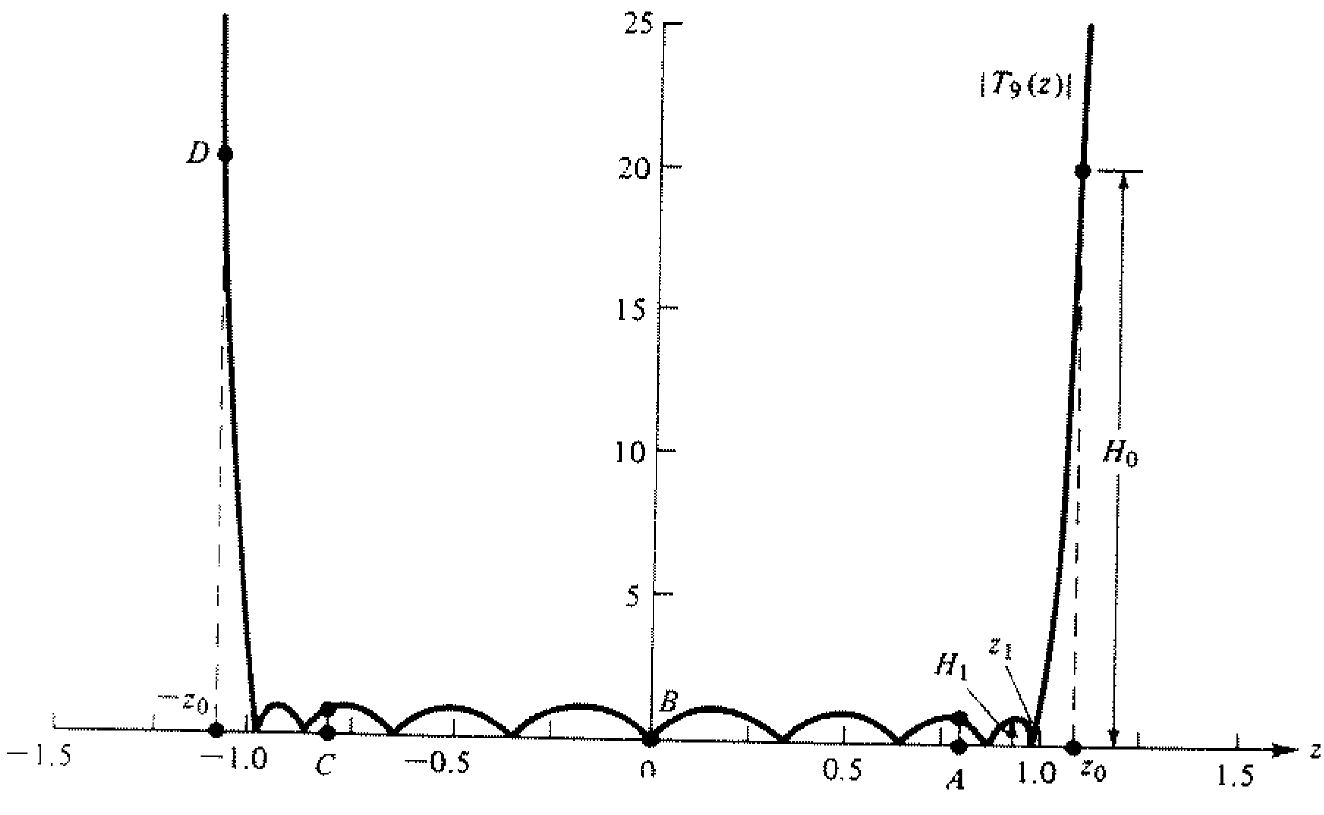


Fig. 6.20b, p. 298, Balanis

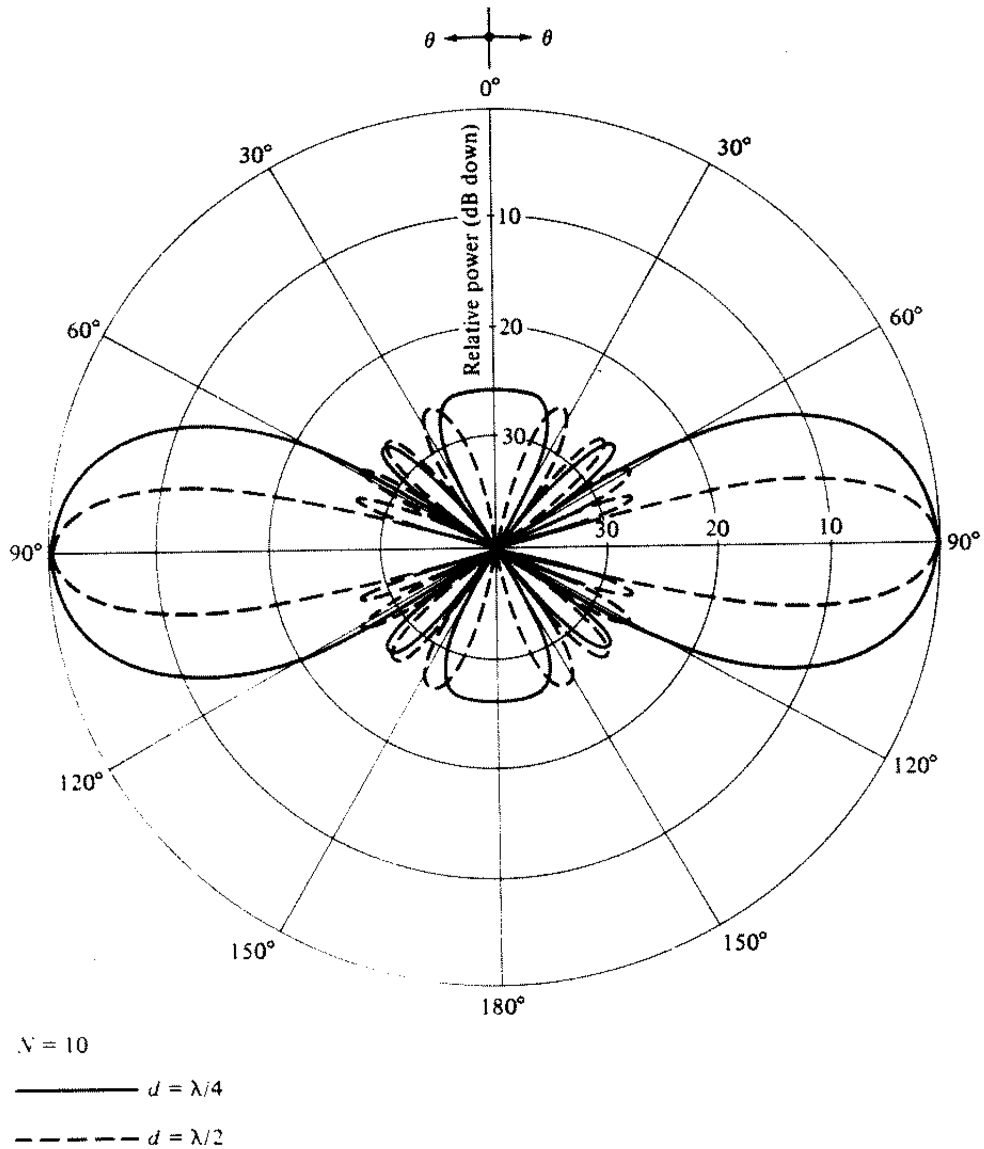


Fig. 6.21, p. 300, Balanis

4.3. Maximum affordable d for Dolph-Chebyshev arrays

This restriction arises from the requirement for a single major lobe – see also equation (15.25), $-1 \le z \le z_0$:

$$z \ge -1, \ \cos u = z / z_0, \ u = \frac{\pi d}{\lambda}\cos\theta,$$

$$\Rightarrow z = z_0 \cos\left(\frac{\pi d}{\lambda}\cos\theta\right) \ge -1. \qquad (15.27)$$

For a broadside array, when θ varies from $0°$ to $180°$, the argument z assumes values

$$\text{from } z_{(\theta=0°)} = z_0 \cos\left(\frac{\pi d}{\lambda}\right) \qquad (15.28)$$

$$\text{to } z_{(\theta=90°)} = z_0 \qquad (15.29)$$

$$\text{back to } z_{(\theta=180°)} = z_0 \cos\left(-\frac{\pi d}{\lambda}\right) = z_{(\theta=0°)}. \qquad (15.30)$$

