Solutions for Problems on Signal Analysis

1. (a)

$$S(\omega) = \int_{-\infty}^{\infty} s(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} s(t)\cos\omega t dt - j\int_{-\infty}^{\infty} s(t)\sin\omega t dt$$

Thus

$$\begin{aligned} Re[S(\omega)] &= \int_{-\infty}^{\infty} s(t)\cos\omega t dt \\ &= \int_{-\infty}^{\infty} (s_e(t) + s_o(t))\cos\omega t dt \\ &= \int_{-\infty}^{\infty} s_e(t)\cos\omega t dt \end{aligned}$$

$$\begin{aligned} \mathcal{F}[s_e(t)] &= \int_{-\infty}^{\infty} s_e(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} s_e(t)\cos\omega t dt - j\int_{-\infty}^{\infty} s_e(t)\sin\omega t dt \\ &= \int_{-\infty}^{\infty} s_e(t)\cos\omega t dt = Re[S(\omega)] \end{aligned}$$

$$\begin{aligned} jIm[S(\omega)] &= -j\int_{-\infty}^{\infty} s(t)\sin\omega t dt \\ &= -j\int_{-\infty}^{\infty} (s_e(t) + s_o(t))\sin\omega t dt \\ &= -j\int_{-\infty}^{\infty} s_o(t)\sin\omega t dt \end{aligned}$$

$$\begin{aligned} \mathcal{F}[s_o(t)] &= \int_{-\infty}^{\infty} s_o(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} s_o(t)\cos\omega t dt - j\int_{-\infty}^{\infty} s_o(t)\sin\omega t dt \\ &= -j\int_{-\infty}^{\infty} s_o(t)\sin\omega t dt = jIm[S(\omega)] \end{aligned}$$

(b)

$$\begin{aligned} s(t) &= s_r(t) + js_i(t) \\ s^*(t) &= s_r(t) - js_i(t) \end{aligned}$$

Thus

$$s_r(t) = \frac{1}{2}[s(t) + s^*(t)]$$

$$s_i(t) = \frac{1}{2}\left[s(t) - s^*(t)\right]$$

$$\begin{aligned}\mathcal{F}[s_r(t)] &= \frac{1}{2}\left[\mathcal{F}[s(t)] + \mathcal{F}[s^*(t)]\right] \\ &= \frac{1}{2}[S(\omega) + S^*(-\omega)]\end{aligned}$$

$$\begin{aligned}\mathcal{F}[s_i(t)] &= \frac{1}{2}\left[\mathcal{F}[s(t)] - \mathcal{F}[s^*(t)]\right] \\ &= \frac{1}{2}[S(\omega) - S^*(-\omega)]\end{aligned}$$

2. (a) $s(t)$ is an even function:

$$\begin{aligned}S(\omega) &= 2\int_0^\infty s(t)\cos\omega t dt \\ &= 2\left[\int_0^{3/2} A(-t+1)\cos\omega t dt + \int_{3/2}^2 A(t-2)\cos\omega t dt\right] \\ &= 2A\left[(-\frac{t\sin\omega t}{\omega} - \frac{\cos\omega t}{\omega^2} + \frac{\sin\omega t}{\omega})|_0^{3/2} + (\frac{t\sin\omega t}{\omega} + \frac{\cos\omega t}{\omega^2} - 2\frac{\sin\omega t}{\omega})|_{3/2}^2\right] \\ &= \frac{2A}{\omega^2}[1 + \cos 2\omega - 2\cos(3/2)\omega]\end{aligned}$$

(b)

$$\begin{aligned}S(\omega) &= \int_{-1}^0 A(t+1)e^{-j\omega t}dt + \int_0^4 A(-\frac{1}{4}t+1)e^{-j\omega t}dt \\ &= A\left\{\frac{e^{-j\omega t}}{-j\omega}|_{-1}^4 + \frac{1}{-j\omega}\left[te^{-j\omega t}|_{-1}^0 - \frac{e^{-j\omega t}}{-j\omega}|_{-1}^0\right] - \frac{1}{-4j\omega}\left[te^{-j\omega t}|_0^4 - \frac{e^{-j\omega t}}{-j\omega}|_0^4\right]\right\} \\ &= \frac{A}{4\omega^2}\left[5 - 4e^{+j\omega} - e^{-4j\omega}\right]\end{aligned}$$

