EE3TR4 Supplementary Mid-Term Test 14 March, 2011

1 a) $\mathcal{F}[e^{j\omega_0 t}] = \lim_{\tau\to\infty} \mathcal{F}[e^{j\omega_0 t} \cdot u(t+\frac{\tau}{2}) \cdot u(-t+\tau/2)]$

$$= \lim_{\tau\to\infty} \int_{-\tau/2}^{\tau/2} e^{j\omega_0 t} e^{-j\omega t}\, dt$$

$$= \lim_{\tau\to\infty} \left[ \frac{e^{-j(\omega-\omega_0)t}}{-j(\omega-\omega_0)} \right]_{-\tau/2}^{\tau/2}$$

$$= \lim_{\tau\to\infty} \frac{\tau}{2} \left\{ \frac{e^{j(\omega-\omega_0)\tau/2} - e^{-j(\omega-\omega_0)\tau/2}}{j(\omega-\omega_0)\tau/2} \right\}$$

$$= \lim_{\tau\to\infty} 2\pi \left[ \frac{(\tau/2)}{\pi} \frac{\sin\frac{\tau}{2}(\omega-\omega_0)}{\frac{\tau}{2}(\omega-\omega_0)} \right] = 2\pi\, \delta(\omega-\omega_0)$$

Since ~~For~~ $\lim_{k\to\infty} \frac{k}{\pi} \frac{\sin kx}{kx} = \delta(x)$

b) $\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega_s t}$ ; $\omega_s = 2\pi/T$

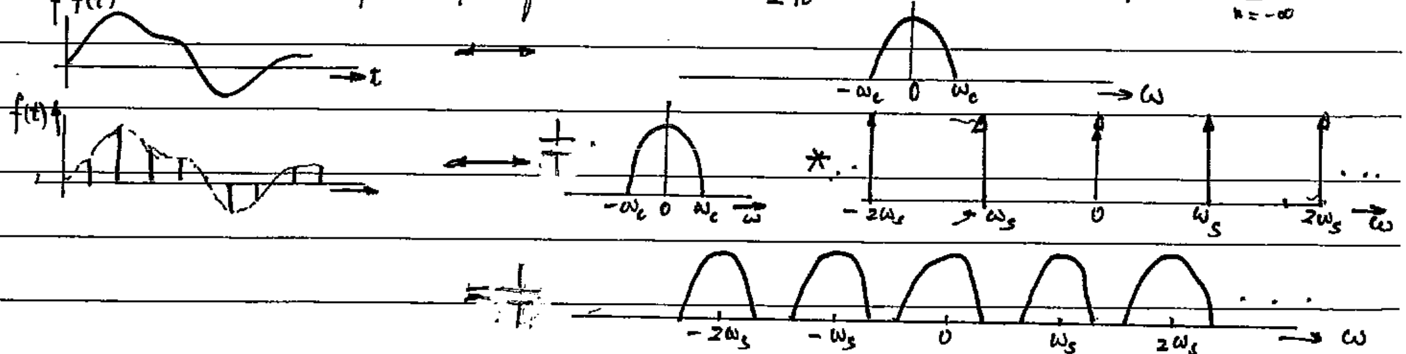
$$\alpha_n = \frac{1}{T} \sum_n \int_{-T/2}^{T/2} \delta(t-nT)\, e^{-jn\omega_s t}\, dt = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t)\, dt = \frac{1}{T}$$

$$\therefore\ \delta_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{jn\omega_s t}$$

$$\therefore\ \mathcal{F}[\delta(t)] = \frac{1}{T} \sum_{n=-\infty}^{\infty} \mathcal{F}[e^{jn\omega_s t}] = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

c) sampling in time: $f(t) \cdot \delta_T(t)$ ; $|F(\omega)| = 0 \quad |\omega| \geq \omega_c$

Fourier transform of sampling in time $\equiv \frac{1}{2\pi} \cdot F(\omega) * \mathcal{F}[\delta(t)] = \frac{1}{T} F(\omega) * \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$



