Solutions for Problems on Transmission Signals

1.

$$H(\omega) = \frac{\frac{1}{\frac{1}{R_2}+j\omega C_2}}{\frac{1}{\frac{1}{R_1}+j\omega C_1} + \frac{1}{\frac{1}{R_2}+j\omega C_2}}$$
$$= \frac{R_2 + j\omega R_1 R_2 C_1}{(R_1+R_2) + j\omega R_1 R_2 (C_1+C_2)} = K$$

so

$$K = \frac{R_2}{R_1+R_2} = \frac{C_1}{C_1+C_2}$$

2.

$$S(\omega) = \frac{2}{1+j\omega}$$

$$H(\omega) = \begin{cases} ke^{-j\omega t_0} & |\omega| \leq 1 rad/sec \\ 0 & |\omega| > 1 rad/sec \end{cases}$$

$$H(\omega) = ke^{-j\omega t_0}$$

$$|Y(\omega)|^2 = |S(\omega)|^2 |H(\omega)|^2 = \frac{4k^2}{1+\omega^2}$$

$$E_{in} = \int_{-\infty}^{\infty} s^2(t)dt = \int_0^{\infty} 4e^{-2t}dt = 2$$

$$E_{out} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-1}^{1} \frac{4k^2}{1+\omega^2}d\omega = \frac{4k^2}{\pi}\arctan 1$$

3. (a)

$$E_{out} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$
$$= \frac{2}{2\pi}\int_0^{2\pi} [-\frac{1}{2\pi}(\omega - 2\pi)]^2 d\omega$$
$$= \frac{1}{4\pi^3}\left[\frac{1}{3}\omega^3 - 2\pi\omega^2 + 4\pi^2\omega\right]|_0^{2\pi} = \frac{2}{3}$$

(b)

$$\frac{1}{4\pi^3}\left[\frac{1}{3}\omega^3 - 2\pi\omega^2 + 4\pi^2\omega\right]|_0^{\omega_1} = \frac{1}{3}$$

therefore,

$$(\omega_1 - 2\pi)^3 = 4\pi^3 \Rightarrow \omega_1 = (2 + \sqrt[3]{4})\pi$$

4.

$$
\begin{aligned}
s(t) &= \cos\omega_0 t + \frac{1}{2}\cos 2\omega_0 t + \frac{1}{4}\sin 4\omega_0 t \\
&= \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t} + \frac{1}{4}e^{j2\omega_0 t} + \frac{1}{4}e^{-j2\omega_0 t} - \frac{j}{8}e^{j4\omega_0 t} + \frac{j}{8}e^{-j4\omega_0 t}
\end{aligned}
$$

$$
\mathcal{S}_s(\omega) = 2\pi \sum_{n=\pm1,\pm2,\pm4} |S_n|^2 \delta(\omega - n\omega_0)
$$

where

$$
S_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} s(t)e^{-jn\omega_0 t}dt
$$

therefore,

$$
\begin{aligned}
\mathcal{S}_s(\omega) = 2\pi \Big[&\frac{1}{4}\delta(\omega-\omega_0) + \frac{1}{4}\delta(\omega+\omega_0) + \frac{1}{16}\delta(\omega-2\omega_0) + \frac{1}{16}\delta(\omega+2\omega_0) \\
&+\frac{1}{64}\delta(\omega-4\omega_0) + \frac{1}{64}\delta(\omega+4\omega_0)\Big]
\end{aligned}
$$

$$
E_s = 2\frac{1}{2\pi}\int_0^\infty \mathcal{S}_s(\omega)d\omega = 2\left(\frac{1}{4}+\frac{1}{16}+\frac{1}{64}\right) = \frac{31}{32}
$$

low pass RC filter:

$$
H(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1+j\omega RC} = \frac{1}{1+j\frac{\omega}{2\omega_0}}
$$

