LECTURE 16: PLANAR ARRAYS AND CIRCULAR ARRAYS

1. Planar Arrays

Planar arrays provide directional beams, symmetrical patterns with low side lobes, much higher directivity (narrow main beam) than that of their individual element. In principle, they can point the main beam toward any direction.

Applications – tracking radars, remote sensing, communications, etc.

A. The array factor of a rectangular planar array in the *xy* plane

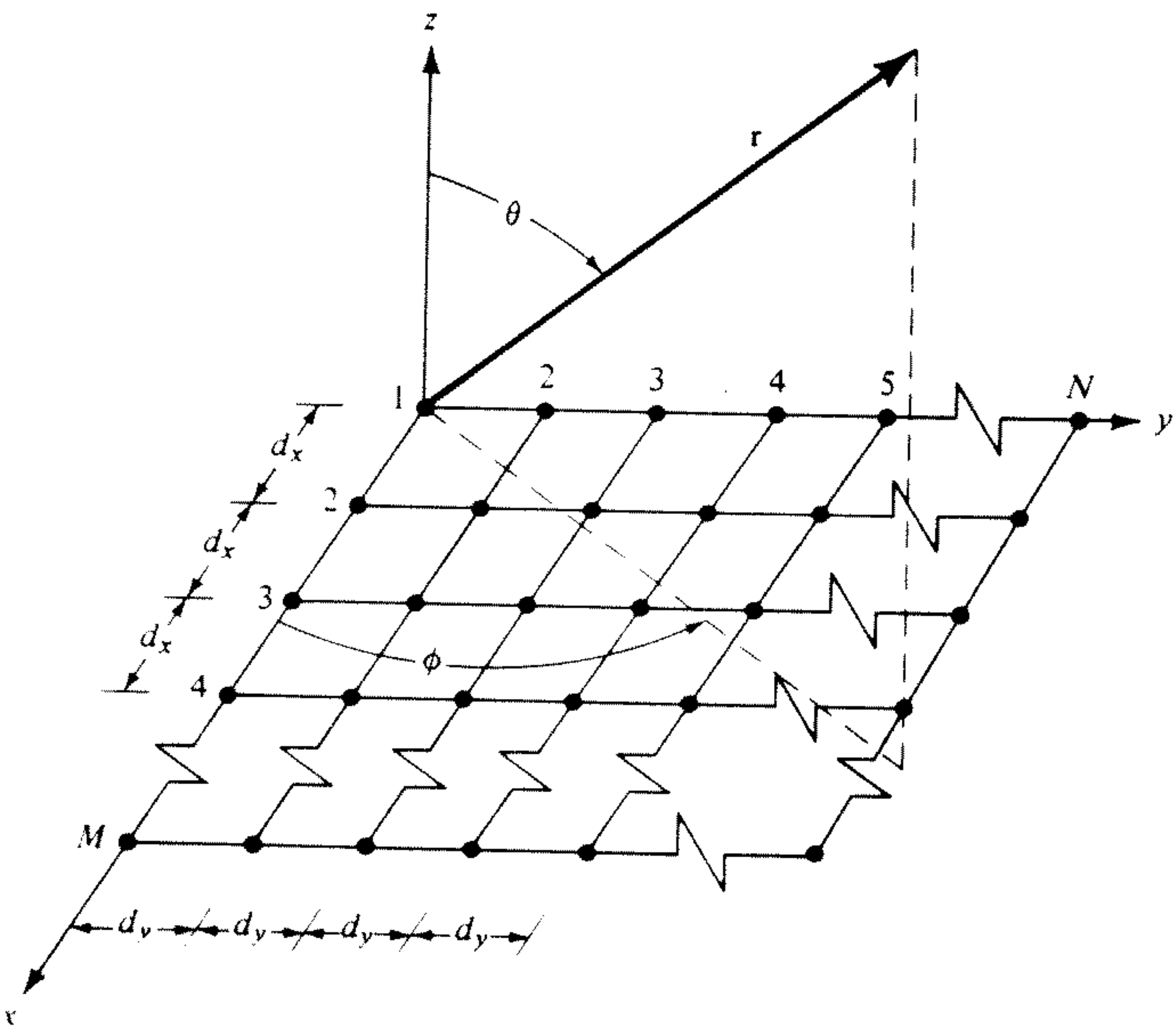


Fig. 6.23b, p. 310, Balanis

The AF of a linear array of M elements along the x-axis is

$$AF_{x1} = \sum_{m=1}^{M} I_{m1} e^{j(m-1)(kd_x \sin\theta\cos\phi + \beta_x)} \tag{16.1}$$

where $\sin\theta\cos\phi = \cos\gamma_x$ is the directional cosine with respect to the x-axis (γ_x is the angle between $\mathbf{r}$ and the x axis). It is assumed that all elements are equispaced with an interval of d_x and a progressive shift β_x. I_{m1} denotes the excitation amplitude of the element at the point with coordinates $x = (m-1)d_x$, $y = 0$. In the figure above, this is the element of the m-th row and the 1st column of the array matrix. Note that the 1st row corresponds to $x = 0$.