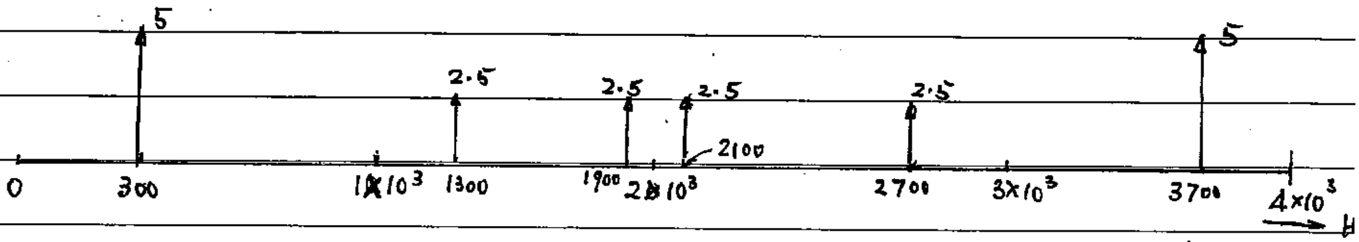

Spectrum of sampled signal $= \frac{1}{T} \sum_{n=-\infty}^{\infty} F(\omega - n\omega_s)$, $\therefore\ \omega_s \geq 2\omega_c$ without aliasing error.

d) $x(t) = 5\cos 600\pi t\,[1 + \cos 3200\pi t]$

$$= 5\cos 600\pi t + 5\left\{\frac{1}{2}\left[\cos 3800\pi t + \cos 2600\pi t\right]\right\}$$

$$= 5\cos 2\pi \times 300 t + 2.5 \cos 2\pi \times 1900 t + 2.5 \cos 2\pi\, 1300 t$$

sampling frequency $f_s = 4000\ Hz$



2 a) Energy signal $x(t)$

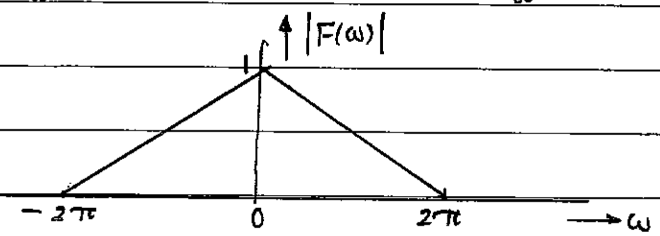
$$\therefore \text{normalized Energy} = \int_{-\infty}^{\infty} x^2(t)\, dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt \; d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, X(-\omega)\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

b)




(i) Normalized energy $E = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} |F(\omega)|^2 d\omega = \frac{2}{2\pi} \int_{0}^{2\pi} |F(\omega)|^2 d\omega$

Now, $|F(\omega)| = \frac{-1}{2\pi} \left[ \omega + 2\pi \right] \qquad 0 \le \omega \le 2\pi$

$$\therefore |F(\omega)|^2 = \frac{1}{4\pi^2} (\omega - 2\pi)^2 = \frac{1}{4\pi^2} (\omega^2 - 2\pi\omega + 4\pi^2)$$

$$\therefore \quad E = \frac{2}{2\pi} \int_0^{2\pi} \frac{1}{4\pi^2} (\omega^2 - 4\pi\omega + 4\pi^2)\, d\omega$$

$$= \frac{1}{4\pi^3} \left[ \frac{\omega^3}{3} - \frac{4\pi\omega^2}{2} + 4\pi^2\omega \right]_0^{2\pi}$$

$$\cancel{= \frac{1}{4\pi^3} \left[ \frac{8\pi^3}{3} - \frac{16\pi^3}{2} + 8\pi^3 \right]} = \frac{1}{4\pi^3} \left[ \frac{8\pi^3}{3} - 8\pi^3 + 8\pi^3 \right] = \frac{2}{3}$$

$$\cancel{= \frac{8\pi^3}{4\pi^3} \left[ \frac{1}{3} - \frac{1}{2} + 1 \right]}$$

(ii) Let $\omega_1$ be the freq. for which half the energy is contained in $|F(\omega)|$, i.e.,

$$\frac{1}{4\pi^3} \left[ \frac{\omega^3}{3} - \frac{4\pi\omega^2}{2} + 4\pi^2\omega \right]_0^{\omega_1} = \frac{1}{3},$$

i.e., $\cancel{\omega_1^3 - 3 \times 2\pi\omega_1^2 + 3 \times 4\pi^2\omega_1 - 4\pi^3 = 0}$;

$$\omega_1^3 - 3 \times 2\pi\omega_1^2 + 3 \times 4\pi^2\omega_1 - 8\pi^3 = 4\pi^3 \implies (\omega_1 - 2\pi)^3 = 4\pi^3$$

$$\omega_1 - 2\pi = \pi\sqrt[3]{4}, \quad \underline{\omega_1 = \pm\pi(2 + \sqrt[3]{4})}$$

