COMPLETE RESPONSE OF LINEAR CIRCUITS WITH 1 or 2 time constants

* These are simple circuits which contain one or two energy storage elements.

* We study them in detail, because many practical circuits are dominated by their linear components, and are often dominated by one or two time constants. Studying them helps us develop intuition, and circuit design methods.

* Response of a circuit to an input consists of
  1) transient component - dies out over time
  2) steady-state component - persists even when input has been applied for a long time

* When calculating the full response we usually calculate

  a) Natural response - which is a function of the circuit only.

  b) Forced response - which depends on the input signal

  Note that one cannot say that a) = 1) and b) = 2) in general, but this does happen in some special cases

* For linear circuits with 1 or 2 time constants, the natural response is easy to calculate. This is not the case for non-linear circuits

* For linear circuits with 1 or 2 time constants, the forced response can still be hard to find. Fortunately, for several useful signals, such as steps, ramps, sinusoids + exponentials the forced response can be calculated. We will focus on these cases

* Now we develop expressions for the complete responses

* This will serve as a guide for learning how to deal with non-linear systems.

* Later we will study Laplace Transforms, which greatly simplify the analysis of linear circuits

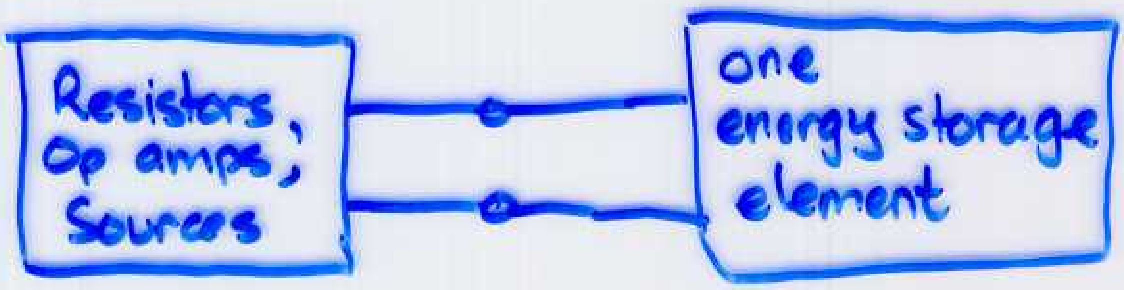
* We concentrate on differential equation methods here, because they can be extended to non-linear circuits (in an albeit complicated way) whereas Laplace transform methods cannot

RL and RC Circuits.

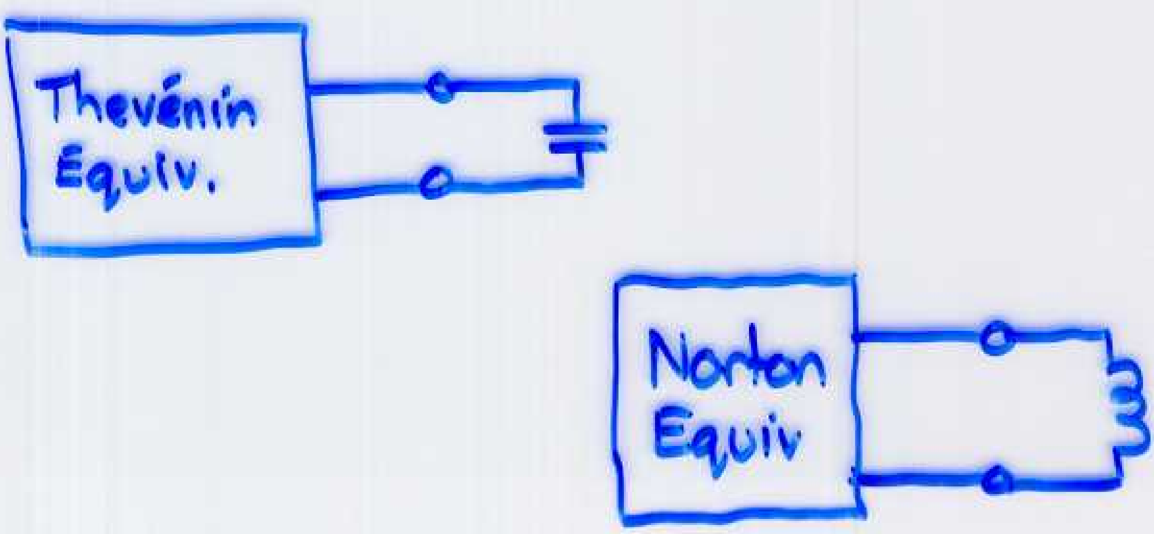
one inductor or capacitor
result in first-order differential equations.
Hence called first-order circuits