If N such arrays are placed at even intervals along the y direction, a rectangular array is formed. We assume again that they are equispaced at a distance d_y and there is a progressive phase shift β_y along each row. We also assume that the normalized current distribution along each of the x-directed arrays is the same but the absolute values correspond to a factor of I_{1n} $(n = 1,...,N)$. Then, the AF of the entire $M\times N$ array is

$$AF = \sum_{m=1}^{M} I_{m1} e^{j(m-1)(kd_x \sin\theta\cos\phi + \beta_x)} \times \sum_{n=1}^{N} I_{1n} e^{j(n-1)(kd_y \sin\theta\sin\phi + \beta_y)}, \tag{16.2}$$

or

$$AF = S_{x_M} S_{y_N}, \tag{16.3}$$

where

$$S_{x_M} = AF_{x1} = \sum_{m=1}^{M} I_{m1} e^{j(m-1)(kd_x \sin\theta\cos\phi + \beta_x)}, \text{ and}$$

$$S_{y_N} = AF_{1y} = \sum_{n=1}^{N} I_{1n} e^{j(n-1)(kd_y \sin\theta\sin\phi + \beta_y)}.$$

In the array factors above,

$$\begin{aligned} \sin\theta\cos\phi &= \hat{\mathbf{x}}\cdot\hat{\mathbf{r}} = \cos\gamma_x, \\ \sin\theta\sin\phi &= \hat{\mathbf{y}}\cdot\hat{\mathbf{r}} = \cos\gamma_y. \end{aligned} \tag{16.4}$$

The pattern of a rectangular array is the product of the array factors of the linear arrays in the x and y directions.

In the case of a ***uniform planar rectangular array***, $I_{m1} = I_{1n} = I_0$ for all m and n, i.e., all elements have the same excitation amplitudes. Thus,

$$AF = I_0 \sum_{m=1}^{M} e^{j(m-1)(kd_x \sin\theta\cos\phi + \beta_x)} \times \sum_{n=1}^{N} e^{j(n-1)(kd_y \sin\theta\sin\phi + \beta_y)}. \tag{16.5}$$

The normalized array factor is obtained as

$$\boxed{AF_n(\theta,\phi) = \left[\frac{\sin\left(M\dfrac{\psi_x}{2}\right)}{M\sin\left(\dfrac{\psi_x}{2}\right)}\right]\left[\frac{\sin\left(N\dfrac{\psi_y}{2}\right)}{N\sin\left(\dfrac{\psi_y}{2}\right)}\right]}, \tag{16.6}$$

where

$$\psi_x = kd_x \sin\theta\cos\phi + \beta_x,$$
$$\psi_y = kd_y \sin\theta\sin\phi + \beta_y.$$

The major lobe (principal maximum) and grating lobes of the terms

$$S_{x_M} = \frac{\sin\left(M\dfrac{\psi_x}{2}\right)}{M\sin\left(\dfrac{\psi_x}{2}\right)} \tag{16.7}$$

and

$$S_{y_N} = \frac{\sin\left(N\dfrac{\psi_y}{2}\right)}{N\sin\left(\dfrac{\psi_y}{2}\right)} \tag{16.8}$$

are located at angles such that

$$kd_x \sin\theta_m \cos\phi_m + \beta_x = \pm 2m\pi, \quad m = 0,1,\ldots, \tag{16.9}$$
$$kd_y \sin\theta_n \sin\phi_n + \beta_y = \pm 2n\pi, \quad n = 0,1,\ldots. \tag{16.10}$$

The principal maximum corresponds to $m = 0$, $n = 0$.

In general, β_x and β_y can be independent from each other. But, if it is required that the main beams of S_{x_M} and S_{y_N} intersect (which is usually the case), then the common main beam is in the direction:

$$\theta = \theta_0 \text{ and } \phi = \phi_0,\ m = n = 0. \tag{16.11}$$

With the principal maximum specified by (θ_0, ϕ_0), the progressive phase shifts β_x and β_y must satisfy

$$\beta_x = -kd_x \sin\theta_0 \cos\phi_0, \tag{16.12}$$

$$\beta_y = -kd_y \sin\theta_0 \sin\phi_0. \tag{16.13}$$

If β_x and β_y are specified, then the direction of the main beam can be found by solving (16.12) and (16.13) as a system of equations:

$$\tan\phi_0 = \frac{\beta_y d_x}{\beta_x d_y}, \tag{16.14}$$

$$\sin\theta_0 = \pm\sqrt{\left(\frac{\beta_x}{kd_x}\right)^2 + \left(\frac{\beta_y}{kd_y}\right)^2}. \tag{16.15}$$

The grating lobes can be located by substituting (16.12) and (16.13) in (16.9) and (16.10):