3 a) Carrier power $P_c = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} A\cos\omega_c t\,dt = \frac{A^2}{2}$ $\qquad T_0 = 2\pi/\omega_c$

Sideband power $P_s = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} f_m^2(t)\cos^2\omega_c t\,dt$

av. to zero

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T/2}^{T/2} f_m^2(t)\left[1+\cos 2\omega_c t\right]dt$$

$$= \frac{1}{2}\overline{f_m^2(t)}$$

Total Power $P_T = P_s + P_c = \frac{1}{2}\left[\overline{f_m^2(t)} + A^2\right]$

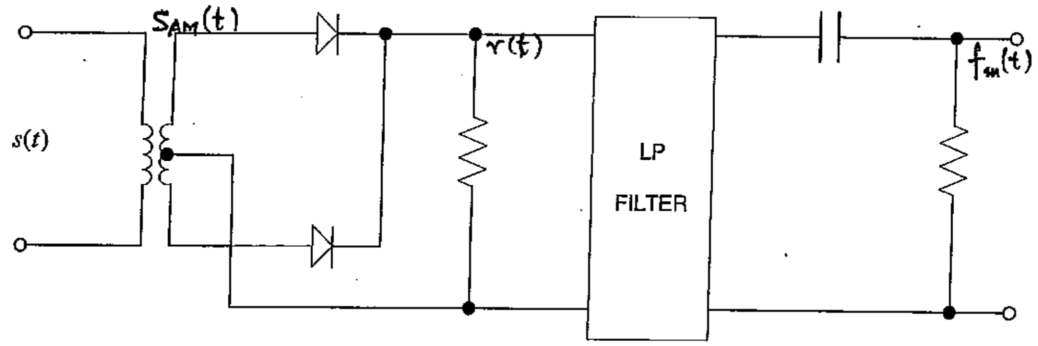
$$\eta = \frac{P_s}{P_T}\times 100\% = \frac{\overline{f_m^2(t)}/2}{\left(\frac{A^2}{2}+\frac{\overline{f_m^2(t)}}{2}\right)}\times 100\%$$

For $f_m(t) = mA\cos\omega_s t$, $P_s = \frac{1}{2}\overline{f_m^2(t)} = \frac{1}{2}\frac{(mA)^2}{2}$

$$\therefore \eta = \frac{(mA)^2/4}{\frac{A^2}{2}+\frac{(mA)^2}{4}} = \frac{m^2}{2+m^2}\times 100\%$$

$$\eta_{max}\Big|_{m=1} = \frac{1}{3}\times 100\% = \underline{33.33\%}$$

b) The full-wave rectifier




Let the rectified signal be $r(t)$. The signals $s_{AM}(t)$, $r(t)$ and $f_m(t)$ are plotted in Figure 5 a), b), c) respectively

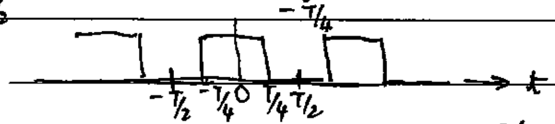
d) $p(t)$ is a periodic function

$$\therefore \; p(t) = \sum_n \alpha_n e^{-jn\omega_e t} \qquad \omega_e = 2\pi/T$$

$$\alpha_n = \frac{1}{T}\int_{-T/2}^{T/2} p(t)\, e^{-jn\omega_e t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{-T/4} -e^{-jn\omega_e t}\, dt + \frac{1}{T}\int_{-T/4}^{T/4} e^{-jn\omega_e t}\, dt + \frac{1}{T}\int_{T/4}^{T/2} -e^{-jn\omega_e t}\, dt \quad \text{(Too long!)}$$