(c) $s(t)$ is an even function:

$$\begin{aligned}S(\omega) &= 2\int_0^\infty s(t)\cos\omega t dt \\ &= 2\int_0^{\pi/2} A\cos t\cos\omega t dt \\ &= A\left[\int_0^{\pi/2} \cos(t+\omega t) + \cos(t-\omega t)dt\right] \\ &= A\left[\frac{\sin(\omega+1)t}{\omega+1}|_0^{\pi/2} + \frac{\sin(\omega-1)t}{\omega-1}|_0^{\pi/2}\right] \\ &= A\left[\frac{\sin\frac{(\omega+1)\pi}{2}}{\omega+1} + \frac{\sin\frac{(\omega-1)\pi}{2}}{\omega-1}|\right] \\ &= \frac{2A}{1-\omega^2}\cos\frac{\omega\pi}{2}\end{aligned}$$

(d)

$$\begin{aligned}
S(\omega) &= \int_0^1 t^2 e^{-j\omega t} dt + \int_2^3 t^2 e^{-j\omega t} dt \\
&= \frac{e^{-j\omega t}}{-j\omega}\left[t^2 - \frac{t}{-j\omega} + \frac{1}{(-j\omega)^2}\right](|_0^1 + |_2^3) \\
&= \frac{1}{j\omega}\left(-e^{-j\omega} + 4e^{-2j\omega} - 9e^{-3j\omega} - \frac{e^{-j\omega} - 2e^{-2j\omega} + 3e^{-3j\omega}}{j\omega} + \frac{e^{-j\omega} - e^{-2j\omega} + e^{-3j\omega}}{\omega^2}\right)
\end{aligned}$$

3.

$$\int_{-\infty}^{\infty} \delta(t-2)\sin t \; dt = \sin 2$$

$$\int_{-\infty}^{\infty} \delta(t+3)e^{-t} \; dt = e^{-3}$$

$$\int_{-\infty}^{\infty} \delta(1-t)(t^3+4) \; dt = 1^3 + 4 = 5$$

4. (a)

$$G_\tau(t) = \begin{cases} 1 & |t| \leq \frac{\tau}{2} \\ 0 & |t| > \frac{\tau}{2} \end{cases}$$

$$G_\tau(t) \leftrightarrow \tau \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}}$$

$$\mathcal{F}[S(t)] = 2\pi s(-\omega) \Rightarrow \mathcal{F}^{-1}[s(-\omega)] = \frac{1}{2\pi}S(t)$$

Now

$$s(-\omega) = AG_W(\omega) \text{ where } W = 2\omega_0$$

therefore,

$$\mathcal{F}^{-1}[AG_W(\omega)] = \frac{A}{2\pi}2\omega_0\frac{\sin\omega_0 t}{\omega_0 t} = A\frac{\omega_0}{\pi}\frac{\sin\omega_0 t}{\omega_0 t}$$

$$\mathcal{F}^{-1}[AG_W(\omega)e^{-j\omega t_0}] = A\frac{\omega_0}{\pi}\frac{\sin\omega_0(t-t_0)}{\omega_0(t-t_0)}$$

(b)

$$|S(\omega)| = AG_W(\omega) \text{ where } W = 2\omega_0$$

$$\theta(\omega) = \begin{cases} -\frac{\pi}{2} & \omega \leq 0 \\ \frac{\pi}{2} & \omega < 0 \end{cases}$$

Hence,

$$\begin{aligned}
s(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{j\omega t}d\omega \\
&= \frac{A}{2\pi}\int_{-\omega_0}^{0} e^{j\frac{\pi}{2}}e^{j\omega t}d\omega + \frac{A}{2\pi}\int_{0}^{\omega_0} e^{-j\frac{\pi}{2}}e^{j\omega t}d\omega \\
&= \frac{A}{2\pi}e^{j\frac{\pi}{2}}\frac{e^{j\omega t}}{jt}|_{-\omega_0}^{0} + \frac{A}{2\pi}e^{-j\frac{\pi}{2}}\frac{e^{j\omega t}}{jt}|_{0}^{\omega_0} \\
&= \frac{A}{2\pi jt}e^{j\frac{\pi}{2}}\left(1 - e^{-j\omega_0 t}\right) + \frac{A}{2\pi jt}e^{-j\frac{\pi}{2}}\left(e^{j\omega_0 t} - 1\right) \\
&= \frac{A}{\pi t}[1 + \sin(-\frac{\pi}{2} + \omega_0 t)] = \frac{A}{\pi t}[1 - \cos\omega_0 t]
\end{aligned}$$

