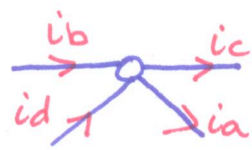
NODE VOLTAGE ANALYSIS

Kirchoff's Current Law

"The algebraic sum of currents into a node is zero"



$$i_b + i_d - i_a - i_c = 0$$

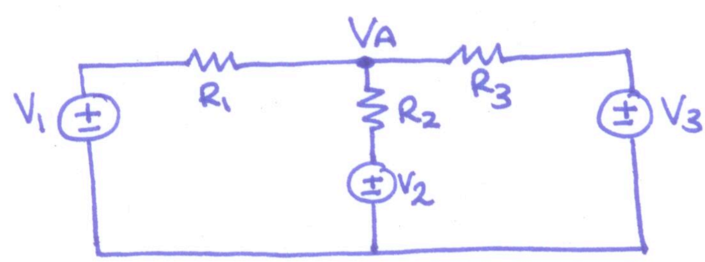
or

$$i_b + i_d = i_a + i_c$$

Why does it hold?

because a perfect conductor cannot store charge

Previous examples: simple circuits, e.g.,



Only one KCL equation needed (that at node A)

What if the circuit contains more nodes?

3, 10, 10,000 ?

We need a structured procedure that does not rely on the engineer's insight

Node Voltage Analysis: systematic application of KCL
Mesh Current Analysis: systematic application of KVL

TERMINOLOGY

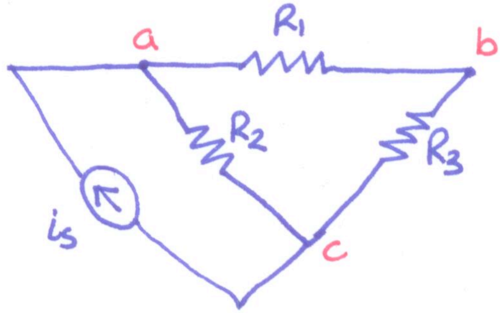
NODE: where circuit elements are joined together

NODE VOLTAGE: potential difference between the node and the node that has been designated as the reference node

* If a node is grounded it is usually chosen as the reference node

* Otherwise, we often choose the "bottom" node as the reference node

Example.



Three nodes: a, b, c

Often node c would be chosen as the reference node

NODE VOLTAGE ANALYSIS: Find the voltage at all nodes in the circuit.

By definition, the voltage at the reference node is zero.

Hence the node voltages are sometimes referred to as relative voltages

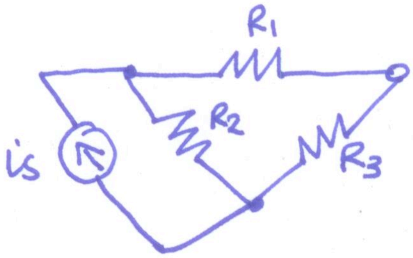
NODE VOLTAGE ANALYSIS: RECIPES

We want a structured/systematic procedure

We will start with the simplest case

NODE VOLTAGE ANALYSIS WITH INDEPENDENT CURRENT SOURCES

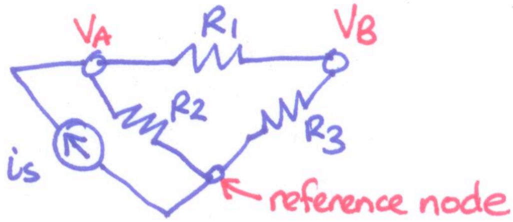
Consider a circuit with no voltage sources and no dependent current source. eg.,



