Chapter4 Solution

1.

$$
\begin{aligned}
P &= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} Ac^2(1+2mf_m(t)+m^2f_m^2(t))\cos^2\omega_c t dt \\
&= \frac{Ac^2}{2}\lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} (1+2mf_m(t)+m^2f_m^2(t)+\cos 2\omega_c t+2mf_m(t)\cos 2\omega_c t+m^2f_m^2(t)\cos 2\omega_c t)dt \\
&= \frac{Ac^2}{2}\lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} (1+2mf_m(t)+m^2f_m^2(t))dt \\
&= \frac{Ac^2}{2}(1+m^2\overline{f_m^2(t)})
\end{aligned}
$$

Note: if $\overline{f_m^2(t)} = 0 \Rightarrow$no dc component

2. (a)

$$|f_m(t)|_{max} = K(2+1+3) = 6K \leq 1 \Rightarrow K \leq \frac{1}{6}$$

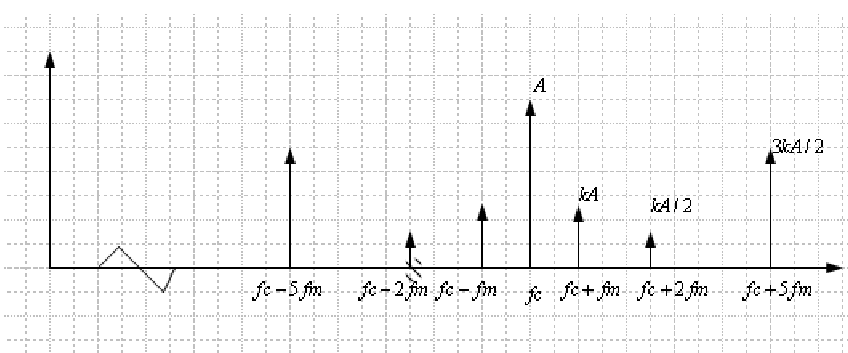


Figure 1:

(b)

$$
\begin{aligned}
Pc &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} A^2\cos^2\omega_c t dt \\
&= \frac{A^2}{2T}\left[t + \frac{\sin 2\omega_c t}{2\omega_c}\right]_{-\frac{T}{2}}^{\frac{T}{2}} \\
&= \frac{A^2}{2T}\left[(\frac{T}{2} + \frac{\sin\omega_c T}{2\omega_c}) - (-\frac{T}{2} + \frac{\sin(-\omega_c T)}{2\omega_c})\right] \\
&= \frac{A^2}{2}
\end{aligned}
$$

Check for any sinousoid wave the power $= \frac{A^2}{2}$

$$
\begin{aligned}
\because P_{SB} &= \frac{1}{2}\overline{f_m^2(t)} \\
\therefore p_{SB} &= \frac{1}{2}(\frac{1}{2}(4K^2 + K^2 + 9K^2)) = \frac{7}{2}K^2 \\
P_T = P_{SB} + Pc &= \frac{7}{2}K^2 + \frac{A^2}{2}
\end{aligned}
$$

Extent the question:

$$
\begin{aligned}
P_{SB}/Pc &= \frac{7}{2}K^2/\frac{A^2}{2} = \frac{7K^2}{A^2} \\
Pc/P_T &= \frac{A^2}{2}/\frac{7K^2 + A^2}{2} = \frac{A^2}{7K^2 + A^2}
\end{aligned}
$$

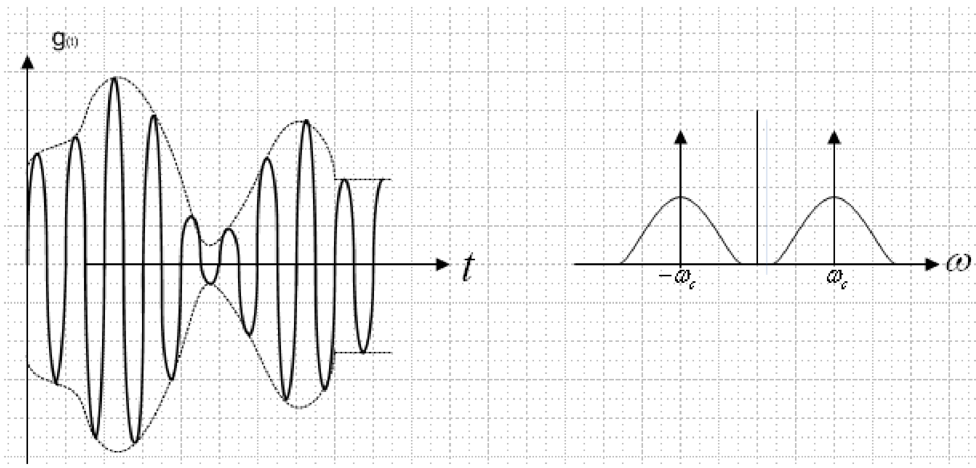
3. (a) Detection of AM signal $g(t) = [A + f_m(t)]\cos\omega_c t$