$$
\begin{aligned}
\mathcal{S}_o(\omega) =& 2\pi \sum_{n=\pm1,\pm2,\pm4} |H(n\omega_0)|^2 |S_n|^2 \delta(\omega - n\omega_0) \\
&= 2\pi\left[\frac{4}{5}\frac{1}{4}\delta(\omega-\omega_0) + \frac{4}{5}\frac{1}{4}\delta(\omega+\omega_0) + \frac{1}{2}\frac{1}{16}\delta(\omega-2\omega_0) + \frac{1}{2}\frac{1}{16}\delta(\omega+2\omega_0)\right. \\
&\left.+\frac{1}{5}\frac{1}{64}\delta(\omega-4\omega_0) + \frac{1}{5}\frac{1}{64}\delta(\omega+4\omega_0)\right] \\
&= 2\pi\left[\frac{1}{5}\delta(\omega-\omega_0) + \frac{1}{5}\delta(\omega+\omega_0) + \frac{1}{32}\delta(\omega-2\omega_0) + \frac{1}{32}\delta(\omega+2\omega_0)\right. \\
&\left.+\frac{1}{320}\delta(\omega-4\omega_0) + \frac{1}{320}\delta(\omega+4\omega_0)\right]
\end{aligned}
$$

$$
E_o = 2\frac{1}{2\pi}\int_0^\infty \mathcal{S}_o(\omega)d\omega = 2\left(\frac{1}{5}+\frac{1}{32}+\frac{1}{320}\right) = \frac{15}{64}
$$

5.

$$
\begin{aligned}
R_{v_i}(\tau) &= \int_0^\infty v_i(t)v_i(t+\tau)dt = \int_0^\infty e^{-\alpha t}e^{-\alpha(t+\tau)}dt \\
&= e^{-\alpha\tau}\int_0^\infty e^{-2\alpha t}dt = -\frac{e^{-\alpha\tau}}{2\alpha}\left[e^{-2\alpha t}\right]_0^\infty = \frac{e^{-\alpha\tau}}{2\alpha}
\end{aligned}
$$

Total normalized energy:

$$E_{v_i} = R_{v_i}(0) = \frac{1}{2\alpha}$$

Fourier transform of the input signal $v_i(t)$:

$$V_i(\omega) = \int_0^{\infty} e^{-\alpha t} e^{\jmath\omega t} dt = -\frac{1}{\alpha + j\omega}\left[e^{-(\alpha+j\omega)t}\right]_0^{\infty} = \frac{1}{\alpha + j\omega}$$

Transfer function of the filter:

$$H(\omega) = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{1 + j\omega RC}$$

therefore,energy spectral density function of the output voltage $v_o(t)$ is

$$|V_o(\omega)|^2 = |V_o(\omega)|^2 |H(\omega)|^2 = \frac{\omega^2 R^2 C^2}{1 + \omega^2 R^2 C^2} \frac{1}{\alpha^2 + \omega^2}$$

if $\alpha = \frac{1}{RC}$, then

$$|V_o(\omega)|^2 = \frac{\omega^2 \frac{1}{\alpha^2}}{1 + \omega^2 \frac{1}{\alpha^2}} \frac{1}{\alpha^2 + \omega^2} = \frac{\omega^2}{(\alpha^2 + \omega^2)^2}$$

Total normalized energy in the output is:

$$E_{v_o} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |V_o(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\omega^2}{(\alpha^2 + \omega^2)^2} d\omega$$

Let $\omega = \alpha \tan\theta$, $1 + \tan^2\theta = \sec^2\theta$, $d\omega = \alpha \sec^2\theta d\theta$ therefore,

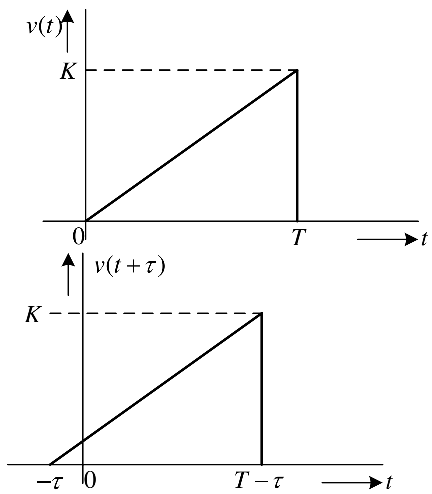
$$\begin{aligned}
E_{v_o} &= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\alpha^2 \tan^2\theta}{(\alpha^2 \sec^2\theta)^2} \alpha \sec^2\theta d\theta \\
&= \frac{1}{2\pi}\frac{1}{\alpha}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\tan^2\theta}{\sec^2\theta} d\theta = \frac{1}{2\pi\alpha}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta d\theta \\
&= \frac{1}{2\pi\alpha}\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \cos 2\theta) d\theta = \frac{1}{4\alpha}
\end{aligned}$$