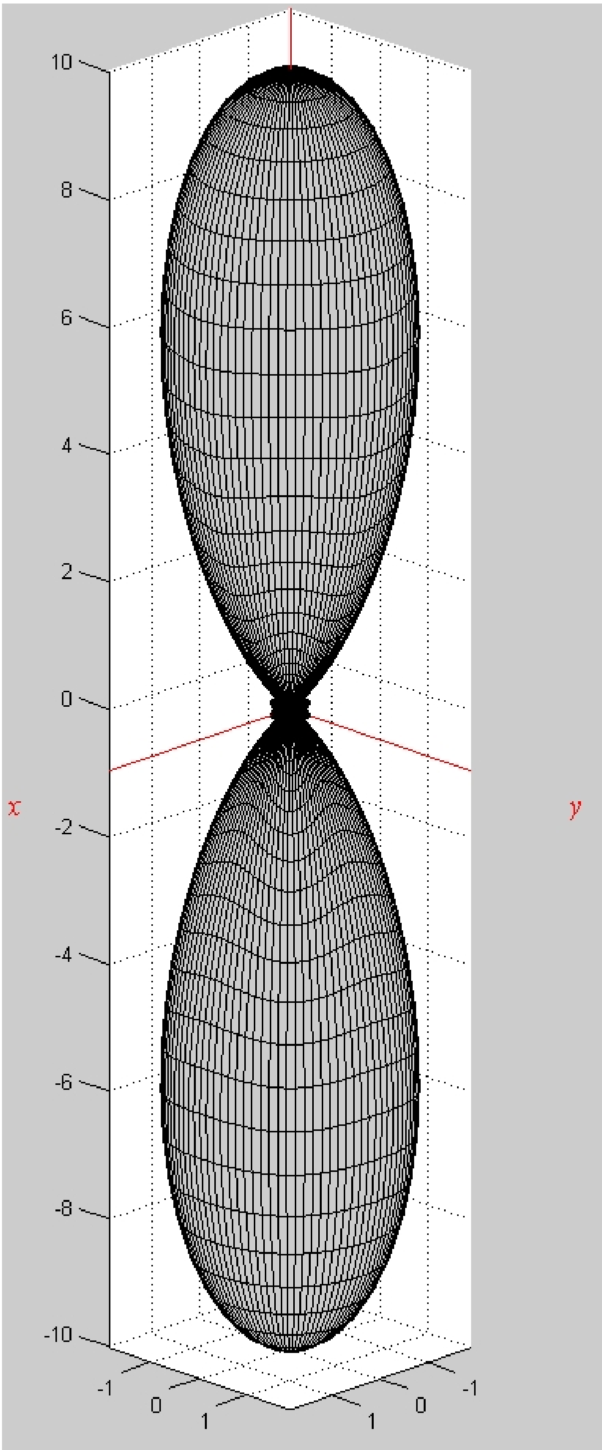
$$\tan\phi_{mn} = \frac{\sin\theta_0 \sin\phi_0 \pm n\lambda/d_y}{\sin\theta_0 \cos\phi_0 \pm m\lambda/d_x}, \tag{16.16}$$

$$\sin\theta_{mn} = \frac{\sin\theta_0 \cos\phi_0 \pm m\lambda/d_x}{\cos\phi_{mn}} = \frac{\sin\theta_0 \sin\phi_0 \pm n\lambda/d_y}{\sin\phi_{mn}}. \tag{16.17}$$

To avoid grating lobes, the spacing between the elements must be less than λ, i.e., $d_x < \lambda$ and $d_y < \lambda$. In order a true grating lobe to occur, both equations (16.16) and (16.17) must have a real solution (θ_{mn}, ϕ_{mn}).

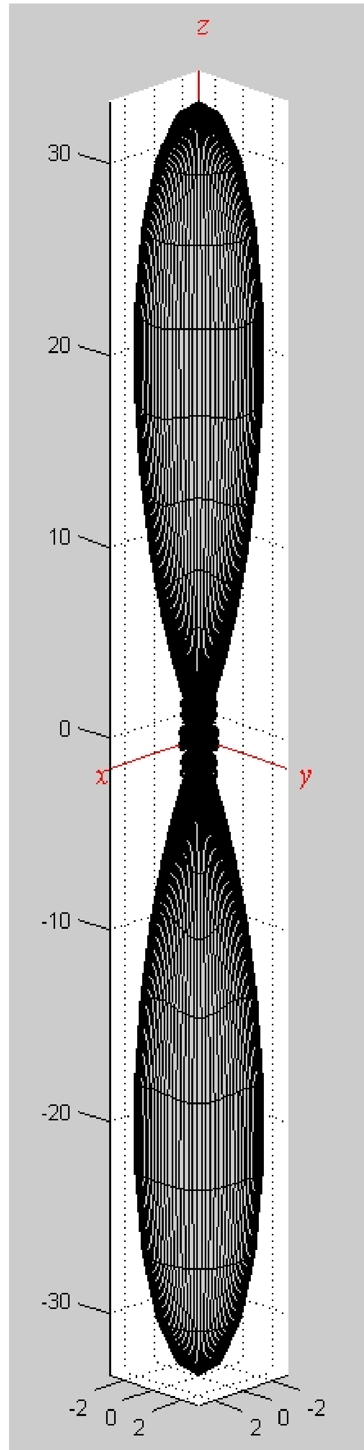
The array factors of a 5 by 5 uniform array are shown below for two spacing values: $d = \lambda/4$ and $d = \lambda/2$. Notice the considerable decrease in the beamwidth as the spacing is increased from $\lambda/4$ to $\lambda/2$.