(b)

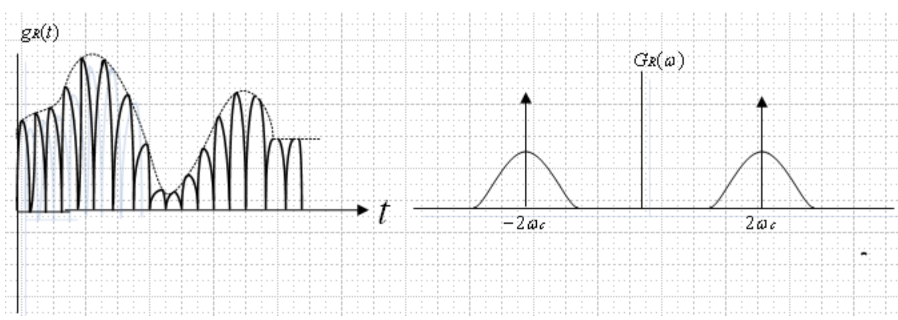
$$
\begin{aligned}
s(t) &= [A + f_m(t)]\cos\omega_c t \\
S(\omega) &= \mathcal{F}[f_m(t)\cos\omega_c t + A\cos\omega_c t] \\
&= \frac{1}{2}[F_m(\omega + \omega_c) + F_m(\omega - \omega_c)] + \pi A[\delta(\omega + \omega_c) + \delta(\omega - \omega_c)]
\end{aligned}
$$

$$
p(t) \leftrightarrow P(\omega) = 2\pi\Sigma_{n=-\infty}^{\infty} P_n\delta(\omega - n\omega_c)
$$

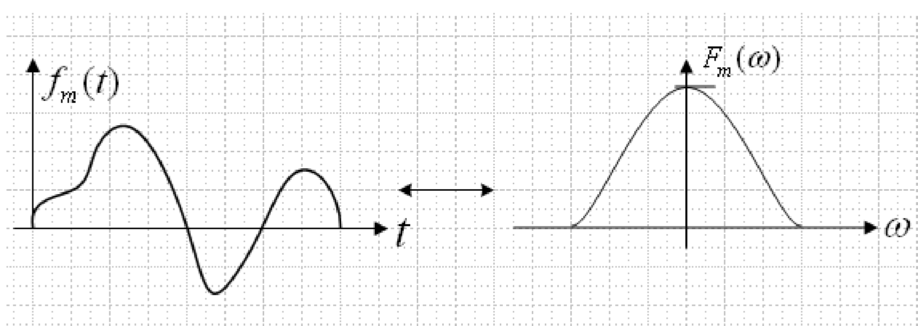
$$
Pn = \begin{cases} \left[\frac{-1^{\frac{n-1}{2}}}{n\pi}\right] & \text{for n odd} \\ \left[\frac{-1^{\frac{n+1}{2}}}{n\pi}\right] & \text{for n even} \\ \frac{1}{2} & \text{for n=0} \end{cases}
$$



(a)



(b)



(c)

Figure 2:

$$
\begin{aligned}
s(t)p(t) &\leftrightarrow \frac{1}{2\pi}[S(\omega) * P(\omega)] \\
&\leftrightarrow \frac{1}{2\pi}S(\omega) * [2\pi \sum_{n=-\infty}^{\infty} P_n\delta(\omega - n\omega_c)] \\
&\leftrightarrow \frac{1}{2}S(\omega) + \sum_{nodd} \frac{(-1)^{(\frac{n-1}{2})}}{n\pi} S(\omega - n\omega_c) + \sum_{neven} \frac{(-1)^{(\frac{n+1}{2})}}{n\pi} S(\omega - n\omega_c)
\end{aligned}
$$

