EE3CL4:
Introduction to Linear Control Systems
Section 9: Design of Lead and Lag Compensators using Frequency Domain Techniques

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Outline

1. Frequency Domain Approach to Compensator Design
2. Lead Compensators
3. Lag Compensators
Frequency domain design

- Analyze closed loop using open loop transfer function
  \( L(s) = G_c(s)G(s)H(s) \).
- We would like closed loop to be stable:
  - Use Nyquist's stability criterion
- We might like to make sure that the closed loop remains stable even if there is an increase in the gain
  - Require a particular gain margin
- We might like to make sure that the closed loop remains stable even if there is additional phase lag
  - Require a particular phase margin
- We might like to make sure that the closed loop remains stable even if there is a combination of increased gain and additional phase lag
  - Require \( \min_{\omega} |L(j\omega) - (-1)| \) to be large enough
Frequency domain design

- We might like to control the damping ratio of the dominant pole pair
  - Use the fact that $\phi_{pm} = f(\zeta)$
- We might like to control the steady-state error constants
  - For step, ramp and parabolic inputs, these constants are related to the behaviour of $L(s)$ around zero; i.e., behaviour near DC
- We might like to influence the settling time
  - Roughly speaking, the settling time decreases with increasing bandwidth
- Once we have a general idea of the shape of the Nyquist diagram, is some of this information available in a more convenient form?
Frequency Domain Approach to Compensator Design

Lead Compensators
Lag Compensators

\[ L(j\omega) = \frac{1}{j\omega(1 + j\omega)(1 + j\omega/5)} \]

- Gain margin \(\approx 15\) dB
- Phase margin \(\approx 43^\circ\)
Compensators and Bode diagram

- We have seen the importance of phase margin.
- If $G(s)$ does not have the desired margin, how should we choose $G_c(s)$ so that $L(s) = G_c(s)G(s)$ does?
- To begin, how does $G_c(s)$ affect the Bode diagram?
- Magnitude:

\[
20 \log_{10} \left( |G_c(j\omega)G(j\omega)| \right) = 20 \log_{10} \left( |G_c(j\omega)| \right) + 20 \log_{10} \left( |G(j\omega)| \right)
\]

- Phase:

\[
\angle G_c(j\omega)G(j\omega) = \angle G_c(j\omega) + \angle G(j\omega)
\]
Lead Compensators

- \( G_c(s) = \frac{K_c(s+z)}{s+p} \), with \(|z| < |p|\), alternatively,
- \( G_c(s) = \frac{K_c}{\alpha} \frac{1+s\alpha\tau}{1+s\tau} \), where \( p = 1/\tau \) and \( \alpha = p/z > 1 \)
- Bode diagram (in the figure, \( K_1 = K_c/\alpha \)): 

\[ \text{Bode diagram} \]
Lead Compensation

- What will lead compensation, do?
- Phase is positive: might be able to increase phase margin $\phi_{pm}$
- Slope is positive: might be able to increase the cross-over frequency, $\omega_c$, (and the bandwidth)
Lead Compensation

\[ G_c(s) = \frac{K_c}{\alpha} \frac{1+sa\tau}{1+s\tau} \]

By making the denom. real, can show that
\[ \angle G_c(j\omega) = \text{atan} \left( \frac{\omega\tau(\alpha-1)}{1+\alpha(\omega\tau)^2} \right) \]

Max. occurs when \( \omega = \omega_m = \frac{1}{\tau\sqrt{\alpha}} = \sqrt{zp} \)

Max. phase angle satisfies \( \tan(\phi_m) = \frac{\alpha-1}{2\sqrt{\alpha}} \)

Equivalently, \( \sin(\phi_m) = \frac{\alpha-1}{\alpha+1} \)

At \( \omega = \omega_m \), we have \( |G_c(j\omega_m)| = \frac{K_c}{\sqrt{\alpha}} \)
Bode Design Principles (lead)

- Set the loop gain so that desired steady-state error constants are obtained
- Insert the compensator to modify the transient properties:
  - Damping: through phase margin
  - Response time: through bandwidth
- Compensate for the attenuation of the lead network, if appropriate

To maximize impact of phase lead, want peak of phase near $\omega_c$ of the *compensated* open loop
Design Guidelines

1. For uncompensated (i.e., proportionally controlled) closed loop, set gain $K_p$ so that steady-state error constants of the closed loop meet specifications.