DIRECTIVITY PATTERNS OF A 5-ELEMENT SQUARE PLANAR UNIFORM ARRAY WITHOUT GRATING LOBES $\beta_x = \beta_y = 0$: (a) $d = \lambda/4$, (b) $d = \lambda/2$



$D_0 = 10.0287$ (10.0125 dB)

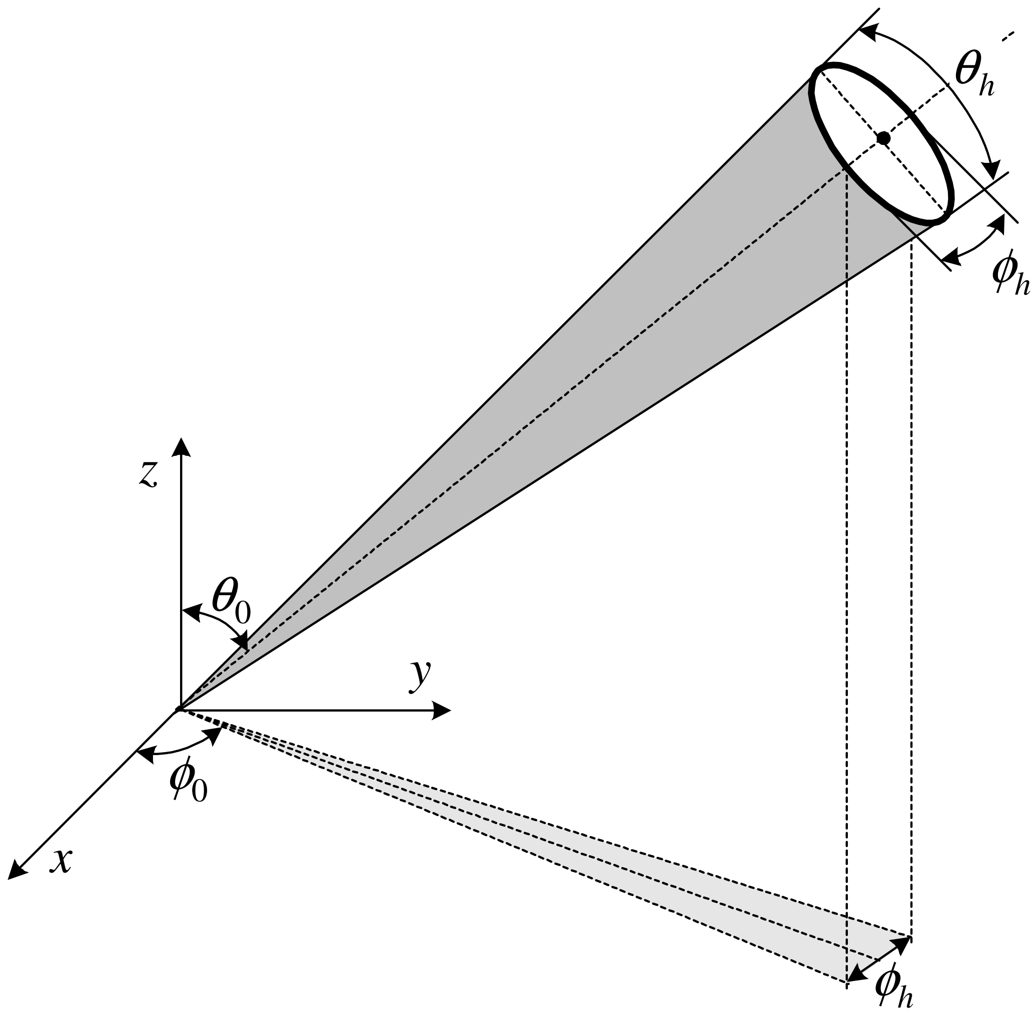
(a)



$D_0 = 33.2458$ (15.2174 dB)

(b)

B. The beamwidth of a planar array



A simple procedure, proposed by R.S. Elliot[1] is outlined below. It is based on the use of the beamwidths of the linear arrays building the planar array.

For a large array, the maximum of which is near the broad side, the ***elevation plane HPBW*** is approximately

$$\theta_h \approx \frac{1}{\cos\theta_0 \sqrt{\Delta\theta_x^{-2}\cos^2\phi_0 + \Delta\theta_y^{-2}\sin^2\phi_0}} \tag{16.18}$$

where

[1] "Beamwidth and directivity of large scanning arrays", *The Microwave Journal*, Jan. 1964, pp.74-82.