$$\therefore E_{v_o} = \frac{1}{2} E_{v_i}$$

6. (a)

$$v(t) = \begin{cases} k\frac{t}{T} & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

$$v(t+\tau) = \begin{cases} k\frac{(t+\tau)}{T} & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

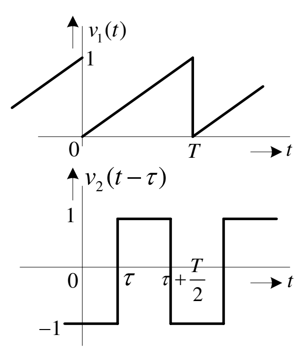


where $T \geq \tau \geq 0$

$$
\begin{aligned}
R_{v_i}(\tau) &= \int_{-\infty}^{\infty} v_i(t) v_i(t+\tau)dt = k^2 \int_0^{T-\tau} \frac{t}{T}\frac{(t+\tau)}{T}dt \\
&= \frac{k^2}{T^2}\left[\frac{t^3}{3} + \frac{1}{2}t^2\tau\right]_0^{T-\tau} \\
&= k^2\left[\frac{T}{3} - \frac{1}{2}\tau + \frac{1}{6}\frac{\tau^3}{T^2}\right] \quad \text{for } T \geq \tau \geq 0
\end{aligned}
$$

Since $R_{v_i}(\tau) = R_{v_i}(-\tau)$.

$$
R_{v_i}(\tau) = \begin{cases} k^2\left[\frac{T}{3} - \frac{1}{2}|\tau| + \frac{1}{6}\frac{|\tau|^3}{T^2}\right] & |\tau| \leq T \\ 0 & |\tau| > T \end{cases}
$$



(b) Consider $0 \leq t \leq T$ and $0 < \tau \leq \frac{T}{2}$.

$$
v_1(t) = t/T
$$

$$
v_2(t-\tau) = \begin{cases} -1 & 0 < t \leq \tau \\ 1 & \tau < t \leq \tau + T/2 \\ -1 & \tau + T/2 < t \leq T \end{cases}
$$

therefore,

$$
\begin{aligned}
R_{v_1v_2}(\tau) = R_{v_1v_2}(-\tau) &= \frac{1}{T}\int_0^T v_1(t)v_2(t-\tau)dt \\
&= \frac{1}{T}\left[\int_0^\tau \frac{t}{T}(-1)dt + \int_\tau^{\tau+T/2}\frac{t}{T}(1)dt + \int_{\tau+T/2}^T \frac{t}{T}(-1)dt\right] \\
&= \frac{1}{T}\left\{\left[-\frac{t^2}{2T}\right]|_0^\tau + \left[\frac{t^2}{2T}\right]|_\tau^{\tau+T/2} + \left[-\frac{t^2}{2T}\right]\Big|_{\tau+T/2}^T\right\} \\
&= -\frac{\tau^2}{2T^2} + \frac{(\tau+T/2)^2-\tau^2}{2T^2} + \frac{(\tau+T/2)^2-T^2}{2T^2} \\
&= \frac{2\tau T - T^2/2}{2T^2} = \frac{\tau}{T} - \frac{1}{4}
\end{aligned}
$$

If $\frac{T}{2} < \tau < T$, then

$$R_{v_1v_2}(\tau) = -\frac{\tau}{T} + \frac{1}{4}$$

7. (a)

$$
\begin{aligned}
R_v(\tau) &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} v(t)v(t+\tau)dt \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin\omega_0 t \sin\omega_0(t+\tau)dt \\
&= \frac{1}{2T}\int_{-\frac{T}{2}}^{\frac{T}{2}} -\cos\omega_0(2t+\tau) + \cos\omega_0\tau dt = \frac{\cos\omega_0\tau}{2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathcal{S}(\omega) = \mathcal{F}[R(\tau)] &= \int_{-\infty}^{\infty}\frac{\cos\omega_0\tau}{2}e^{-j\omega\tau}d\tau \\
&= \frac{1}{4}\int_{-\infty}^{\infty}\left(e^{j\omega_0\tau}+e^{-j\omega_0\tau}\right)e^{-j\omega\tau}d\tau \\
&= \frac{\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]}{2}
\end{aligned}
$$