After the low pass filter, we only have the low frequency part which is centered at $\omega = 0$. This low frequency part is given by $n = \pm 1$

$$
\begin{aligned}
V_0(t) &\leftrightarrow \frac{1}{\pi}S(\omega - \omega_c) + S(\omega + \omega_c) \\
&\leftrightarrow \frac{1}{\pi}[F_m(\omega) + 2A\delta(\omega)]
\end{aligned}
$$

i.e

$$
V_0(t) = \frac{1}{\pi}[f_m(t) + A]
$$

After the capacitor we filtered out the d.c component the signal becomes

$$
V_0(t) = \frac{1}{\pi}f_m(t)
$$

4. The rectified wave below

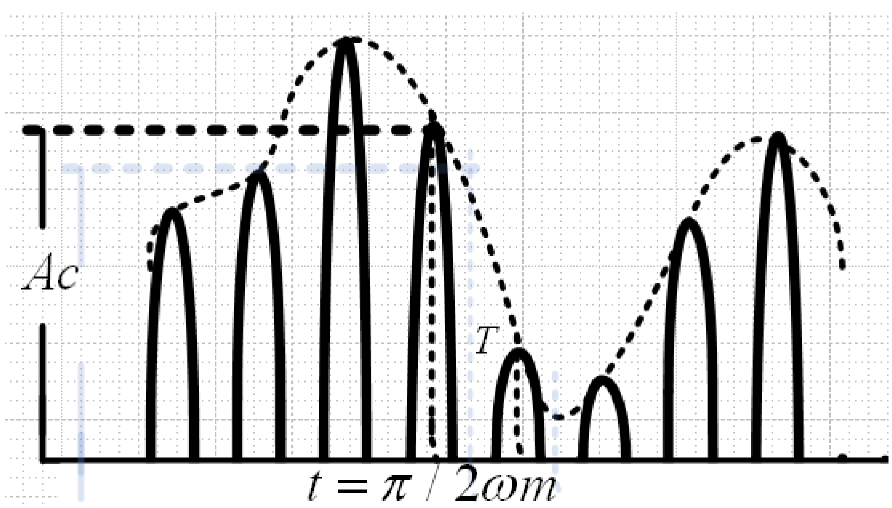


Figure 3:

Let $\tau = RC$
the fastest drop that the RC circuit has to deal with is when the modulating envelop is at frequency ω_m. The envelop is given by $e(t) = A(1 + m\cos\omega_m t)$ where m is the modulation index. At this frequency ω_m, the slop of the envelop is given by $\frac{\partial e(t)}{\partial t} = -mA\omega_m \sin\omega_m t$ The slope is steepest at $\sin\omega_m t = 1$ i.e $\omega_m t = \frac{\pi}{2}$ If the hight of

the rectified carrier pulse at this instant is Ac. Then T seconds later another rectified carrier pulse appears with amplitude a.
where

$$a = A\left[1 + m\cos\omega_m(\frac{\pi}{2\omega_m} + T)\right]$$

where

$$A_c = A(1 + m\cos\omega_m\frac{\pi}{2\omega_m}) = A$$
$$T = \frac{2\pi}{\omega_c}$$

Therefore

$$\begin{aligned} a &= A_c\left[1 + m\cos\omega_m(\frac{\pi}{2\omega_m} + T)\right] \\ &= A_c[1 - m\sin\omega_m T] \\ &= A_c[1 - \sin\frac{2\pi\omega_m}{\omega_c}] \end{aligned}$$

Inspecting the discharge voltage across the capacitor at this instant, the actual discharge curve is given by:

$$v(t) = A_c e^{\frac{-t}{\tau}}$$

and we have the following condition

$$A_c e^{\frac{-t}{\tau}} \le a = A_c[1 - \sin\frac{2\pi\omega_m}{\omega_c}]$$

Now LHS is:

$$A_c e^{\frac{-T}{\tau}} = A_c\left(1 - \frac{T}{\tau} + \frac{T^2}{2\tau^2} - \cdots\right)$$

RHS is:

$$A_c[1 - \sin\frac{2\pi\omega_m}{\omega_c}] = A_c\left[1 - \frac{2\pi\omega_m}{\omega_c} + \frac{1}{3!}(\frac{2\pi\omega_m}{\omega_c})^3 - \cdots\right]$$

Since $T \ll \tau$ and $\omega_m \ll \omega_c$, thus ignoring quadratic and higher terms. We have

$$\begin{aligned} A_c\left(1 - \frac{T}{\tau}\right) &\le A_c\left[1 - \frac{2\pi\omega_m}{\omega_c}\right] \\ \Rightarrow \frac{T}{\tau} &\ge \frac{2\pi\omega_m}{\omega_c} \end{aligned}$$

$$\because T = \frac{2\pi}{\omega_c}$$
$$\therefore \tau \le \frac{1}{\omega_m}$$

