

Tim Davidson

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Step 5 (approx'd)

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Example

Parameter
Design

“Negative”
Root Locus

EE3CL4: Introduction to Linear Control Systems

Section 5: Root Locus Procedure

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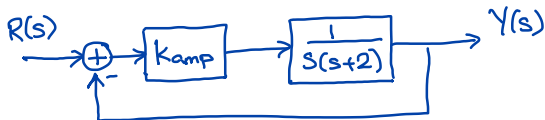
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Root LocusOpen loop transfer function: $K_{\text{amp}}G(s)$ Closed loop transfer function $T(s) = \frac{K_{\text{amp}}G(s)}{1+K_{\text{amp}}G(s)}$ Char. eqn: $s^2 + 2s + K_{\text{amp}} = 0$ Closed-loop poles: $s_1, s_2 = -1 \pm \sqrt{1 - K_{\text{amp}}}$ What paths do these closed-loop poles take as K_{amp} goes from 0 to $+\infty$?

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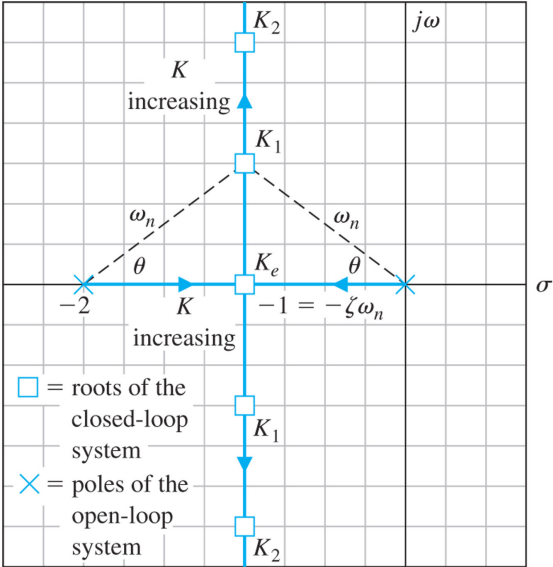
Compensator design for VTOL aircraft

Sketching the Root Locus, Steps 5-7

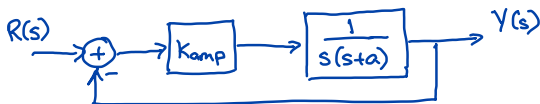
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Another example



Closed loop transfer function $T(s) = \frac{K_{amp}G(s)}{1+K_{amp}G(s)}$

Consider K_{amp} to be fixed

Char. eqn: $s^2 + as + K_{amp} = 0$

Closed-loop poles: $s_1, s_2 = \left(-a \pm \sqrt{a^2 - 4K_{amp}}\right)/2$

What paths do these closed-loop poles take as a goes from 0 to $+\infty$?

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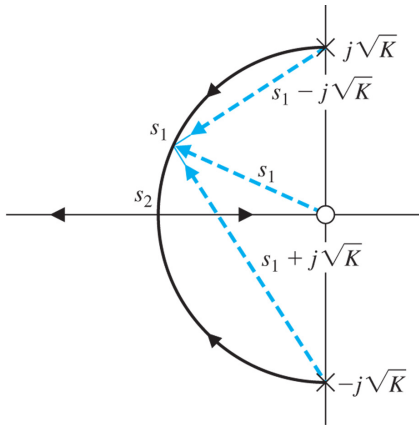
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Another example



What to do in the general case?

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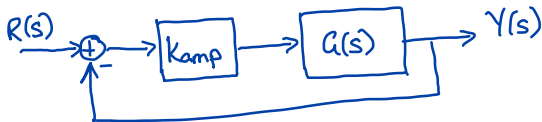
Parameter Design

“Negative” Root Locus

In the previous examples we exploited the simple factorization of second order polynomials

However, it would be very useful to be able to draw the paths that the closed-loop poles take as K_{amp} increases for more general open-loop systems

Principles of general procedure



$$\text{Closed loop transfer function } T(s) = \frac{K_{\text{amp}}G(s)}{1+K_{\text{amp}}G(s)} = \frac{p(s)}{q(s)}$$

Closed loop poles are solutions to $q(s) = 0$

These are also sol'ns to $1 + K_{\text{amp}}G(s) = 0$; i.e., $K_{\text{amp}}G(s) = -1 + j0$
 In polar form, $|K_{\text{amp}}G(s)| \angle K_{\text{amp}}G(s) = 1 \angle (180^\circ + \ell 360^\circ)$