Another way:

$$v(t) = \sin\omega_0 t = \frac{j}{2}e^{-j\omega_0 t} - \frac{j}{2}e^{j\omega_0 t}$$

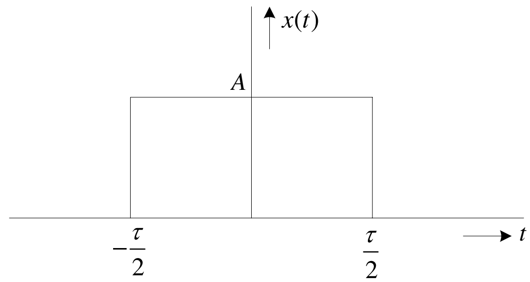
$$\mathcal{S}(\omega) = 2\pi\sum_{n=-\infty}^{\infty}|S_n|^2\delta(\omega - n\omega_0)$$

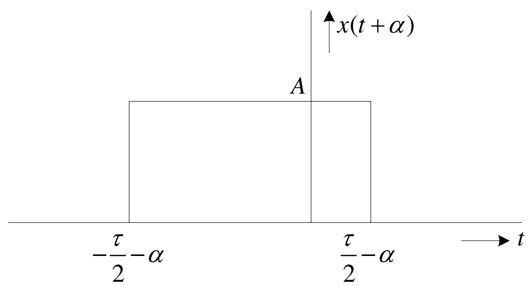
where

$$S_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} s(t)e^{-jn\omega_0 t}dt$$

therefore,

$$\begin{aligned}\mathcal{S}_s(\omega) &= 2\pi\left[\frac{1}{4}\delta(\omega-\omega_0)+\frac{1}{4}\delta(\omega+\omega_0)\right]\\ &= \frac{\pi}{2}\delta(\omega-\omega_0)+\frac{\pi}{2}\delta(\omega+\omega_0)\end{aligned}$$





8. (a) when $0 \le \alpha \le \tau$,

$$R_x(\alpha) = \int_{\infty}^{\infty} x(t)x(t+\alpha)dt = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}-\alpha} A^2 dt = A^2(\tau-\alpha)$$

when $-\tau \le \alpha < 0$,

$$R_x(\alpha) = \int_{\infty}^{\infty} x(t)x(t+\alpha)dt = \int_{-\frac{\tau}{2}-\alpha}^{\frac{\tau}{2}} A^2 dt = A^2(\tau+\alpha)$$

so

$$R_x(\alpha) == \begin{cases} A^2(\tau-|\alpha|) & \text{for } |\alpha| \le \tau \\ 0 & \text{for } |\alpha| > \tau \end{cases}$$

(b)

$$\begin{aligned}
\mathcal{S}_x(\omega) = \mathcal{F}[R_x(\alpha)] &= \int_{-\infty}^{\infty} R_x(\alpha) e^{-j\omega\alpha} d\alpha \\
&= 2\int_0^{\tau} A^2\tau(1-\alpha/\tau)\cos\omega\alpha d\alpha \\
&= -2A^2\int_0^{\tau} \alpha\cos\omega\alpha d\alpha \\
&= -2A^2\int_0^{\tau} \alpha/\omega d\sin\omega\alpha \\
&= \frac{2A^2}{\omega}\int_0^{\tau} \sin\omega\alpha d\alpha \\
&= \frac{2A^2}{\omega}\frac{1-\cos\omega\tau}{\omega} = \frac{2A^2}{\omega^2}\left(\sin\frac{\omega\tau}{2}\right)^2
\end{aligned}$$