ANALYSIS PLAN

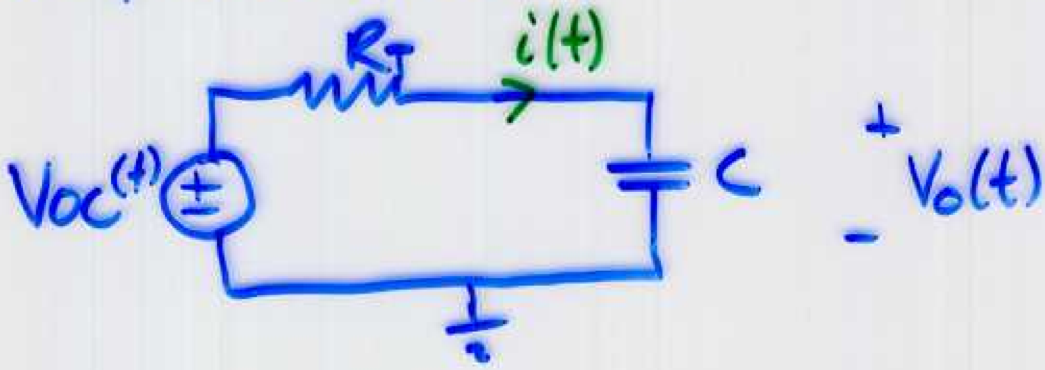
use Thevénin + Norton equivalents



If Capacitor use Thevénin.
If inductor use Norton.
That is.



* Capacitor circuits

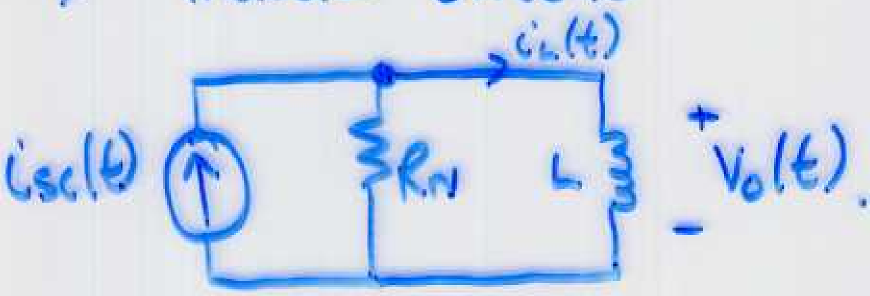


KVL $\quad V_{oc}(t) = R_T\, i(t) + v_o(t)$

Capacitor $\quad i(t) = C \dfrac{dV(t)}{dt}.$

$$\Rightarrow \quad \frac{dV(t)}{dt} + \frac{V(t)}{R_T C} = \frac{V_{oc}(t)}{R_T C}$$

* Inductor circuits



KCL $\quad i_{sc}(t) = \dfrac{v_o(t)}{R_N} + i_L(t)$

Inductor $\quad v_o(t) = L \dfrac{di_L(t)}{dt}$

$$\Rightarrow \quad \frac{di_L(t)}{dt} + \frac{R_N}{L} i_L(t) = \frac{R_N}{L} i_{sc}(t)$$

Both These equations have the form

$$\frac{d\,x(t)}{dt} + \frac{x(t)}{\tau} = f(t) \qquad (*)$$

- τ is called a "time-constant"
- to find $x(t)$ precisely, we need some "initial conditions". That is, knowledge of $x(t_0)$ for some given t_0.

The solution to $(*)$ is of the form.

$$x(t) = x_n(t) + x_f(t).$$

$x_n(t)$ is the natural response.

That is, response to initial conditions, when $f(t)=0$

This is given by. (we will prove this with Laplace transforms)

$$x_n(t) = K e^{-t/\tau}.$$

K can be found from initial conditions

[Show that this satisfies $(*)$ with $f(t)=0$]

Using the "integrating factor" method, it can be shown that

$$x_f(t) = e^{-t/\tau} \int_{-\infty}^{t} f(\lambda)\, e^{\lambda/\tau}\, d\lambda$$

This can be difficult to compute, but for certain simple, but useful signals, it has the same form as $f(t)$.