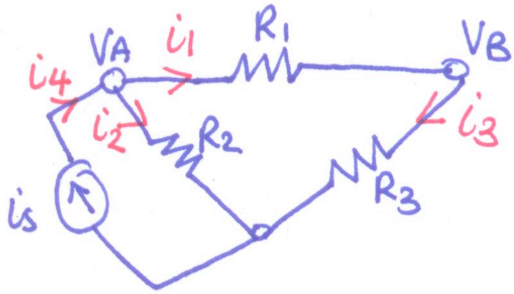
STEPS

① Pick a reference node

② Give the voltage at each node a unique label



③ Give current in each branch a label.
The direction can be chosen arbitrarily

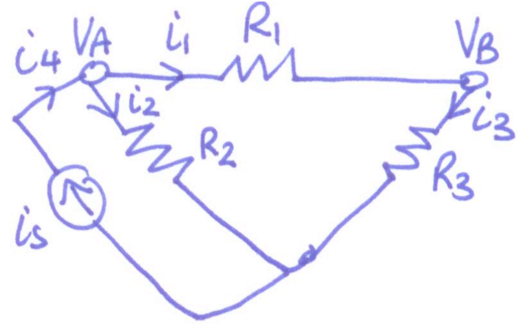


④ Write the KCL equation for each node of the circuit, except the reference node, in terms of the current variables

KCL at node A: $i_4 = i_1 + i_2$

KCL at node B: $i_1 = i_3$

(5) For branches that consist of resistors, use Ohm's Law to relate branch currents to Node voltages.
At this stage it is critical that the way that you measure voltage is chosen in accordance with the direction that you chose for current, and the passive Sign convention



$$i_1 = \frac{V_A - V_B}{R_1}$$

$$i_2 = \frac{V_A - 0}{R_2}$$

$$i_3 = \frac{V_B - 0}{R_3}$$

(6) For branches that consist of current sources, equate the label for that branch current to the source current

$$i_4 = i_s$$

(7) Make sure that the number of linear independent equations is equal to the number of unknowns

Unknowns: V_A, V_B
i_1, i_2, i_3, i_4

KCLs: $i_4 = i_1 + i_2$
$i_1 = i_3$

ohm: $i_1 = \frac{V_A - V_B}{R_1}$

$i_2 = \frac{V_A}{R_2}$

(8) Solve the linear system

$i_3 = \frac{V_B}{R_3}$

Source $i_4 = i_s$

Completing example

Substituting Ohm's Law and source equations into KCLs,

$$i_S = \frac{V_A - V_B}{R_1} + \frac{V_A}{R_2} \qquad (A)$$

$$\frac{V_A - V_B}{R_1} = \frac{V_B}{R_3} \qquad (B)$$

Two linearly independent linear equations in two unknowns.

This case is simple enough to solve by hand.

$$(B) \Rightarrow \quad V_B = \frac{R_3}{R_1 + R_3} V_A \qquad (C)$$

Substitute into (A)

$$\Rightarrow \quad i_S = V_A \left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{R_3}{R_1+R_3} \right)$$

$$\Rightarrow \quad V_A = \frac{i_S}{\frac{1}{R_1} + \frac{1}{R_2} - \frac{R_3}{R_1+R_3}}$$

Substitute back into (C)

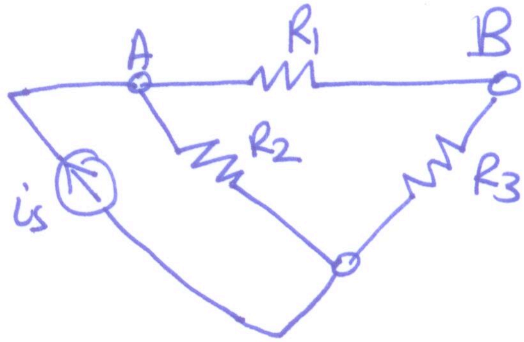
$$\Rightarrow V_B = \frac{i_S}{\frac{(R_1+R_2)(R_1+R_3)}{R_1 R_2 R_3} - 1}$$

If $i_S = 4A$, $R_1 = 1\Omega$ and $R_2 = R_3 = 0.5\Omega$,

$$V_A = \frac{3}{2} V \quad \text{and} \quad V_B = \frac{1}{2} V$$

Revisit Example

Two key equations



$$i_s = \frac{V_A - V_B}{R_{\cancel{3}1}} + \frac{V_A}{R_2}$$

$$\frac{V_A - V_B}{R_1} = \frac{V_B}{R_3}$$

Rewrite in ~~[illegible]~~ matrix/vector form.

$$\begin{array}{l} \text{KCL@A} \\[2ex] \text{KCL@B} \end{array} \begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_1} \\ -\frac{1}{R_1} & \frac{1}{R_1} + \frac{1}{R_3} \end{bmatrix} \begin{bmatrix} V_A \\ V_B \end{bmatrix} = \begin{bmatrix} i_s \\ 0 \end{bmatrix}.$$

$$\mathbf{G} \qquad \mathbf{v} = \mathbf{i}$$

Structure of $\mathbf{G}$.