(c)

$$\begin{aligned}
\mathcal{S}_x(\omega) &= |X(\omega)|^2 = |\mathcal{F}[x(t)]|^2 \\
&= \left|\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} Ae^{-j\omega t}dt\right|^2 = \left|\frac{Ae^{-j\omega\tau/2} - Ae^{j\omega\tau/2}}{-j\omega}\right|^2 \\
&= \frac{2A^2}{\omega^2}\left(\sin\frac{\omega\tau}{2}\right)^2
\end{aligned}$$

9. (a)

$$H(\omega) = 1/(1+j\omega RC)$$

power spectral density of the output noise

$$\mathcal{S}_{n_o}(\omega) = \mathcal{S}(\omega)|1/(1+j\omega RC)|^2 = \frac{1}{2}K\left(\frac{1}{1+\omega^2R^2C^2}\right)$$

(b)

$$\begin{aligned}
R_{n_o}(\tau) =& \mathcal{F}^{-1}[\mathcal{S}_{n_o}(\omega)] \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2}K\left(\frac{1}{1+\omega^2R^2C^2}\right)e^{j\omega t}d\omega \\
&= \frac{1}{2}K\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{R^2C^2}\left[\frac{1}{\frac{1}{R^2C^2}+\omega^2}\right]e^{j\omega t}d\omega\right\} \\
&= \frac{K}{2R^2C^2}\mathcal{F}^{-1}\left[\frac{1}{\alpha^2+\omega^2}\right]
\end{aligned}$$

where $\alpha = \frac{1}{RC}$.
To find $\mathcal{F}^{-1}\left[\frac{1}{\alpha^2+\omega^2}\right]$:
From Problem 5, if $f(t) = e^{-\alpha t}u(t)$,

then $F(\omega) = \frac{1}{\alpha+j\omega} = \frac{\alpha}{\alpha^2+\omega^2} - j\frac{\omega}{\alpha^2+\omega^2}$. But

$$f(t) = \frac{1}{2}\left[e^{-\alpha|t|} + e^{-\alpha|t|}\mathrm{sgn}(t)\right] = f_e(t) + f_o(t)$$

and

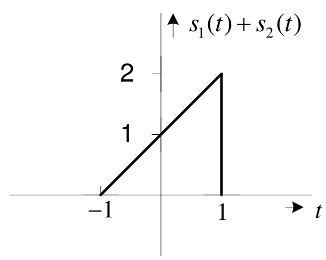
$$f_e(t) \leftrightarrow Re[F(\omega)]$$

$$f_o(t) \leftrightarrow Im[F(\omega)]$$

$$\therefore \frac{1}{2}e^{-\alpha|t|} \leftrightarrow \frac{\alpha}{\alpha^2+\omega^2}$$

$$\therefore \mathcal{F}^{-1}\left[\frac{1}{\alpha^2+\omega^2}\right] = \frac{1}{2\alpha}e^{-\alpha|t|}$$

$$\therefore R_{n_o}(\tau) = \frac{K}{2R^2C^2}\mathcal{F}^{-1}\left[\frac{1}{\alpha^2+\omega^2}\right] = \frac{K}{2R^2C^2}\left[\frac{RC}{2}e^{-|t|/RC}\right] = \frac{K}{4RC}e^{-|t|/RC}$$



10.

$$s_1(t) = 1 \qquad -1 \le t \le 1$$
$$s_2(t) = t \qquad -1 \le t \le 1$$

$$R_{s_1 s_2}(0) = \int_{-1}^{1} s_1(t)s_2(t)dt = \int_{-1}^{1} tdt = \frac{1}{2}t^2|_{-1}^{1} = 0$$

$$\int |s_1(t)+s_2(t)|^2 dt = \int_{-1}^{1} |1+t|^2 dt = \int_{-1}^{1} 1+2t+t^2 dt = 2 + \frac{2}{3} = \frac{8}{3}$$

$$\int |s_1(t)|^2 dt = \int_{-1}^{1} dt = 2$$

$$\int |s_2(t)|^2 dt = \int_{-1}^{1} t^2 dt = \frac{2}{3}$$

$$\therefore \int |s_1(t)+s_2(t)|^2 dt = \int |s_1(t)|^2 dt + \int |s_2(t)|^2 dt$$