2. Evaluate the phase margin, and the amount of phase lead required.

3. Add a little “safety margin” to the amount of phase lead.

4. From this, determine $\alpha$ using $\sin(\phi_m) = \frac{\alpha-1}{\alpha+1}$.

5. To maintain steady-state error const’s, set $K_c = K_p \alpha$.

6. Determine (or approximate) the frequency at which $K_p G(j\omega)$ has magnitude $-10 \log_{10}(\alpha)$.

7. If we set $\omega_m$ of the compensator to be this frequency, then $G_c(j\omega_m)G(j\omega_m) = 1$ (or $\approx 1$). Hence, the compensator will provide its maximum phase contribution at the appropriate frequency.

8. Choose $\tau = 1/(\omega_m \sqrt{\alpha})$. Hence, $p = \omega_m \sqrt{\alpha}$.

9. Set $z = p/\alpha$.

10. Compensator: $G_c(s) = \frac{K_c(s+z)}{s+p}$. 


Example

- Type 1 plant of order 2: \( G(s) = \frac{5}{s(s+2)} \)
- Design goals:
  - Steady-state error due to a ramp input less than 5% of velocity of ramp
  - Phase margin at least 45° (implies a damping ratio)
- Steady state error requirement implies \( K_v = 20 \).
- For prop. controlled Type 1 plant: \( K_v = \lim_{s \to 0} sK_p G(s) \).
  Hence \( K_p = 8 \).
- To find phase margin of prop. controlled loop we need to find \( \omega_c \), where \( |K_p G(j\omega_c)| = \left| \frac{40}{j\omega_c(j\omega_c+2)} \right| = 1 \)
- \( \omega_c \approx 6.2 \text{ rad/s} \)
- Evaluate \( \angle K_p G(j\omega) = -90° - \tan(\omega/2) \) at \( \omega = \omega_c \)
- Hence \( \phi_{pm, \text{prop}} = 18° \)
Example

- $\phi_{pm, \text{prop}} = 18^\circ$. Hence, need $27^\circ$ of phase lead
- Let's go for a little more, say $30^\circ$
- So, want peak phase of lead comp. to be $30^\circ$
- Solving $\frac{\alpha - 1}{\alpha + 1} = \sin(30^\circ)$ yields $\alpha = 3$. Set $K_c = 3 \times 8$
- Since $10 \log_{10}(3) = 4.8$ dB we should choose $\omega_m$ to be where $20 \log_{10}(\left|\frac{40}{j\omega_m(j\omega_m + 2)}\right|) = -4.8$ dB
- Solving this equation yields $\omega_m = 8.4$ rad/s
- Therefore $z = \omega_m/\sqrt{\alpha} = 4.8$, $p = \alpha z = 14.4$
- $G_c(s) = \frac{24(s+4.8)}{s+14.4}$
- $G_c(s)G(s) = \frac{120(s+4.8)}{s(s+2)(s+14.4)}$, actual $\phi_{pm} = 43.6^\circ$
- Goal can be achieved by using a larger target for additional phase, e.g., $\alpha = 3.5$
Bode Diagram
Frequency Domain Approach to Compensator Design

Lead Compensators

Lag Compensators

Step Response
Ramp Response

Lead Compensators

Lag Compensators
Ramp Response, detail
Lag Compensators

- \( G_c(s) = \frac{K_c(s+z)}{s+p} \), with \(|p| < |z|\), alternatively,
- \( G_c(s) = \frac{K_c\alpha(1+s\tau)}{1+s\alpha\tau} \), where \( z = 1/\tau \) and \( \alpha = z/p > 1 \)
- Bode diagrams of lag compensators for two different \( \alpha \)s, in the case where \( K_c = 1/\alpha \)
What will lag compensation do?