Now, $p_1(t)$




$$p_1(t) = \sum_n a_n e^{-jn\omega_e t}, \quad a_n = \frac{1}{T}\int_{-T/4}^{T/4} e^{-jn\omega_e t}\, dt$$

$$\therefore \; a_n = \frac{1}{T}\left.\frac{e^{-jn\omega_e t}}{-jn\omega_e}\right|_{-T/4}^{T/4} = \frac{1}{T}\left[\frac{e^{jn\omega_e T/4} - e^{-jn\omega_e T/4}}{jn\omega_e}\right] = \frac{1}{2}\left(\frac{\sin \frac{n\omega_e T}{4}}{\frac{n\omega_e T}{4}}\right)$$

$$a_n = \begin{cases} \frac{1}{2}, & n = 0 \\ 0 & n \text{ even} \\ (-1)^{\frac{n-1}{2}}/n\pi & n \text{ odd} \end{cases}$$

Now, $p(t) = 2\left(p_1(t) - \frac{1}{2}\right)$

$$\therefore \; \alpha_n = \begin{cases} 0 & n = 0 \\ 0 & n \text{ even} \\ \frac{2(-1)^{n-1/2}}{n\pi} & n \text{ odd} \end{cases}$$

Since $e^{-jn\omega_o t} \leftrightarrow 2\pi\,\delta(\omega - n\omega_o)$

$$p(t) \leftrightarrow P(\omega) = \sum_n 2\pi \alpha_n\, \delta(\omega - n\omega_e)$$

Hence $\quad s_{AM}(t) \cdot p(t) \leftrightarrow \frac{1}{2\pi}\left[S_{AM}(\omega) * P(\omega)\right]$

$$= \sum_n S_{AM}(\omega - n\omega_e)$$

$$S_{AM}(\omega) = F(\omega + \omega_c) + A\delta(\omega + \omega_c) + F(\omega - \omega_c) + A\delta(\omega - \omega_c)$$

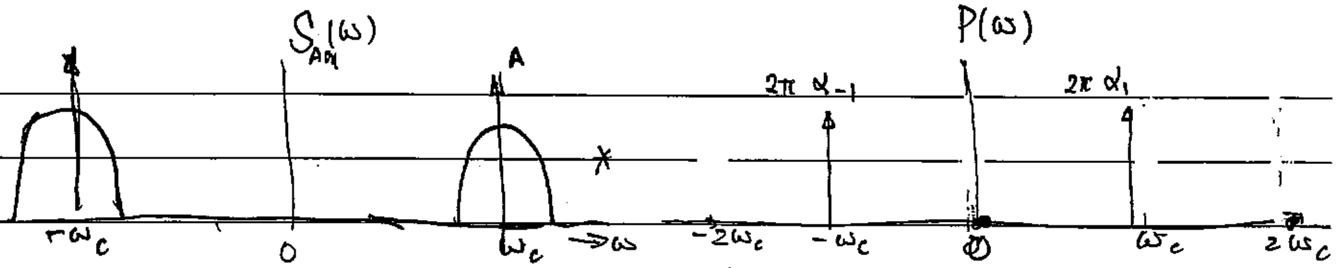
$$\therefore \; s_{AM}(t) \cdot p(t) \leftrightarrow \sum_n \alpha_n \left[A\,\delta(\omega \pm (n \pm 1)\omega_c) + F(\omega \pm (n \pm 1)\omega_c)\right]$$

After low-pass filter, we select the original signal + carrier, i.e, $n = \pm 1$

For $n = -1$, ~~and~~ ~~an~~ we have $\left.\begin{array}{l} \alpha_{-1}\left[A\,\delta(\omega) + F(\omega)\right] \\ \alpha_1\left[A\,\delta(\omega) + F(\omega)\right] \end{array}\right\} = (\alpha_1 + \alpha_1)\left[A\delta(\omega) + F(\omega)\right]$

$n = +1$

d)

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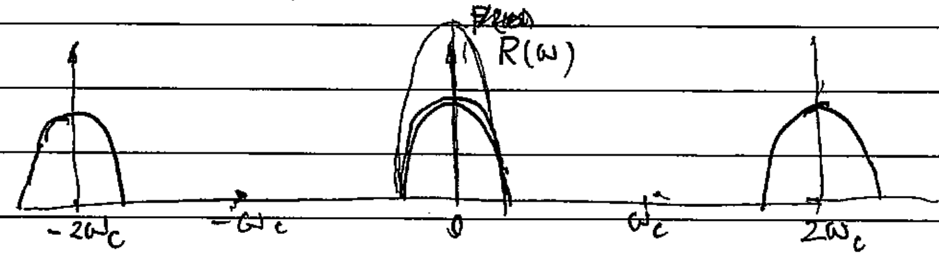


After L-P filtering and d-c block