Note $\tau \gg T$, say 10 times

$$\Rightarrow \frac{1}{\omega_m} \geq 10T = 10\frac{2\pi}{\omega_c}$$
$$\Rightarrow \frac{\omega_c}{\omega_m} \geq 20\pi$$

5.
$$S(t) = f_m(t)\cos\omega_c t$$

let the output of the demodulator be $x(t)$

$$\begin{aligned} x(t) &= S(t)\cos(\omega_c t + \Delta\phi) \\ &= f_m(t)\cos\omega_c t \cdot \cos(\omega_c t + \Delta\phi) \\ &= \frac{1}{2} f_m(t)[\cos(2\omega_c t + \phi) + \cos\Delta\phi] \end{aligned}$$

After filtering the output is :

$$y(t) = \frac{1}{2} f_m(t)\cos\Delta\phi$$

6. (a) The characteristic of the 90^O−phase shift circuit is

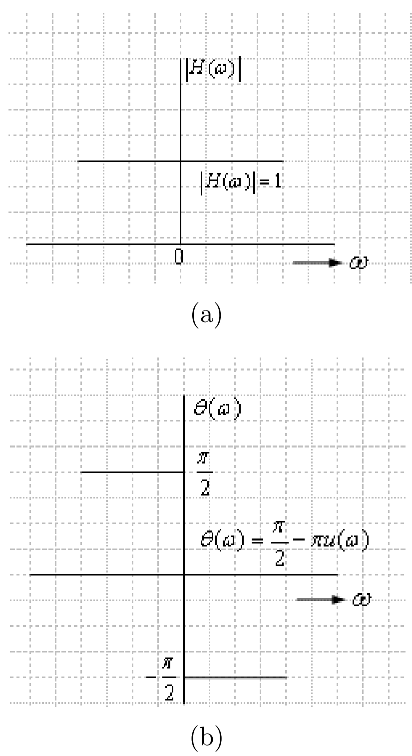


Figure 4:

$$H(\omega) = |H(\omega)|e^{\jmath\theta(\omega)}$$

Where

$$|H(\omega)| = 1$$
$$e^{\jmath\theta(\omega)} = e^{\jmath(\frac{\pi}{2} - \pi u(\omega))} = \jmath e^{-\jmath\pi u(\omega)}$$

$$v(t) \leftrightarrow V(\omega)$$
$$\hat{v}(t) \leftrightarrow \hat{V}(\omega)$$

$$\Rightarrow \hat{V}(\omega) = V(\omega)H(\omega) = \jmath V(\omega)e^{-\jmath\pi u(\omega)}$$

Where

$$u(\omega) = \begin{cases} 1 & f \geq 0 \\ 0 & f < 0 \end{cases}$$

(b) Hilbert transform is defined as

$$v_h(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{v(\tau)}{t-\tau} d\tau = \frac{1}{\pi}\left[v(t) * \frac{1}{t}\right]$$

Now

$$\frac{\jmath}{\pi t} \leftrightarrow \text{sgn}(\omega)$$
$$\frac{1}{\pi t} \leftrightarrow -\jmath\text{sgn}(\omega)$$
$$v_h(t) \leftrightarrow V(\omega)(-\jmath\text{sgn}(\omega))$$

$$\begin{aligned} V(\omega)(-\jmath\text{sgn}(\omega)) &= -\jmath V(\omega)\text{sgn}(\omega) \\ &= \jmath V(\omega)e^{-\jmath\pi u(\omega)} \\ &= \hat{V}(\omega) \end{aligned}$$