Therefore, for an arbitrary point on the complex plane s_0 to be a closed-loop pole for a given value of K_{amp} the following equations must be satisfied

$$|K_{\text{amp}}G(s_0)| = 1 \quad \text{and} \quad \angle K_{\text{amp}}G(s_0) = (180^\circ + \ell 360^\circ)$$

where ℓ is any integer. (Note: book uses k , but we will use ℓ to avoid confusion with K)

We will also keep in mind that $R(s)$ and $Y(s)$ correspond to real signals. Hence, closed-loop poles are either real or occur in complex-conjugate pairs

In terms of poles and zeros

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For s_0 to be a closed-loop pole, we must have

$$|K_{\text{amp}}G(s_0)| = 1 \quad \text{and} \quad \angle K_{\text{amp}}G(s_0) = \angle(180^\circ + \ell 360^\circ)$$

Write $G(s) = \frac{K_G \prod_{i=1}^M (s+z_i)}{\prod_{j=1}^n (s+p_j)}$, which means that the open loop zeros are $-z_i$'s; open loop poles are $-p_j$'s

For s_0 to be a closed-loop pole

$$\frac{|K_{\text{amp}}K_G| \prod_{i=1}^M |s_0 + z_i|}{\prod_{j=1}^n |s_0 + p_j|} = 1$$

$$\angle K_{\text{amp}} + \angle K_G + \sum_{i=1}^M \angle(s_0 + z_i) - \sum_{j=1}^n \angle(s_0 + p_j) = 180^\circ + \ell 360^\circ$$

(From the definition of the factorization of $G(s)$, when $M = 0$ the terms related to the zeros “disappear” a natural way)

Can we interpret these expressions in a geometric way?

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Vector difference

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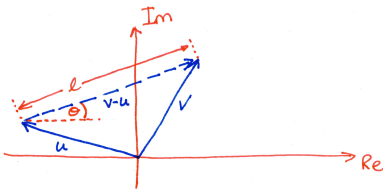
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Parameter Design

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- Let u and v be complex numbers.
- Can you describe $v - u$ in geometric terms?
- Use the fact that $v = u + (v - u)$.
- That means that $v - u$ is the vector from u to v



- $v - u = l e^{j\theta}$. That is,
 - $|v - u|$ is the length of the vector from u to v .
 - $\angle(v - u)$ is the angle of the vector from u to v
- In our expressions we have terms of the form $s_0 + z_i = s_0 - (-z_i)$ and $s_0 + p_j = s_0 - (-p_j)$

Geometric interpretation

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Magnitude criterion:

$$\frac{|K_{\text{amp}} K_G| \prod_{i=1}^M |s_0 + z_i|}{\prod_{j=1}^n |s_0 + p_j|} = 1$$

$$\frac{|K_{\text{amp}} K_G| \prod_{i=1}^M \text{distances from zeros (if any) of } G(s) \text{ to } s_0}{\prod_{j=1}^n \text{distances from poles of } G(s) \text{ to } s_0} = 1$$

Phase criterion:

$$\angle K_{\text{amp}} + \angle K_G + \sum_{i=1}^M \angle(s_0 + z_i) - \sum_{j=1}^n \angle(s_0 + p_j) = 180^\circ + \ell 360^\circ$$

$$\begin{aligned} \angle K_{\text{amp}} + \angle K_G + \sum_{i=1}^M \text{angles from zeros (if any) of } G(s) \text{ to } s_0 \\ - \sum_{j=1}^n \text{angles from poles of } G(s) \text{ to } s_0 \\ = 180^\circ + \ell 360^\circ \end{aligned}$$

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Now for the challenge

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- Can we build on these geometric interpretations of the equations in the simple case of amplifier gains to develop a broadly applicable approach to control system design?
- The first step will be to develop a formal procedure for sketching the paths that the closed-loop poles take as a design parameter (often an amplifier gain) changes. These are called the root loci.
- We will develop the formal procedure in a slightly more general setting than what we have seen so far

Preparing for formal procedure

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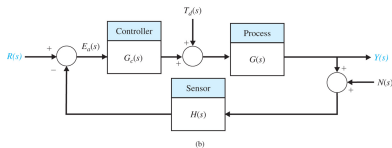
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$$Y(s) = \frac{G_c(s)G(s)}{1 + H(s)G_c(s)G(s)} R(s) + \frac{G(s)}{1 + H(s)G_c(s)G(s)} T_d(s) - \frac{H(s)G_c(s)G(s)}{1 + H(s)G_c(s)G(s)} N(s)$$

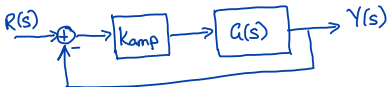
- Note that all transfer functions have the same denominator
- Note that the form of the denominator is $1 + F(s)$
- The closed loop poles are the solutions to $1 + F(s) = 0$