The extreme value of z to the left on the abscissa corresponds to the end-fire directions of the AF ($\theta = 0, 180°$). This value must not go beyond $z = -1$. Otherwise, end-fire lobes of levels higher than 1 (higher than R_0) will appear. Therefore, the inequality (15.27) must hold for $\theta = 0°$ or $180°$:

$$z_0 \cos\left(\frac{\pi d}{\lambda}\right) \ge -1 \Rightarrow \cos\left(\frac{\pi d}{\lambda}\right) \ge -\frac{1}{z_0}. \qquad (15.31)$$

Let the angle γ be such that $\cos\gamma = 1 / z_0$ (see figure below). Then, $\gamma = \arccos\left(z_0^{-1}\right)$ and

$$\cos(\pi d / \lambda) \ge -\cos\gamma. \qquad (15.32)$$

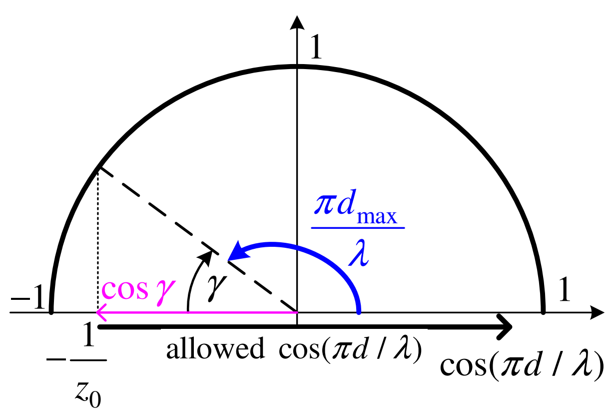


ILLUSTRATION OF EQUATION (15.31) AND THE REQUIREMENT IN (15.32) AND (15.33)

Remember that $z_0 > 1$; thus γ is a real-positive angle. Then, from (15.32),

$$\pi d / \lambda < \pi - \gamma = \pi - \arccos\left(z_0^{-1}\right) \tag{15.33}$$

or

$$\frac{\pi d_{\max}}{\lambda} = \pi - \underbrace{\arccos\left(z_0^{-1}\right)}_{\gamma} \Rightarrow \boxed{\frac{d_{\max}}{\lambda} = 1 - \frac{1}{\pi}\arccos\left(\frac{1}{z_0}\right)} \tag{15.34}$$

For the case of the previous example,

$$\frac{d}{\lambda} < 1 - \frac{1}{\pi}\arccos\left(\frac{1}{1.08515}\right) = 1 - \frac{0.39879}{\pi} = 0.873,$$

$$d_{\max} = 0.873\lambda .$$

5. Directivity of Non-uniform Arrays

It is difficult to derive closed form expressions for the directivity of non-uniform arrays. Here, we derive expressions in the form of series in the most general case of a linear array when the excitation coefficients are known.

The non-normalized array factor is

$$AF = \sum_{n=0}^{N-1} a_n e^{j\beta_n} e^{jkz_n \cos\theta}, \tag{15.35}$$

where

a_n is the amplitude of the excitation of the n-th element;

β_n is the phase angle of the excitation of the n-th element;

z_n is the z-coordinate of the n-th element.

The maximum AF is

$$AF_{\max} = \sum_{n=0}^{N-1} a_n . \tag{15.36}$$

The normalized AF is

$$AF_n = \frac{AF}{AF_{\max}} = \frac{\sum_{n=0}^{N-1} a_n e^{j\beta_n} e^{jkz_n \cos\theta}}{\sum_{n=0}^{N-1} a_n}. \tag{15.37}$$