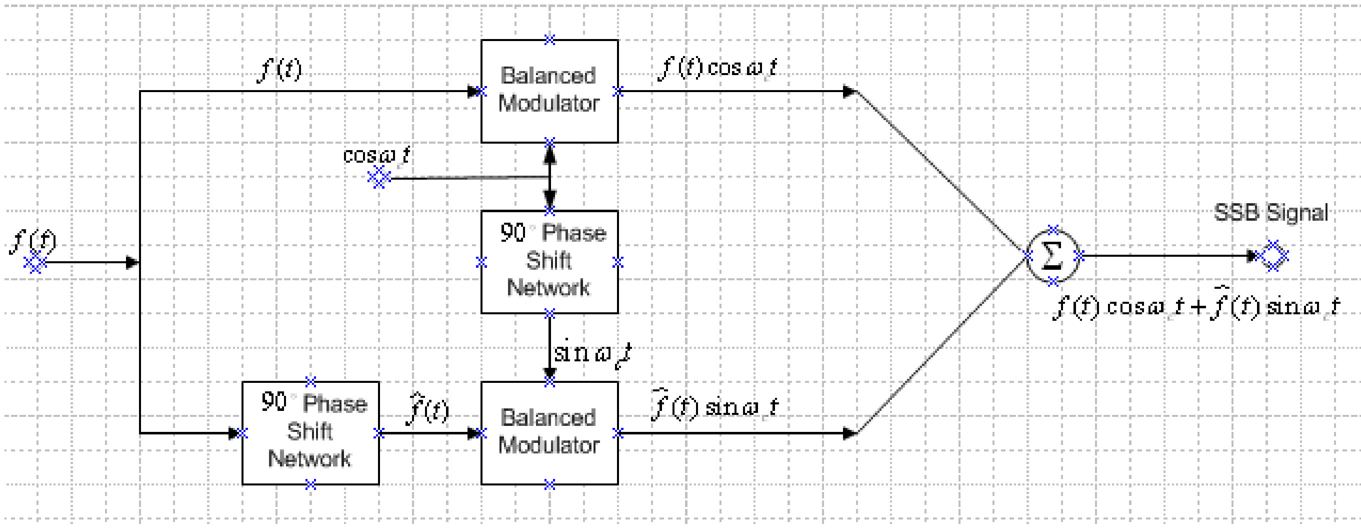
$$\therefore v_h(t) = \hat{v}(t)$$

Which means a 90^o phase shifter is known as Helbert transform

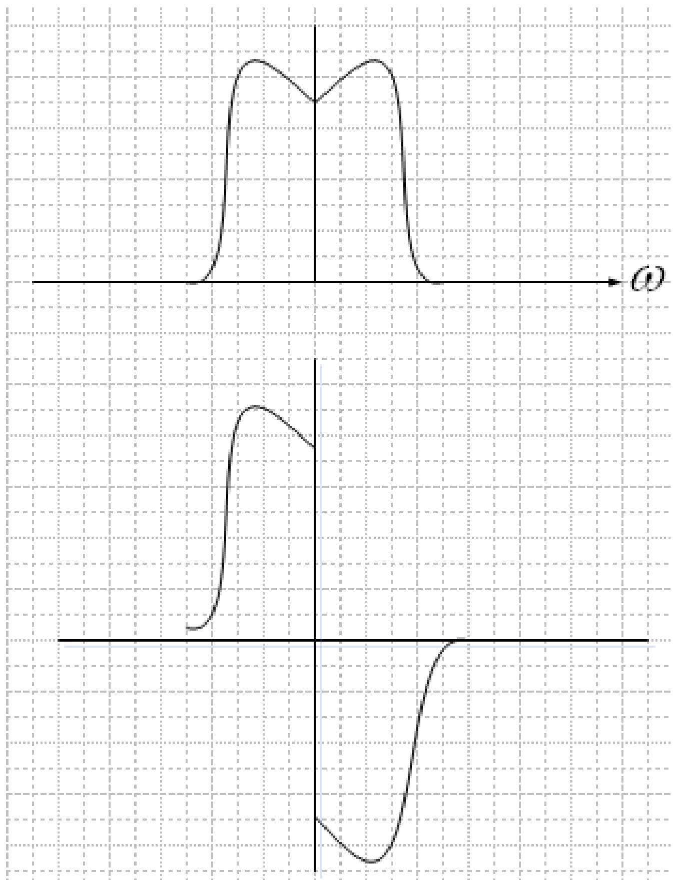
(c) SSB Generation by phase-shift method
For example:

$$f(t) = \cos\omega_s t$$
$$\hat{f}(t) = \sin\omega_s t$$

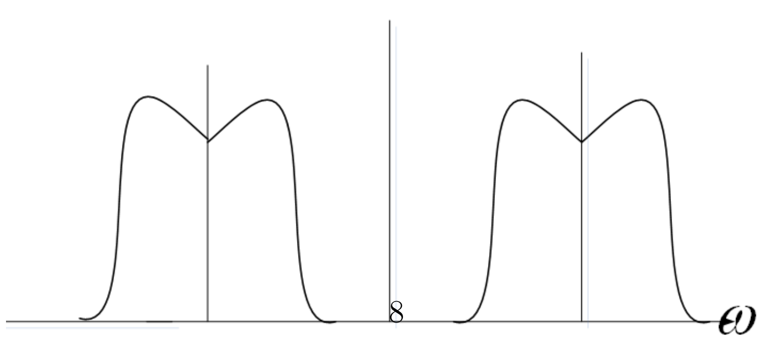
$$\begin{aligned} S_{SSB}(t) &= \cos\omega_s t\cos\omega_c t + \sin\omega_s t\sin\omega_c t \\ &= \cos(\omega_c - \omega_s)t \end{aligned}$$