$f(t)$	$x_f(t)$
M	N
Me^{-bt}	Ne^{-bt}
$M\sin(\omega_0 t+\phi)$	$N\sin(\omega_0 t+\theta)$ $= A\sin\omega_0 t + B\cos\omega_0 t$

We can find N, θ using initial conditions

So for a differential equation

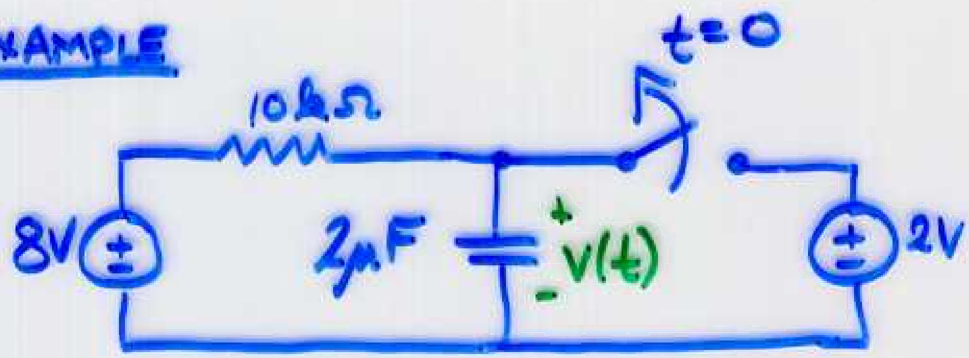
$$\frac{dx(t)}{dt} + \frac{x(t)}{\tau} = f(t)$$

we have.

$$x(t) = Ke^{-t/\tau} + x_f(t)$$

The constant K, and the constants in $x_f(t)$ can be found using additional information from the circuit.

EXAMPLE



Find $v(t)$ for $t \geq 0$

When the switch is closed, $v(t) = 2V$

$$\Rightarrow \quad v(0) = 2V$$

For $t \geq 0$ we have

$$\frac{dv(t)}{dt} + \frac{v(t)}{20\times10^{-3}} = \frac{8}{20\times10^{-3}}$$

$$\Rightarrow \quad v(t) = Ke^{-\frac{t}{20\times10^{-3}}} + N$$

Now find K and N

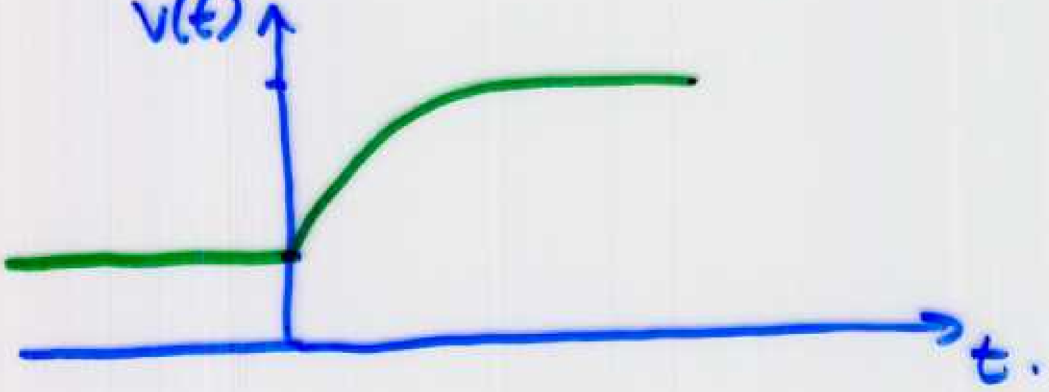
Using $v(0)=2 \quad \Rightarrow \quad K+N=2$