Diagonal elements : sum of conductances attached to the node

off diagonal : negative of conductance between appropriate node pairs

Also, $\mathbf{G}$ is symmetric

Structure of $\mathbf{i}$: each element is current injected into the node

These observations are true in general.

Great for checking derivations, and constructing software

STREAMLINING THE PROCEDURE

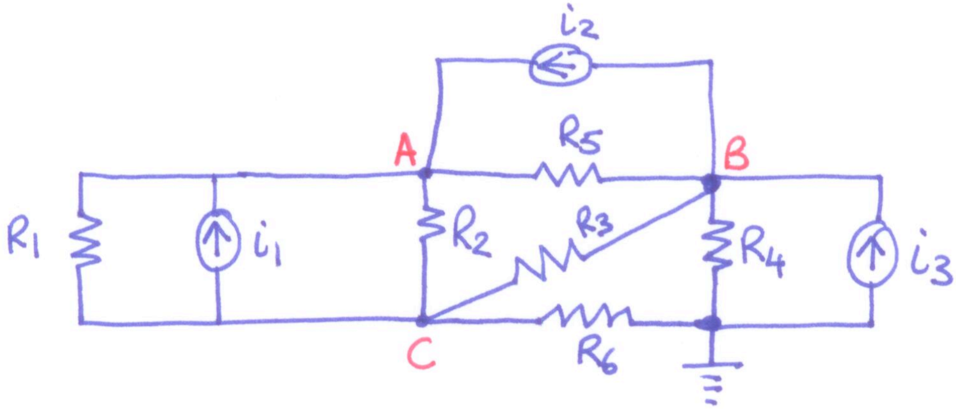
Recall steps 3 → 6

3. Label currents
4. KCL in terms of currents
5. Ohm's Law to relate branch currents to node voltages.
6. Dealing with sources.

Can't we just do this all at once?

Sure — once you feel confident

MORE COMPLICATED EXAMPLE



KCL at A $\quad i_1 + i_2 = \frac{V_A - V_C}{R_1} + \frac{V_A - V_C}{R_2} + \frac{V_A - V_B}{R_5}$

KCL at B $\quad \frac{V_A - V_B}{R_5} + \frac{V_C - V_B}{R_3} + i_3 = i_2 + \frac{V_B}{R_4}$

KCL at C $\quad 0 = \frac{V_C - V_A}{R_1} + i_1 + \frac{V_C - V_A}{R_2} + \frac{V_C - V_B}{R_3} + \frac{V_C}{R_6}$

Unknowns: V_A, V_B, V_C

Equations: 3

Sanity check.

$$
\begin{array}{l}
\text{KCL at A} \\ \\ \text{KCL at B} \\ \\ \text{KCL at C}
\end{array}
\begin{bmatrix}
\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} & -\frac{1}{R_5} & -\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \\
-\frac{1}{R_5} & \frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5} & -\frac{1}{R_3} \\
-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) & -\frac{1}{R_3} & \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_6}
\end{bmatrix}
\begin{bmatrix} V_A \\ V_B \\ V_C \end{bmatrix}
=
\begin{bmatrix} i_1 + i_2 \\ i_3 - i_2 \\ -i_1 \end{bmatrix}
$$

In the case that

$$i_1 = 1A, \quad i_2 = 2A, \quad i_3 = 3A$$

$$R_1 = 5\Omega, \quad R_2 = 2\Omega, \quad R_3 = 10\Omega$$

$$R_4 = 4\Omega, \quad R_5 = 5\Omega, \quad R_6 = 2\Omega$$

we have

$$\begin{bmatrix} 0.9 & -0.2 & -0.7 \\ -0.2 & 0.55 & -0.1 \\ -0.7 & -0.1 & 1.3 \end{bmatrix} \begin{bmatrix} V_A \\ V_B \\ V_C \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

Solving numerically (eg Gaussian elimination)

$$\begin{bmatrix} V_A \\ V_B \\ V_C \end{bmatrix} \approx \begin{bmatrix} 7.16 \\ 5.05 \\ 3.47 \end{bmatrix}$$