5. (a)

$$S(\omega) = AG_a(\omega) * [\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$$

$$s(t) = 2\pi A\frac{a}{\pi}\frac{\sin at}{at}\cdot(\frac{1}{\pi}\cos\omega_0 t) = \frac{2Aa}{\pi}\frac{\sin at}{at}\cos\omega_0 t$$

(b) $W = 2a$

$$s(t) = 2\pi\cdot\frac{1}{2\pi}2Aa\left[\frac{\sin at}{at}\right]^2\cdot(\frac{1}{\pi}\cos\omega_0 t) = \frac{2Aa}{\pi}\frac{\sin^2 at}{(at)^2}\cos\omega_0 t$$

6. (a)

$$s(t) = AG_{\frac{2\pi}{5}}(t)\cdot\cos 20t$$

$$S(\omega) = \frac{1}{2\pi}A\frac{2\pi}{5}\frac{\sin\frac{\pi}{5}\omega}{\frac{\pi}{5}\omega} * \pi[\delta(\omega+20)+\delta(\omega-20)] = \frac{A\pi}{5}\left[\text{sinc}\left(\frac{\omega+20}{5}\right) + \text{sinc}\left(\frac{\omega-20}{5}\right)\right]$$

(b)

$$\begin{aligned}
\mathcal{F}[tG_{\frac{9\pi}{20}}(t)] &= j\frac{d(\frac{9\pi}{20}\frac{\sin\frac{9\pi}{40}\omega}{\frac{9\pi}{40}\omega})}{d\omega} = 2j\frac{d(\frac{\sin\frac{9\pi}{40}\omega}{\omega})}{d\omega} \\
&= \frac{2j}{\omega^2}\left[\frac{9\pi}{40}\omega\cos\frac{9\pi}{40}\omega - \sin\frac{9\pi}{40}\omega\right]
\end{aligned}$$

$$\begin{aligned}
S(\omega) &= \frac{1}{2\pi}(-\frac{40A}{9\pi})\frac{2j}{\omega^2}\left[\frac{9\pi}{40}\omega\cos\frac{9\pi}{40}\omega - \sin\frac{9\pi}{40}\omega\right] * \pi[\delta(\omega+20)+\delta(\omega-20)] \\
&= \frac{40Aj}{9\pi}\left[\frac{\sin\frac{9\pi}{40}(\omega+20) - \frac{9\pi}{40}(\omega+20)\cos\frac{9\pi}{40}(\omega+20)}{(\omega+20)^2}\right. \\
&\left. + \frac{\sin\frac{9\pi}{40}(\omega-20) - \frac{9\pi}{40}(\omega-20)\cos\frac{9\pi}{40}(\omega-20)}{(\omega-20)^2}\right]
\end{aligned}$$

(c)

$$s(t) = AG_2(t)\cos 200\pi t * \sum_{-\infty}^{\infty}\delta(t-4n)$$

$$\begin{aligned}
S(\omega) &= \left[\frac{1}{2\pi}2A\frac{\sin\omega}{\omega} * \pi[\delta(\omega+200\pi)+\delta(\omega-200\pi)]\right]\cdot\left[\frac{2\pi}{4}\sum_{-\infty}^{\infty}\delta(\omega-\frac{2\pi}{4}n)\right] \\
&= \frac{2\pi}{4}A\left[\frac{\sin(\omega+200\pi)}{\omega+200\pi}+\frac{\sin(\omega-200\pi)}{\omega-200\pi}\right] * \sum_{-\infty}^{\infty}\delta(\omega-\frac{2\pi}{4}n) \\
&= \frac{\pi A}{2}\sum_{-\infty}^{\infty}\left[\left(\frac{\sin(200\pi+\frac{n\pi}{2})}{200\pi+\frac{n\pi}{2}}+\frac{\sin(-200\pi+\frac{n\pi}{2})}{-200\pi+\frac{n\pi}{2}}\right)\delta(\omega-\frac{n\pi}{2})\right]
\end{aligned}$$