(θ_0, ϕ_0) specifies the main-beam direction;

$\Delta\theta_x$ is the HPBW of a linear BSA of M elements and an amplitude distribution which is the same as that of the x-axis linear arrays building the planar array;

$\Delta\theta_y$ is the HPBW of a linear BSA of N elements and amplitude distribution which is the same as that of the y-axis linear arrays building the planar array.

The azimuth HPBW is the HPBW in the plane orthogonal to the elevation plane and contains the maximum. It is

$$\phi_h \approx \sqrt{\frac{1}{\Delta\theta_x^{-2}\sin^2\phi_0 + \Delta\theta_y^{-2}\cos^2\phi_0}}\,. \tag{16.19}$$

For a square array $(M = N)$ with the same amplitude distributions along the x and y axes, equations (16.18) and (16.19) reduce to

$$\theta_h = \frac{\Delta\theta_x}{\cos\theta_0} = \frac{\Delta\theta_y}{\cos\theta_0}\,, \tag{16.20}$$

$$\phi_h = \Delta\theta_x = \Delta\theta_y\,. \tag{16.21}$$

From (16.20), it is obvious that the HPBW in the elevation plane depends on the elevation angle θ_0 of the main beam whereas the HPBW in the azimuthal plane ϕ_h does not.

The antenna solid angle of the planar array can be approximated by

$$\Omega_A \approx \theta_h \phi_h\,, \tag{16.22}$$

where θ_h and ϕ_h are in radians. Substituting (16.18) and (16.19), yields

$$\boxed{\Omega_A = \frac{\Delta\theta_x \Delta\theta_y}{\cos\theta_0 \sqrt{\left[\sin^2\phi_0 + \dfrac{\Delta\theta_y^2}{\Delta\theta_x^2}\cos^2\phi_0\right]\left[\sin^2\phi_0 + \dfrac{\Delta\theta_x^2}{\Delta\theta_y^2}\cos^2\phi_0\right]}}}\,. \tag{16.23}$$

C. Directivity of planar rectangular array

The general expression for the calculation of the directivity of an array is

$$D_0 = 4\pi \frac{|AF(\theta_0,\phi_0)|^2}{\int_0^{2\pi}\int_0^{\pi} |AF(\theta,\phi)|^2 \sin\theta d\theta d\phi}. \tag{16.24}$$

For large planar arrays, which are nearly broadside, (16.24) reduces to

$$D_0 = \pi D_x D_y \cos\theta_0 \tag{16.25}$$

where

D_x is the directivity of the respective linear BSA, x-axis;

D_y is the directivity of the respective linear BSA, y-axis.

We can also use the array solid angle Ω_A in (16.23) to calculate the approximate directivity of a nearly broadside planar array:

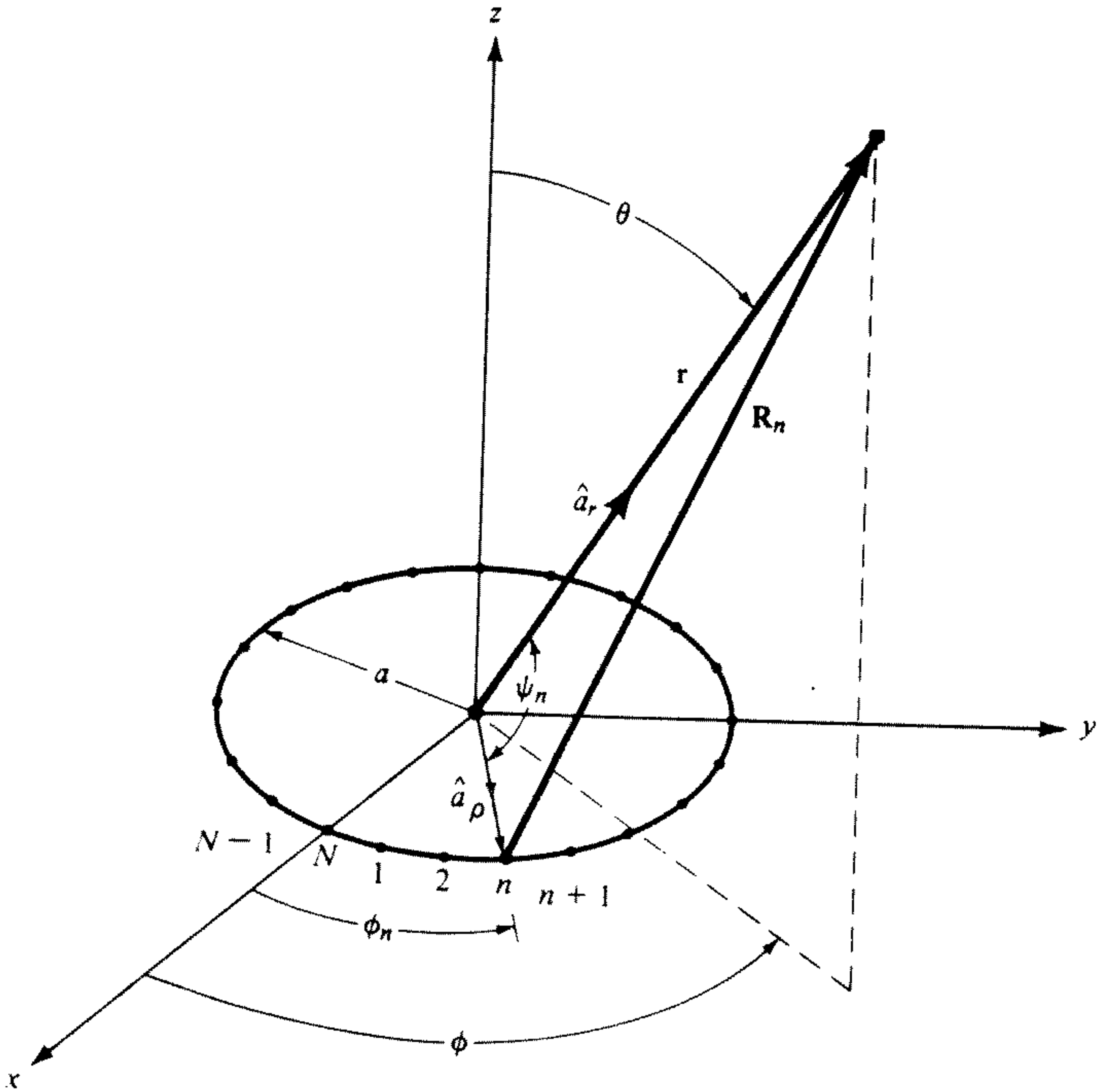
$$D_0 \approx \frac{\pi^2}{\Omega_{A[\mathrm{rad}^2]}} \approx \frac{32400}{\Omega_{A[\mathrm{deg}^2]}}. \tag{16.26}$$[2]