Evaluating diff eqn at $t=0$ gives

$$v'(0) + \frac{v(0)}{20\times10^{-3}} = \frac{8}{20\times10^{-3}}$$

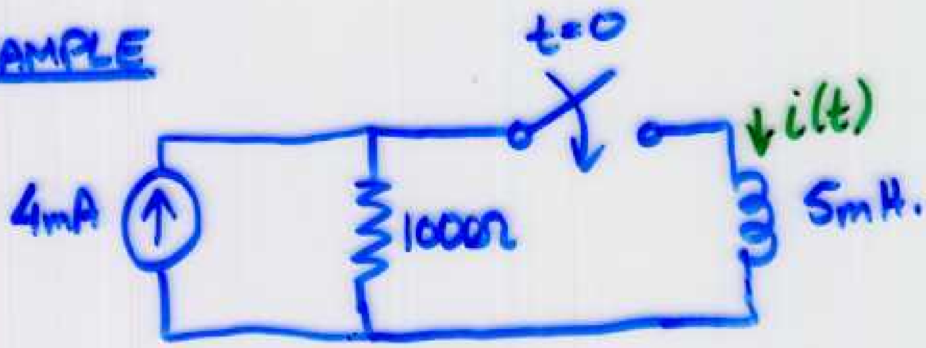
$$\Rightarrow \quad -K + 2 = 8 \quad \Rightarrow K=-6, \; N=8$$

$$\Rightarrow \quad v(t) = 8 - 6e^{-t/20\times10^{-3}}$$



\#

EXAMPLE



Find $i(t)$ for $t \geq 0$.

For $t<0$ $\quad i(t)=0 \Rightarrow i(0)=0$

(Current in an inductor is continuous)

For $t \geq 0$, KCL + inductor property give

$$\frac{di(t)}{dt} + \frac{1000}{5\times10^{-3}}\, i(t) = \frac{1000}{5\times10^{-3}} \cdot 4\times10^{-3}$$

$$\Rightarrow \quad i(t) = K e^{-t/5\times10^{-6}} + N$$

Since $i(0)=0 \Rightarrow K+N=0$

at $t=0$ $\quad \left.\frac{d\, i(t)}{dt}\right|_{t=0} = \frac{4\times10^3}{5}$

$$\Rightarrow \quad -K/5\times10^{-6} = \frac{4\times10^3}{5}$$

$$\Rightarrow \quad K = -4\times10^{-3}.$$

$$\Rightarrow \quad i(t) = 4 - 4e^{-t/5\times10^{-6}} \text{ mA.}$$

#

STABILITY OF FIRST-ORDER CIRCUITS

Natural response is of the form.

$$x_n(t) = K e^{-t/\tau}$$

- If $\tau > 0$ $\quad x_n(t)$ decays as t increases, and. approaches zero as $t \to \infty$
- In that case circuit is said to be stable.
- If $\tau < 0$, $\quad x_n(t)$ grows as t increases, eventually dominating the forced response
- Such circuits are said to be unstable

- The growth of $x(t)$ is often checked by non-linearities in the circuit (which were ignored when we formed the differential eqn.) However, in most applications instability is undesirable