(d)

$$\begin{aligned}
S(\omega) &= \left[\frac{1}{2\pi}A\left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^2 * \pi[\delta(\omega+100\pi)+\delta(\omega-100\pi)]\right]\cdot\left[\frac{2\pi}{4}\sum_{-\infty}^{\infty}\delta(\omega-\frac{2\pi}{4}n)\right] \\
&= \frac{A}{2}\frac{2\pi}{4}\left[\left(\frac{\sin\frac{(\omega+100\pi)}{2}}{\frac{\omega+100\pi}{2}}\right)^2+\left(\frac{\sin\frac{(\omega-100\pi)}{2}}{\frac{\omega-100\pi}{2}}\right)^2\right]\cdot\sum_{-\infty}^{\infty}\delta(\omega-\frac{2\pi}{4}n) \\
&= \frac{\pi A}{4}\left[\sum_{-\infty}^{\infty}\left(\frac{\sin\frac{(100\pi+\frac{n\pi}{2})}{2}}{\frac{100\pi+\frac{n\pi}{2}}{2}}\right)^2\delta(\omega-\frac{n\pi}{2})+\sum_{-\infty}^{\infty}\left(\frac{\sin\frac{(-100\pi+\frac{n\pi}{2})}{2}}{\frac{-100\pi+\frac{n\pi}{2}}{2}}\right)^2\delta(\omega-\frac{n\pi}{2})\right]
\end{aligned}$$

7. If

$$s(t)\leftrightarrow S(\omega)$$

determine the Fourier transform of the following:

(a) Let $s_1(t)=ts(t)$, then according to frequency differentiation,

$$\mathcal{F}[s_1(t)]=\mathcal{F}[ts(t)]=j\frac{dS(\omega)}{d\omega}=S_1(\omega)$$

Time scaling,

$$\mathcal{F}[s_1(2t)]=\mathcal{F}[2ts(2t)]=\frac{1}{2}S_1(\frac{\omega}{2})=j\frac{dS(\frac{\omega}{2})}{d\omega}$$

Therefore,

$$\mathcal{F}[ts(2t)]=\frac{j}{2}\frac{dS(\frac{\omega}{2})}{d\omega}$$

(b)

$$\mathcal{F}[(t-2)s(t)]=\mathcal{F}[ts(t)]-\mathcal{F}[2s(t)]=j\frac{dS(\omega)}{d\omega}-2S(\omega)$$

(c)

$$\begin{aligned}\mathcal{F}[(t-2)s(-2t)] &= -\mathcal{F}[(-t)s(-2t)] - 2\mathcal{F}[s(-2t)] \\ &= \frac{j}{2}\frac{dS(-\frac{\omega}{2})}{d\omega} - S(-\frac{\omega}{2})\end{aligned}$$

(d)

$$t\frac{ds}{dt} = \frac{d[ts(t)]}{dt} - s(t)$$

therefore,

$$\begin{aligned}\mathcal{F}[t\frac{ds}{dt}] &= \mathcal{F}[\frac{d[ts(t)]}{dt}] - \mathcal{F}[s(t)] \\ &= j\omega j\frac{dS(\omega)}{d\omega} - S(\omega) = -\omega\frac{dS(\omega)}{d\omega} - S(\omega)\end{aligned}$$

(e) Let $s_1(t) = s(t+1)$

$$\mathcal{F}[s_1(t)] = \mathcal{F}[s(t+1)] = S(\omega)e^{j\omega}$$

$$\mathcal{F}[s(1-t)] = \mathcal{F}[s_1(-t)] = S(-\omega)e^{-j\omega}$$

(f) Let $s_1(t) = ts(t)$

$$\mathcal{F}[s_1(t)] = \mathcal{F}[ts(t)] = j\frac{dS(\omega)}{d\omega} = S_1(\omega)$$

$$\mathcal{F}[(1-t)s(1-t)] = \mathcal{F}[s_1(1-t)] = S_1(-\omega)e^{-j\omega} = -j\frac{dS(-\omega)}{d\omega}e^{-j\omega}$$

8. Evaluate the following convolution integrals:

(a)

$$u(t) * u(t) = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau = \int_0^t 1d\tau = tu(t)$$

(b)

$$u(t) * e^{-t}u(t) = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)u(t-\tau)d\tau = \int_0^t e^{-\tau}d\tau = (1-e^{-t})u(t)$$

Verify:

$$\mathcal{F}[u(t) * e^{-t}u(t)] = [\pi\delta(\omega) + \frac{1}{j\omega}] \cdot \frac{1}{1+j\omega} = \pi\delta(\omega) + \frac{1}{j\omega} - \frac{1}{1+j\omega} \leftrightarrow u(t) - e^{-t}u(t)$$