Remember:

1) The main beam direction is controlled through the phase shifts, β_x and β_y.
2) The beamwidth and side-lobe levels are controlled through the amplitude distribution.

[2] A *steradian* relates to square degrees as 1 sr = $(180/\pi)^2 \approx 3282.80635$ deg. Note that this formula is only approximate and the relationship between the exact values of D_0 and Ω_A is $D_0 = 4\pi/\Omega_A$.

2. Circular Array



A. Array factor of circular array

The normalized field can be written as

$$E(r,\theta,\phi)=\sum_{n=1}^{N}a_n\frac{e^{-jkR_n}}{R_n}, \tag{16.27}$$

where

$$R_n=\sqrt{r^2+a^2-2ar\cos\psi_n}\,. \tag{16.28}$$

For $r\gg a$, (16.28) reduces to

$$R_n\approx r-a\cos\psi_n=r-a(\hat{\mathbf{a}}_{\rho,n}\cdot\hat{\mathbf{r}})\,. \tag{16.29}$$

In a rectangular coordinate system,

$$\left|\begin{array}{l}\hat{\mathbf{a}}_{\rho,n}=\hat{\mathbf{x}}\cos\phi_n+\hat{\mathbf{y}}\sin\phi_n\\ \hat{\mathbf{r}}=\hat{\mathbf{x}}\sin\theta\cos\phi+\hat{\mathbf{y}}\sin\theta\sin\phi+\hat{\mathbf{z}}\cos\theta.\end{array}\right.$$

Therefore,

$$R_n\approx r-a\sin\theta\left(\cos\phi_n\cos\phi+\sin\phi_n\sin\phi\right), \tag{16.30}$$

or

$$R_n\approx r-a\sin\theta\cos\left(\phi-\phi_n\right). \tag{16.31}$$

For the amplitude term, the far-zone approximation

$$\frac{1}{R_n}\approx\frac{1}{r},\ \text{all}\ n \tag{16.32}$$

is made.

With the approximations in (16.31) and (16.32), the far-zone array field is obtained as:

$$E(r,\theta,\phi)=\frac{e^{-jkr}}{r}\sum_{n=1}^{N}a_n e^{jka\sin\theta\cos(\phi-\phi_n)}, \tag{16.33}$$

where

a_n is the complex excitation coefficient (amplitude and phase);

$\phi_n=2\pi n/N$ is the angular position of the n-th element.

In general, the excitation coefficient can be represented as

$$a_n = I_n e^{j\alpha_n}, \tag{16.34}$$

where I_n is the amplitude and α_n is the phase of the excitation of the n-th element relative to a chosen array element of zero phase. Substituting (16.34) into (16.33) leads to:

$$\Rightarrow E(r,\theta,\phi) = \frac{e^{-jkr}}{r} \sum_{n=1}^{N} I_n e^{j\left[ka\sin\theta\cos(\phi-\phi_n)+\alpha_n\right]}. \tag{16.35}$$

The AF is then

$$\boxed{AF(\theta,\phi) = \sum_{n=1}^{N} I_n e^{j\left[ka\sin\theta\cos(\phi-\phi_n)+\alpha_n\right]}}. \tag{16.36}$$