- Since zero and pole are typically close to the origin, phase lag aspect is not really used.
- What is useful is the attenuation above $\omega = 1/\tau$: gain is $-20 \log_{10}(\alpha)$, with little phase lag.
- Can reduce cross-over frequency, $\omega_c$, without adding much phase lag.
- Tends to reduce bandwidth.
Qualitative example

- Uncompensated system has small phase margin
- Phase lag of compensator does not play a large role
- Attenuation of compensator does:  
  \( \omega_c \) reduced by about a factor of a bit more than 3
- Increased phase margin is due to the natural phase characteristic of the plant
Bode Design Principles (lag)

For lag compensators:

- Set the loop gain so that desired steady-state error constants are obtained
- Insert the compensator to modify the phase margin:
  - Do this by reducing the cross-over frequency
  - Observe the impact on response time

Basic principle: Set attenuation to reduce $\omega_c$ far enough so that uncompensated open loop has desired phase margin
Design Guidelines

1. For uncompensated (i.e., proportionally controlled) closed loop, set gain $K_p$ so that steady-state error constants of the closed loop meet specifications.

2. Evaluate the phase margin, analytically, or using a Bode diagram. If that is insufficient...

3. Determine $\omega_c'$, the frequency at which the uncompensated open loop, $K_p G(j \omega)$, has a phase margin equal to the desired phase margin plus $5\degree$.

4. Design a lag comp. so that the gain of the compensated open loop, $G_c(j \omega) G(j \omega)$, at $\omega = \omega_c'$ is $0 \, \text{dB}$
   - Choose $K_c = K_p / \alpha$ so that steady-state error const’s are maintained
   - Place zero of the comp. around $\omega_c'/10$ so that at $\omega_c'$ we get almost all the attenuation available from the comp.
   - Choose $\alpha$ so that $20 \log_{10}(\alpha) = 20 \log_{10}(|K_p G(j \omega_c')|)$. With that choice and $K_c = K_p / \alpha$, $|G_c(j \omega_c') G(j \omega_c')| \approx 1$
   - Place the pole at $p = z / \alpha$
   - Compensator: $G_c(s) = \frac{K_c(s + z)}{s + p}$
Example, same set up as lead design

- Type 1 plant of order 2: $G(s) = \frac{5}{s(s+2)}$

Design goals:
- Steady-state error due to a ramp input less than 5% of velocity of ramp
- Phase margin at least 45° (implies a damping ratio)

- Steady state error requirement implies $K_v = 20$.
- For prop. controlled Type 1 plant: $K_v = \lim_{s \to 0} sK_p G(s)$. Hence $K_p = 8$.
- To find phase margin of prop. controlled loop we need to find $\omega_c$, where $|K_p G(j\omega_c)| = \left| \frac{40}{j\omega_c(j\omega_c+2)} \right| = 1$
- $\omega_c \approx 6.2$ rad/s
- Evaluate $\angle K_p G(j\omega) = -90^\circ - \tan(\omega/2)$ at $\omega = \omega_c$
- Hence $\phi_{pm, prop} = 18^\circ$
Since want phase margin to be $45^\circ$, we set $\omega'_c$ such that
\[ \angle G(j\omega'_c) = -180^\circ + 45^\circ + 5^\circ = -130^\circ. \quad \Rightarrow \quad \omega'_c \approx 1.5 \]

To make the open loop gain at this frequency equal to 0 dB, the required attenuation is 20 dB. Actual curves are around 2 dB lower than the straight line approximation shown.

Hence $\alpha = 10$. Set $K_c = K_p/\alpha = 0.8$

Zero set to be one decade below $\omega'_c$; $z = 0.15$

Pole is $z/\alpha = 0.015$.

Hence $G_c(s) = \frac{0.8(s+0.15)}{s+0.015}$
Example: Comp’d open loop

- Compensated open loop: $G_c(s)G(s) = \frac{4(s+0.15)}{s(s+2)(s+0.015)}$

- Numerical evaluation:
  - new $\omega_c = 1.58$
  - new phase margin $= 46.8^\circ$
  - By design, $K_v$ remains 20
Frequency Domain Approach to Compensator Design

Lead Compensators

Lag Compensators

Step Response

- Input
- Output, P
- Output, Lead
- Output, Lag

Graph showing step responses of different compensator types.
Ramp Response
Ramp Response, detail