(c)

$$e^{-2t}u(t) * e^{-t}u(t) = \int_{-\infty}^{\infty} e^{-2\tau}u(\tau)e^{-(t-\tau)}u(t-\tau)d\tau = \int_0^t e^{-(t+\tau)}d\tau = (e^{-t}-e^{-2t})u(t)$$

Verify:

$$\mathcal{F}[e^{-2t}u(t) * e^{-t}u(t)] = [\frac{1}{2+j\omega}] \cdot \frac{1}{1+j\omega} = \frac{1}{1+j\omega} - \frac{1}{2+j\omega} \leftrightarrow e^{-t}u(t) - e^{-2t}u(t)$$

(d)

$$u(t) * tu(t) = \int_{-\infty}^{\infty} \tau u(\tau)u(t-\tau)d\tau = \int_{0}^{t} \tau d\tau = \frac{1}{2}t^2u(t)$$

Since Fourier transform of $tu(t)$ contains differential of $\delta(t)$, you can not verify this problem using present knowledge.

(e)

$$tu(t) * e^{-t}u(t) = \int_{-\infty}^{\infty} \tau u(\tau)e^{-(t-\tau)}u(t-\tau)d\tau = \int_{0}^{t} \tau e^{-(t-\tau)}d\tau = (e^{-t}+t-1)u(t)$$

Verify:

$$\begin{aligned}\mathcal{F}[tu(t) * e^{-t}u(t)] &= [\frac{1}{(j\omega)^2}] \cdot \frac{1}{1+j\omega} = \frac{1}{j\omega}[\frac{1}{j\omega} - \frac{1}{1+j\omega}] \\ &= \frac{1}{(j\omega)^2} - [\frac{1}{j\omega} - \frac{1}{1+j\omega}] \leftrightarrow (t-1+e^{-t})u(t)\end{aligned}$$

(f)

$$e^{-t} * e^{-2t}u(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)}e^{-2\tau}u(\tau)d\tau = \int_{0}^{\infty} e^{-(t+\tau)}d\tau = e^{-t}$$

Since Fourier transform of e^{-t} is not convergent, you can not verify this problem.

9. Determine the minimum sampling rate and the Nyquist interval for the following signals:

   (a) $\sin(100t)/100t$: $\omega_0 \geq 200rad/sec$

   (b) $\sin^2(100t)/(100t)^2$: $\omega_0 \geq 400rad/sec$

   (c) $\sin(100t)/100t + \sin(50t)/50t$: $\omega_0 \geq 200rad/sec$

   (d) $\sin(100t)/100t + \sin^2(60t)/(60t)^2$: $\omega_0 \geq 240rad/sec$

10. (a) $h(t) = 0$ for all $t < 0$.

$$h(t) = h_e(t) + h_o(t)$$
$$h_e(t) \leftrightarrow R(\omega) \text{ and } h_o(t) \leftrightarrow jX(\omega)$$
$$\text{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$
$$h_e(t) = h_o(t)\text{sgn}(t)$$
$$h_o(t) = h_e(t)\text{sgn}(t)$$

$$\begin{aligned}R(\omega) = \mathcal{F}[h_o(t)\text{sgn}(t)] &= \frac{1}{2\pi}\left\{\mathcal{F}[h_o(t)] * \mathcal{F}[\text{sgn}(t)]\right\} \\ &= \frac{1}{2\pi}\left\{jX(\omega) * \frac{2}{j\omega}\right\} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{X(y)}{\omega - y}dy\end{aligned}$$

similarly,

$$
\begin{aligned}
X(\omega) = -j\mathcal{F}[h_e(t)\mathrm{sgn}(t)] &= \frac{-j}{2\pi}\left\{\mathcal{F}[h_e(t)] * \mathcal{F}[\mathrm{sgn}(t)]\right\} \\
&= \frac{-j}{2\pi}\left\{R(\omega) * \frac{2}{j\omega}\right\} = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R(y)}{\omega - y}dy
\end{aligned}
$$

(b) $h(t) = 0$ for all $t > 0$.

$$
\begin{aligned}
h_e(t) &= -h_o(t)\mathrm{sgn}(t) \\
h_o(t) &= -h_e(t)\mathrm{sgn}(t)
\end{aligned}
$$

therefore,

$$
R(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{X(y)}{\omega - y}dy
$$

and

$$
X(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R(y)}{\omega - y}dy
$$