The beam solid angle of a linear array along z is

$$\Omega_A = 2\pi \int_0^{\pi} \left| AF_n(\theta) \right|^2 \sin\theta d\theta,$$

$$\Omega_A = \frac{2\pi}{\left(\sum_{n=0}^{N-1} a_n \right)^2} \sum_{m=0}^{N-1} \sum_{p=0}^{N-1} a_m a_p e^{j(\beta_m - \beta_p)} \int_0^{\pi} e^{jk(z_m - z_p)\cos\theta} \sin\theta d\theta, \tag{15.38}$$

where

$$\int_0^{\pi} e^{jk(z_m - z_p)\cos\theta} \sin\theta d\theta = \frac{2\sin\left[k(z_m - z_p) \right]}{k(z_m - z_p)}.$$

$$\Rightarrow \Omega_A = \frac{4\pi}{\left(\sum_{n=0}^{N-1} a_n \right)^2} \sum_{m=0}^{N-1} \sum_{p=0}^{N-1} a_m a_p e^{j(\beta_m - \beta_p)} \cdot \frac{\sin\left[k(z_m - z_p) \right]}{k(z_m - z_p)}. \tag{15.39}$$

From

$$D_0 = 4\pi / \Omega_A,$$

we obtain

$$\Rightarrow D_0 = \frac{\left(\sum_{n=0}^{N-1} a_n \right)^2}{\sum_{m=0}^{N-1} \sum_{p=0}^{N-1} a_m a_p e^{j(\beta_m - \beta_p)} \cdot \frac{\sin\left[k(z_m - z_p) \right]}{k(z_m - z_p)}}. \tag{15.40}$$

For equispaced linear ($z_n = nd$) arrays, (15.40) reduces to

$$D_0 = \frac{\left(\sum_{n=0}^{N-1} a_n\right)^2}{\sum_{m=0}^{N-1}\sum_{p=0}^{N-1} a_m a_p e^{j(\beta_m-\beta_p)} \cdot \frac{\sin\left[(m-p)kd\right]}{(m-p)kd}}. \tag{15.41}$$

For equispaced broadside arrays, where $\beta_m = \beta_p$ for any (m,p), (15.41) reduces to

$$D_0 = \frac{\left(\sum_{n=0}^{N-1} a_n\right)^2}{\sum_{m=0}^{N-1}\sum_{p=0}^{N-1} a_m a_p \cdot \frac{\sin\left[(m-p)kd\right]}{(m-p)kd}}. \tag{15.42}$$

For equispaced broadside uniform arrays,

$$D_0 = \frac{N^2}{\sum_{m=0}^{N-1}\sum_{p=0}^{N-1} \frac{\sin\left[(m-p)kd\right]}{(m-p)kd}}. \tag{15.43}$$

When the spacing d is a multiple of $\lambda/2$, equation (15.42) reduces to

$$D_0 = \frac{\left(\sum_{n=0}^{N-1} a_n\right)^2}{\sum_{n=0}^{N-1} (a_n)^2}, \quad d = \frac{\lambda}{2}, \lambda, \ldots. \tag{15.44}$$

<u>Example</u>: Calculate the directivity of the Dolph–Chebyshev array designed in the previous example if $d = \lambda/2$.

The 10-element DCA has the following amplitude distribution:

$a_5 = 1$; $a_4 = 1.357$; $a_3 = 1.974$; $a_2 = 2.496$; $a_1 = 2.798$.

We make use of (15.44):

$$D_0 = \frac{4\left(\sum_{n=1}^{5} a_n\right)^2}{2\sum_{n=1}^{5}(a_n)^2} = 2\cdot\frac{(9.625)^2}{20.797} = 8.9090 \ \ (9.5 \ \text{dB}).$$

Output from ARRAYS.m: $D_0 = 8.9276$.

6. Half-power Beamwidth of a Broadside DCA

For large DCAs with side lobes in the range from –20 dB to –60 dB, the HPBW $HPBW_{DCA}$ can be found from the HPBW of a uniform array $HPBW_{UA}$ by introducing a beam-broadening factor *f* given by

$$f = 1 + 0.636\left\{\frac{2}{R_0}\cosh\left[\sqrt{(\text{arccosh} R_0)^2 - \pi^2}\right]\right\}^2, \tag{15.45}$$

so that

$$HPBW_{DCA} = f \times HPBW_{UA}. \tag{15.46}$$

In (15.45), R_0 denotes the side-lobe level (the voltage ratio).