- In the case of first-order circuits, $\tau = R_T C$ or $\tau = L/R_N$

- Hence we require $R_T > 0$ and $R_N > 0$ for stability

- For circuits with resistors + independent sources, $R_T > 0$ and $R_N > 0$

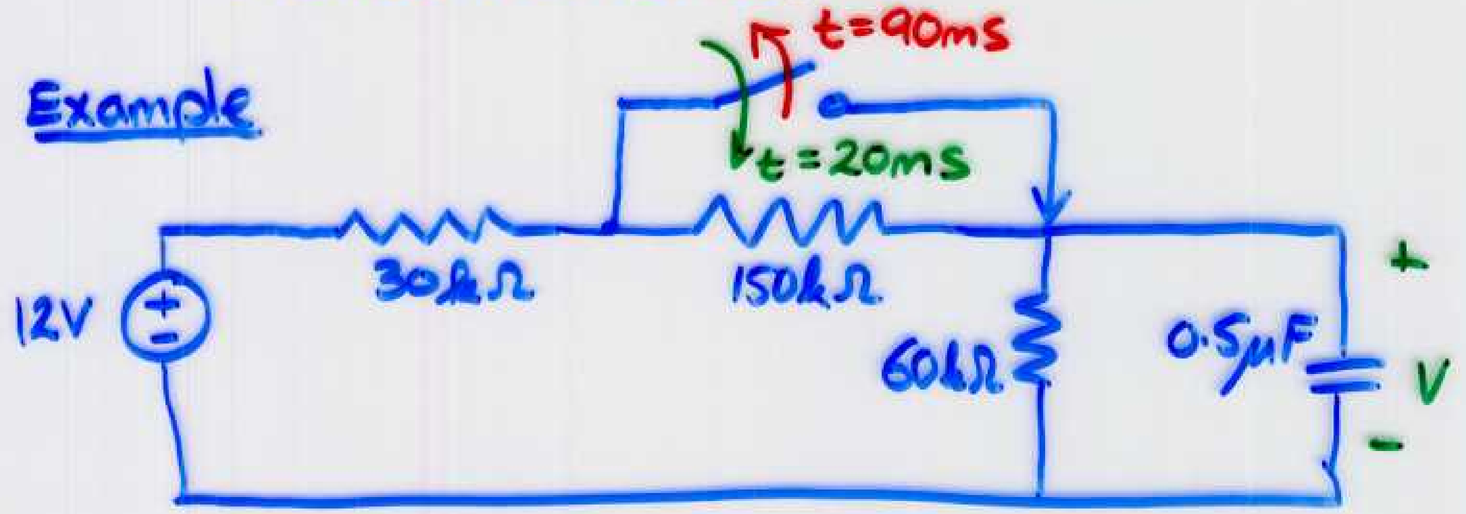
- However, if the circuit contains op-amps or dependent sources then it is possible to have negative equivalent resistances

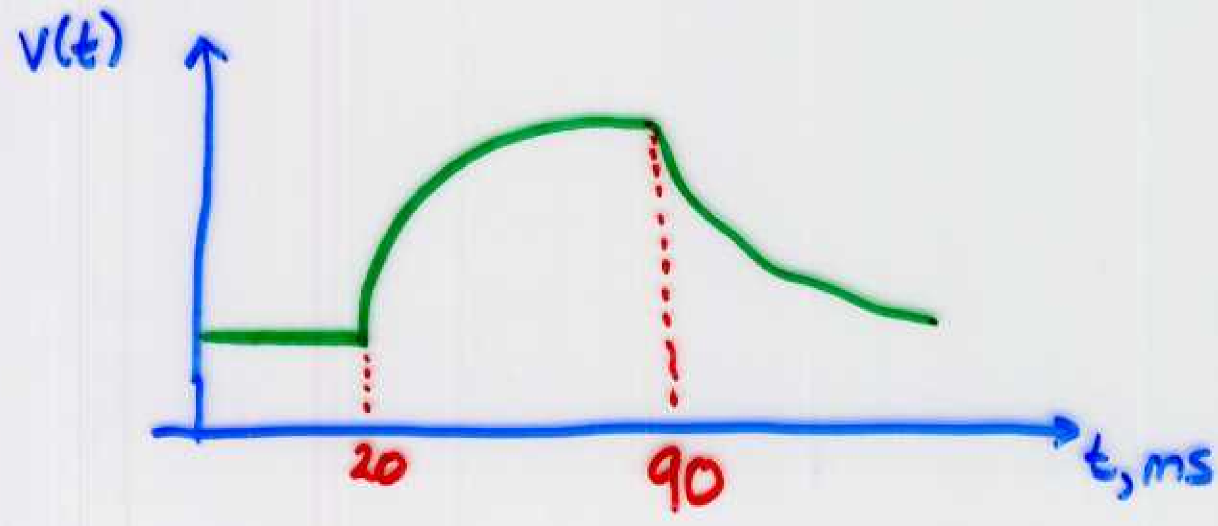
SANITY CHECKS + SUPERVISORY ENGINEERING.

- How can you tell if you, or a subordinate, or even your boss, has done the analysis correctly?

1) DRAW THE RESPONSE
2) Do a sanity check
   - check steady-state values
   - check time constants.

Example





STEADY STATE

* Since source is constant $V(t)$ will be constant in steady state

* For constant voltages capacitor acts like an open circuit

* By voltage division, at steady state,

  if switch open $V = 3V$

  if switch closed $V = 8V$

  checks out

TIME CONSTANTS

if switch open $R_T = 45k\Omega$.

$\Rightarrow \tau = 22.5ms$.

if switch closed $R_T = 20k\Omega$

$\Rightarrow \tau = 10ms$.

Since charge up time is faster than discharge things look o.k. These numbers can be confirmed from gradients at switching instants