$f(t)$
$f(t)$
Balanced Modulator
$f(t)\cos\omega_c t$
$\cos\omega_c t$
90° Phase Shift Network
SSB Signal
$\sin\omega_c t$
90° Phase Shift Network
$\hat{f}(t)$
Balanced Modulator
$\hat{f}(t)\sin\omega_c t$
$f(t)\cos\omega_c t+\hat{f}(t)\sin\omega_c t$

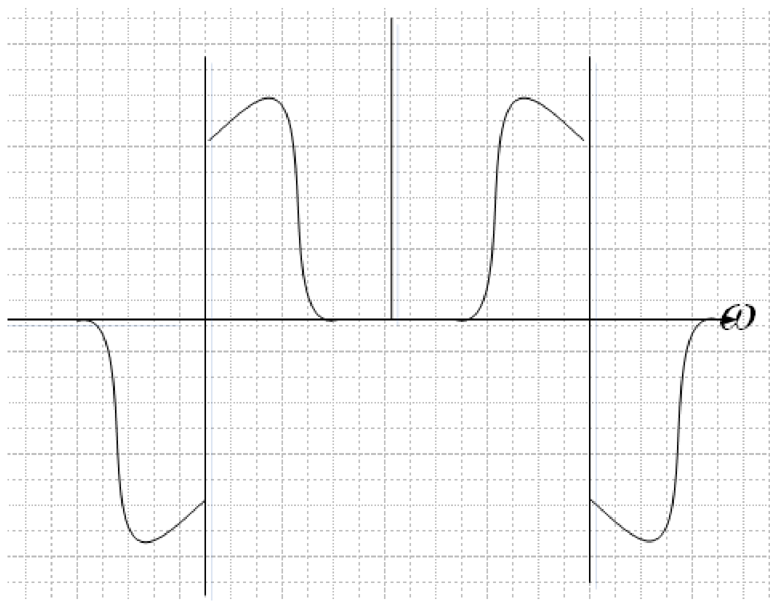
(a)


ω

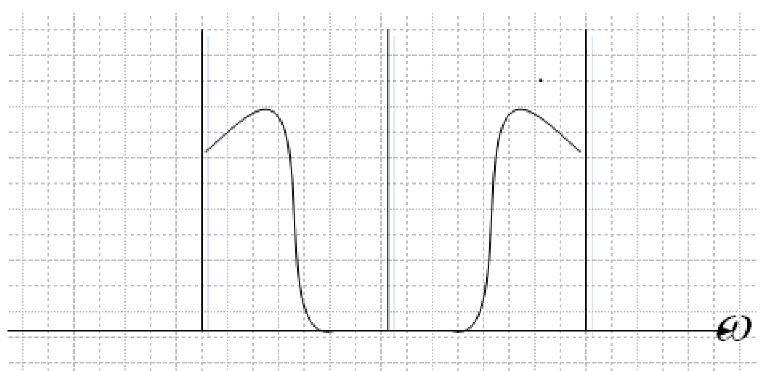
(b)


ω

(c)



(d)



(e)

Figure 5:

7.

$$v(t) = \cos\omega_m t$$
$$\hat{v}(t) = \sin\omega_m t$$

The output signal is:

$$\begin{aligned}
& v(t)\cos\omega_c t + \hat{v}(t)\sin(\omega_c t + \alpha) \\
=& \cos\omega_m t\cos\omega_c t + \sin\omega_m t\sin(\omega_c t + \alpha) \\
=& \frac{1}{2}[\cos(\omega_m + \omega_c)t + \cos(\omega_m - \omega_c)t] + \frac{1}{2}[\cos[(\omega_m + \omega_c)t + \alpha] - \cos[(\omega_m - \omega_c)t - \alpha]] \\
=& \frac{1}{2}[\cos(\omega_m + \omega_c)t + \cos[(\omega_m + \omega_c)t + \alpha] + \cos(\omega_m - \omega_c)t - \cos[(\omega_m - \omega_c)t - \alpha]]
\end{aligned}$$

Apparently it is not a SSB

8.

$$v(t) = \cos\omega_m t$$
$$\hat{v}(t) = \sin(\omega_m t + \alpha)$$

The output signal is:

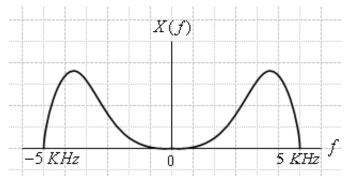
$$\begin{aligned}
& v(t)\cos\omega_c t + \hat{v}(t)\sin\omega_c t \\
=& \cos\omega_m t\cos\omega_c t + \sin(\omega_m t + \alpha)\sin\omega_c t \\
=& \frac{1}{2}[\cos(\omega_m + \omega_c)t + \cos(\omega_m - \omega_c)t] + \frac{1}{2}[\cos[(\omega_m + \omega_c)t + \alpha] - \cos[(\omega_m - \omega_c)t + \alpha]] \\
=& \frac{1}{2}[\cos(\omega_m + \omega_c)t + \cos[(\omega_m + \omega_c)t + \alpha] + \cos(\omega_m - \omega_c)t - \cos[(\omega_m - \omega_c)t + \alpha]]
\end{aligned}$$

The output signal is not a SSB signal

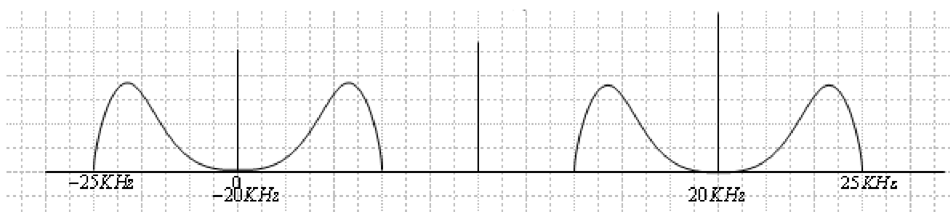
9. Refer to figure 6
Note that high and low frequency in $X(f)$ have been interchanged. So signal is unintelligible. Passing through a second scrambler will again interchange high and low frequency. Thereby restoring original signal.

10. (a)

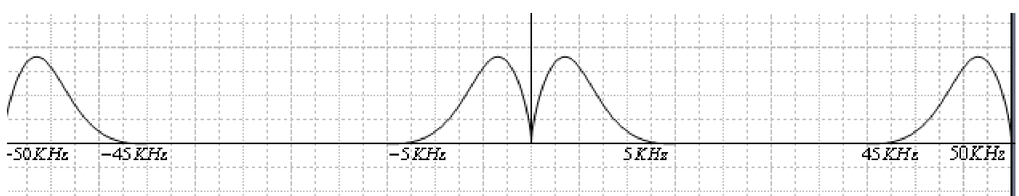
$$\begin{aligned}
x_b &= x_1\cos 2\pi f_1 t + x_2\cos 2\pi(f_1 + 2W)t \\
x_b^2 &= \frac{1}{2}x_1^2 + \frac{1}{2}x_1^2\cos 2\pi 2f_1 t \\
&+ x_1x_2\cos 2\pi(2f_1 + 2W)t + x_1x_2\cos 2\pi 2Wt \\
&+ \frac{1}{2}x_2^2 + \frac{1}{2}x_2^2\cos 2\pi(2f_1 + 4W)t
\end{aligned}$$



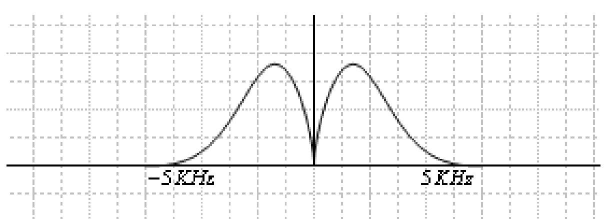
(a)



(b)



(c)



(d)

Figure 6:

The frequency range for each component in x_b^2 is:

$$\begin{aligned}
&\frac{1}{2}x_1^2 \longrightarrow \pm 2W \\
&\frac{1}{2}x_1^2 \cos 2\pi 2 f_1 t \longrightarrow 2f_1 \pm 2W \\
&x_1 x_2 \cos 2\pi(2f_1 + 2W)t \longrightarrow 2f_1 + 2W \pm 2W \\
&x_1 x_2 \cos 2\pi 2Wt \longrightarrow 2W \pm 2W \\
&\frac{1}{2}x_2^2 \longrightarrow \pm 2W \\
&\frac{1}{2}x_2^2 \cos 2\pi(2f_1 + 4W)t \longrightarrow 2f_1 + 4W \pm 2W
\end{aligned}$$