Expression (16.36) represents the AF of a circular array of N equispaced elements. The maximum of the AF occurs when all exponential terms in (16.36) equal unity, or,

$$ka\sin\theta\cos(\phi-\phi_n)+\alpha_n = 2m\pi,\ \ m = 0,\pm1,\pm2,\ \text{all}\ n. \tag{16.37}$$

The principal maximum ($m=0$) is defined by the direction (θ_0,ϕ_0), for which

$$\alpha_n = -ka\sin\theta_0 \cdot \cos(\phi_0-\phi_n),\ \ n = 1,2,\ldots,N. \tag{16.38}$$

For example, for maximum radiation along $\theta_0 = 0°, 180°$ (along the axis of the circular array), all elements must be fed in-phase, i.e., $\alpha_n = 0$ for all n. For maximum radiation along $\theta_0 = 90°$ and $\phi_0 = 0$ (along the x axis in the array's plane), $\alpha_n = -ka\cos(\phi_n)$, $n = 1,2,\ldots,N$.

If a circular array is required to have maximum radiation along (θ_0,ϕ_0), then the phases of its excitations have to fulfil (16.38). Substituting (16.38) into (16.36) shows that the AF of such an array is

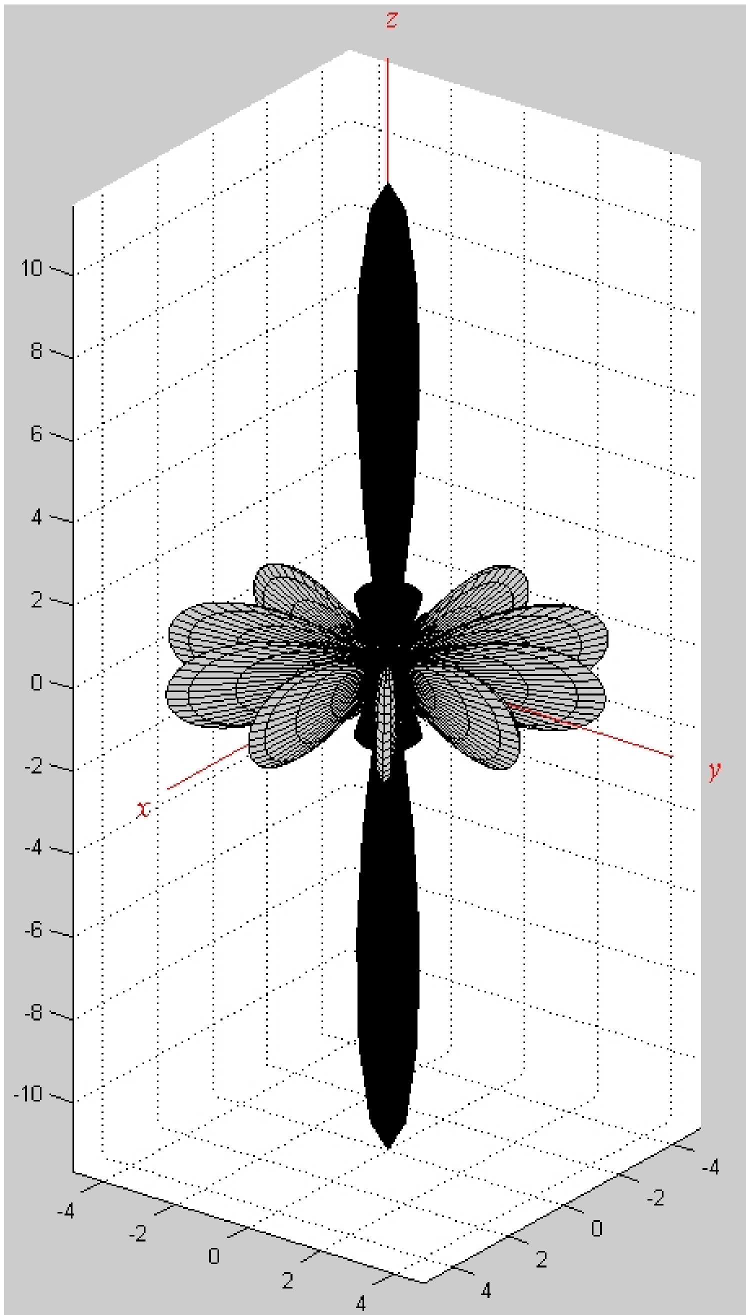
$$AF(\theta,\phi) = \sum_{n=1}^{N} I_n e^{jka\left[\sin\theta\cos(\phi-\phi_n)-\sin\theta_0\cos(\phi_0-\phi_n)\right]}, \tag{16.39}$$

$$\Rightarrow AF(\theta,\phi) = \sum_{n=1}^{N} I_n e^{jka(\cos\psi_n-\cos\psi_{0n})}. \tag{16.40}$$

Here, $\psi_n = \arccos[\sin\theta\cos(\phi-\phi_n)]$ is the angle between $\hat{\mathbf{r}}$ and $\hat{\mathbf{a}}_{\rho,n}$ whereas $\psi_{0,n} = \arccos[\sin\theta_0\cos(\phi_0-\phi_n)]$ is the angle between $\hat{\mathbf{a}}_{\rho,n}$ and $\hat{\mathbf{r}}_0$, where $\hat{\mathbf{r}}_0$ points in the direction of maximum radiation.

