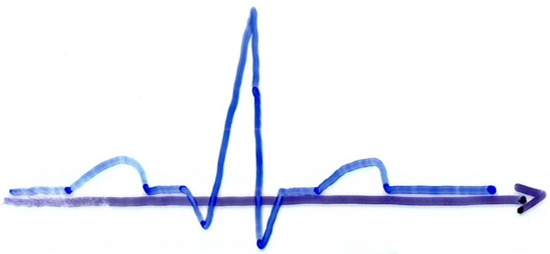
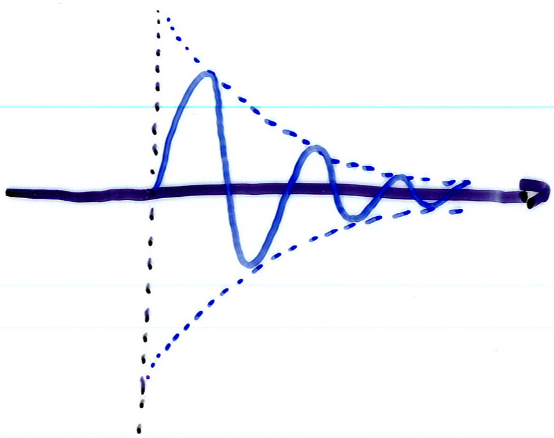
A Motivating Example



To build a defibrillator, we would like to mimic this waveform

Let's try: $e^{-\sigma t}\sin \omega_d t$

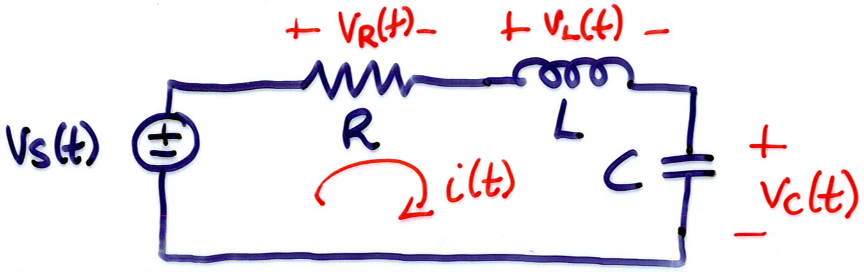


Not too bad.

Furthermore, we know how to synthesize this waveform, in a portable device!

Well, after this lecture we will know

More Formal Analysis



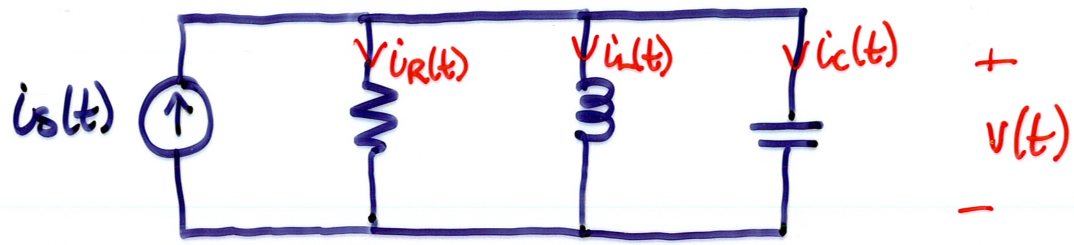
$$-V_S(t) + V_R(t) + V_L(t) + V_C(t) = 0$$

$$\Rightarrow \quad R\,i(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\overset{x}{\cancel{t}})\,d\overset{x}{\cancel{t}} = V_S(t)$$

Differentiate

$$R\frac{di(t)}{dt} + L\frac{d^2i(t)}{dt} + \frac{1}{C}i(t) = \frac{dV_S(t)}{dt}$$

$$\Rightarrow \quad \frac{d^2i(t)}{dt} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = \frac{1}{L}\frac{dV_S(t)}{dt}$$



KCL

$$i_s(t) = i_R(t) + i_L(t) + i_C(t)$$

$$i_s(t) = \frac{v(t)}{R} + \frac{1}{L}\int_{-\infty}^{t} v(x)\,dx + C\frac{dv(t)}{dt}$$

Differentiate and divide by C

$$\frac{d^2v(t)}{dt} + \frac{1}{RC}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = \frac{1}{C}\frac{di_s(t)}{dt}$$

Solution of Second Order Differential Equations

$$\frac{d^2x(t)}{dt} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = f(t)$$

As in the first order case, natural and forced components

$$x(t) = x_n(t) + x_f(t)$$

Natural Response

Solution when $f(t)=0$; ie.

$$\frac{d^2x(t)}{dt} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = 0 \qquad (*)$$

$\Rightarrow$ need a function that has the same shape as its first and second derivatives

The exponential Ae^{st} looks like a good candidate

Here we will allow s to be real-valued or complex-valued.

Let's try. Substitute Ae^{st} into (*)

$$[redacted] As^2e^{st} + a_1 Ase^{st} + a_0 Ae^{st} = 0.$$

$$\Rightarrow \quad Ae^{st}\left([redacted] s^2 + a_1 s + a_0\right) = 0$$

- $A=0$ is a solution, but is unlikely to satisfy initial conditions
- Therefore we seek solutions to the "characteristic equation"

$$[redacted] s^2 + a_1 s + a_0 = 0$$

- Using standard formula, solutions are

$$s_1, s_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4 [redacted] a_0}}{2 [redacted]}$$

- Three regimes

  * $a_1^2 - 4 [redacted] a_0 > 0$: overdamped
  * $a_1^2 - 4 [redacted] a_0 = 0$: critically damped
  * $a_1^2 - 4 [redacted] a_0 < 0$: underdamped.

SUPERPOSITION

- In general there are two solutions to the characteristic equation
- Which one should we use?
- Since the differential equation is linear, we can use both

SIMPLIFING CONCEPTS

Equation: $s^2 + a_1 s + a_0 = 0$

Define

undamped natural frequency: $\omega_0 = \sqrt{a_0}$

damping ratio $\zeta = \dfrac{a_1}{2\omega_0}$

Therefore: $s_1, s_2 = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$

and

$\zeta > 1 \Rightarrow$ overdamped. [s_1, s_2 real]

$\zeta = 1 \Rightarrow$ critically damped [$s_1 = s_2$]

$\zeta < 1 \Rightarrow$ under damped [s_1, s_2 complex]

Overdamped: $\zeta > 1$, s_1, s_2 real

$$x_n(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

$$s_1, s_2 = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$
$$= \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}$$

Underdamped: $\zeta < 1$, s_1, s_2 complex

Complex exponentials correspond to exponentially decaying sinusoids

$$x_n(t) = A_1 e^{-\sigma t}\cos(\omega_d t) + A_2 e^{-\sigma t}\sin(\omega_d t)$$

where

$$\sigma = \zeta\omega_0$$
$$\omega_d = \omega_0\sqrt{1-\zeta^2}$$ damped natural frequency

Critically damped: $\zeta = 1$ $s_1 = s_2 = -\zeta\omega_0 = \sigma$

$$x_n(t) = B_1 e^{-\sigma t} + B_2 t e^{-\sigma t}$$

Finding constants: once the form of $x_f(t)$ also known

Constants typically found from

$x(t)\big|_{t=0}$; $\frac{dx(t)}{dt}\Big|_{t=0}$; eqn itself; if $f(t)$ is constant also $x(\infty)$