$$\begin{aligned}
x_b^3 = {} & \frac{3}{4}x_1^3 \cos 2\pi f_1 t + \frac{3}{4}x_1^3 \cos 2\pi 3 f_1 t + \frac{3}{2}x_1^2 x_2 \cos 2\pi(f_1 + W)t \\
& + \frac{3}{4}x_1^2 x_2 \cos 2\pi(3f_1 + 2W)t + \frac{3}{4}x_1^2 x_2 \cos 2\pi(f_1 - 2W)t \\
& + \frac{3}{4}x_2^3 \cos 2\pi(f_1 + 2W)t + \frac{1}{4}x_2^3 \cos 2\pi(3f_1 + 6W)t \\
& + \frac{3}{2}x_1 x_2^2 \cos 2\pi f_1 t + \frac{3}{4}x_1 x_2^2 \cos 2\pi(3f_1 + 2W)t \\
& + \frac{3}{4}x_1 x_2^2 \cos 2\pi(f_1 + 4W)t
\end{aligned}$$

The frequency range for each component in x_b^3 are:

$$\begin{aligned}
&\frac{3}{4}x_1^3 \cos 2\pi f_1 t \longrightarrow f_1 \pm 3W \\
&\frac{3}{4}x_1^3 \cos 2\pi 3 f_1 t \longrightarrow 3f_1 \pm 3W \\
&\frac{3}{2}x_1^2 x_2 \cos 2\pi(f_1 + W)t \longrightarrow f_1 + W \pm 3W \\
&\frac{3}{4}x_1^2 x_2 \cos 2\pi(3f_1 + 2W)t \longrightarrow 3f_1 + 2W \pm 3W \\
&\frac{3}{4}x_1^2 x_2 \cos 2\pi(f_1 - 2W)t \longrightarrow f_1 - 2W \pm 3W \\
&\frac{3}{4}x_2^3 \cos 2\pi(f_1 + 2W)t \longrightarrow f_1 + 2W \pm 3W \\
&\frac{1}{4}x_2^3 \cos 2\pi(3f_1 + 6W)t \longrightarrow 3f_1 + 6W \pm 3W \\
&\frac{3}{2}x_1 x_2^2 \cos 2\pi f_1 t \longrightarrow f_1 \pm 3W \\
&\frac{3}{4}x_1 x_2^2 \cos 2\pi(3f_1 + 2W)t \longrightarrow 3f_1 + 2W \pm 3W \\
&\frac{3}{4}x_1 x_2^2 \cos 2\pi(f_1 + 4W)t \longrightarrow f_1 + 4W \pm 3W
\end{aligned}$$

All terms in x_b^2 are out of band, but $f_1 \pm 3W$ include portions of

$a3[\frac{3}{4}x_1^3 \cos 2\pi f_1 t + \frac{3}{2}x_1^2 x_2 \cos 2\pi(f_1 + W)t + \frac{3}{4}x_1^2 x_2 \cos 2\pi(f_1 - 2W)t + \frac{3}{4}x_2^3 \cos 2\pi(f_1 + 2W)t + \frac{3}{2}x_1 x_2^2 \cos 2\pi f_1 t]$

The first and forth terms are 3rd harmonic distortions, the remain terms are all nonintelligible cross talk. Similar for $f_1 + 2W \pm W$, thus all distortions and cross talk are proportional to a3.

(b) Take $m = 1$,

$$x_b = (1 + x_1) \cos 2\pi f_1 t + (1 + x_2) \cos 2\pi(f_1 + 2W)t$$

Replacing x_1 by $1 + x_1$ and replacing x_2 by $1 + x_2$ in above results yields all of the above terms plus portions of the following over $f_1 \pm W$

$a3[\frac{3}{4}(1 + 3x_1 + 3x_1^2) \cos 2\pi f_1 t + \frac{3}{2}(1 + 2x_1 + x_1^2 + x_2 + 2x_1 x_2) \cos 2\pi(f_1 + W)t + \frac{3}{4}(1 + 2x_1 + x_1^2 + 2x_1 x_2) \cos 2\pi(f_1 - 2W)t + \frac{3}{4}(1 + 3x_2 + 3x_2^2) \cos 2\pi(f_1 + 2W)t + \frac{3}{2}(1 + 2x_2 + x_1 + x_2^2 + 2x_1 x_2) \cos 2\pi f_1 t]$

Similar for $f_1 + 2W \pm W$, there are 2nd and 3rd harmonic distortions and nonintelligible and intelligible cross talk all proportional to a3