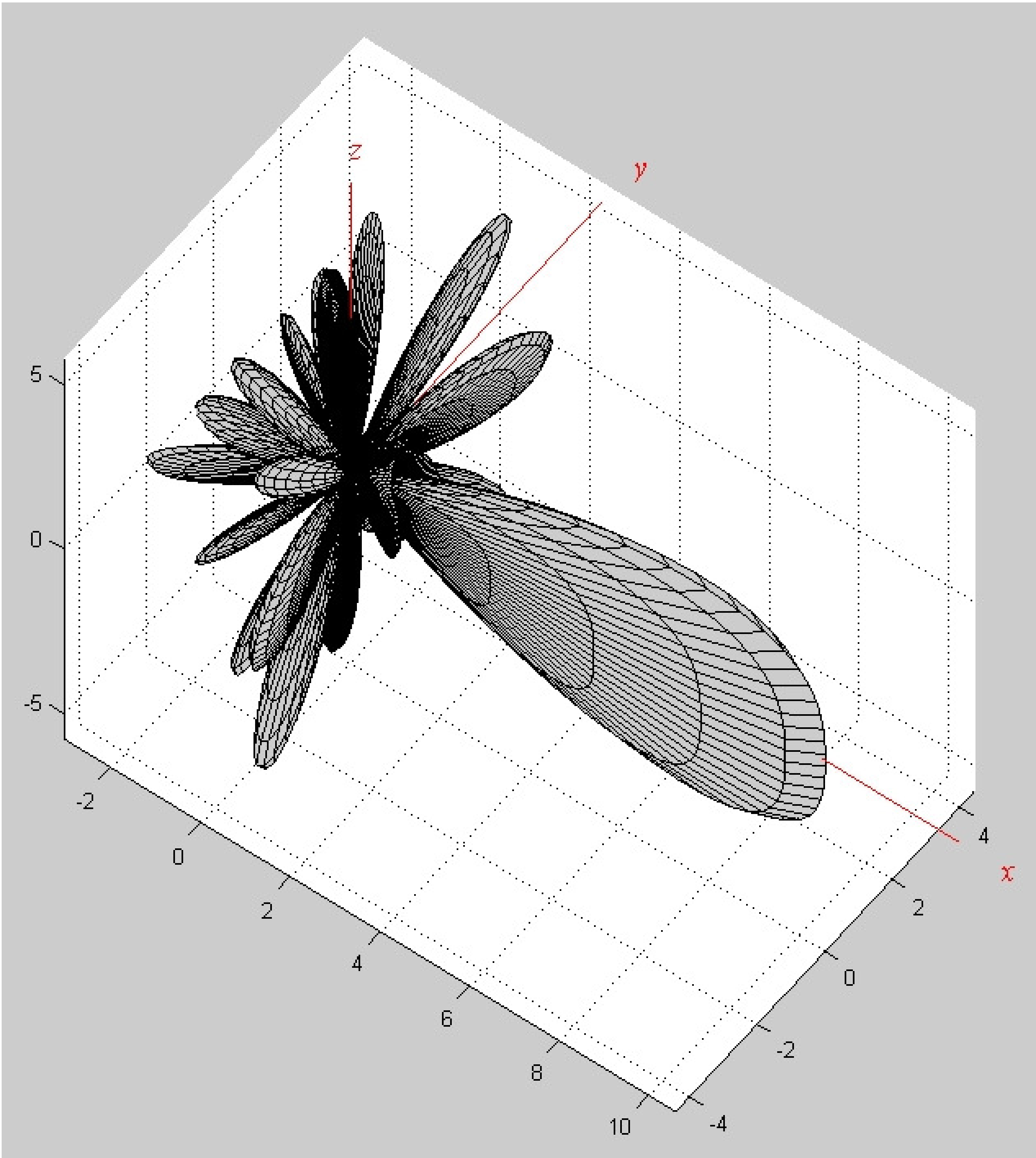
As the radius of the array a becomes large compared to λ, the directivity of the uniform circular array ($I_n = I_0$, all n) approaches the value of N.

UNIFORM CIRCULAR ARRAY 3-D PATTERN ($N = 10$, $ka = 2\pi a/\lambda = 10$): MAXIMUM AT $\theta = 0°, 180°$ ($\alpha_n = 0$ for all n)



$D_0 = 11.6881$ (10.6775 dB)

UNIFORM CIRCULAR ARRAY 3-D PATTERN ($N = 10,\ ka = 2\pi a / \lambda = 10$): MAXIMUM AT $\theta = 90°, \phi = 0°$



$D_0 = 10.589$ (10.2485 dB)