Preparing for the formal procedure, II

- Closed-loop poles are solutions to $1 + F(s) = 0$, where $F(s) = H(s)G_c(s)G(s)$
- We would like to know what paths (loci) the closed-loop poles take as we change a design parameter that is embedded in $G_c(s)$
- Our techniques will work for cases where we can rearrange the equation $1 + F(s) = 0$ into the form $1 + KP(s) = 0$, where
 - K is the design parameter, or a function thereof
 - the numerator and denominator of $P(s)$ are monic polynomials (coefficient of highest power of s is 1)

Examples of $P(s)$

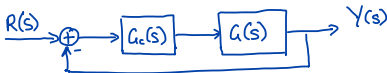
- Pure proportional control



- $F(s) = K_{\text{amp}}G(s)$, where $G(s) = \frac{K_G \prod_{i=1}^M (s+z_i)}{\prod_{j=1}^n (s+p_j)}$.
 - That means that $K = K_{\text{amp}}K_G$ and $P(s) = \frac{\prod_{i=1}^M (s+z_i)}{\prod_{j=1}^n (s+p_j)}$
- We will discover that shape of the root locus (of the closed-loop poles) is highly dependent on the positions of the open-loop poles and zeros
 - Is there a way that we add open-loop poles and zeros to change the shape of the root locus, and hence put closed-loop poles in desirable positions?

Examples of $P(s)$

- Cascade compensation



- Change the shape of the root locus using thoughtfully positioned poles and zeros of the controller transfer function

$$G_c(s) = \frac{K_c \prod_{i=1}^{M_c} (s+z_{c_i})}{\prod_{j=1}^{n_c} (s+p_{c_j})}$$

- In that case,

- $F(s) = G_c(s)G(s),$

- $K = K_c K_G$ and $P(s) = \frac{\prod_{i=1}^{M_c} (s+z_{c_i}) \prod_{i=1}^M (s+z_i)}{\prod_{j=1}^{n_c} (s+p_{c_j}) \prod_{j=1}^n (s+p_j)}$

Formal Procedure

- When closed loop poles are solutions to $1 + KP(s) = 0$,
- what paths (loci) do the poles move along as K goes from 0 to ∞

Table 7.2 Seven Steps for Sketching a Root Locus

| Step | Related Equation or Rule |
|--|--|
| 1. Prepare the root locus sketch. | |
| (a) Write the characteristic equation so that the parameter of interest, K , appears as a multiplier. | $1 + KP(s) = 0.$ |
| (b) Factor $P(s)$ in terms of n poles and M zeros. | $1 + K \frac{\prod_{i=1}^M (s + z_i)}{\prod_{j=1}^n (s + p_j)} = 0.$ |
| (c) Locate the open-loop poles and zeros of $P(s)$ in the s -plane with selected symbols. | $\times = \text{poles}, \circ = \text{zeros}$ Locus begins at a pole and ends at a zero. |
| (d) Determine the number of separate loci, SL . | $SL = n$ when $n \geq M$; $n =$ number of finite poles, $M =$ number of finite zeros. |
| (e) The root loci are symmetrical with respect to the horizontal real axis. | |
| 2. Locate the segments of the real axis that are root loci. | Locus lies to the left of an odd number of poles and zeros. |
| 3. The loci proceed to the zeros at infinity along asymptotes centered at σ_A and with angles ϕ_A . | $\sigma_A = \frac{\sum(-p_j) - \sum(-z_i)}{n - M},$ $\phi_A = \frac{2k + 1}{n - M} 180^\circ, k = 0, 1, 2, \dots, (n - M - 1).$ |
| 4. Determine the points at which the locus crosses the imaginary axis (if it does so). | Use Routh–Hurwitz criterion (see Section 6.2). |
| 5. Determine the breakaway point on the real axis (if any). | a) Set $K = p(s).$ b) Determine roots of $dp(s)/ds = 0$ or use graphical method to find maximum of $p(s).$ |
| 6. Determine the angle of locus departure from complex poles and the angle of locus arrival at complex zeros, using the phase criterion. | $\angle P(s) = 180^\circ + k360^\circ$ at $s = -p_j$ or $-z_r.$ |
| 7. Complete the root locus sketch. | |

- In this course we will focus on Steps 1–4 and 7

Step 1

- Write the characteristic equation as $1 + F(s) = 0$
- Rearrange so that the parameter of interest is contained in the multiplier K in an expr'n of the form

$$1 + KP(s) = 0,$$

where the numerator and denominator of $P(s)$ are monic polynomials

- Factorize $P(s)$ into poles and zeros, $P(s) = \frac{\prod_{i=1}^M (s+z_i)}{\prod_{j=1}^n (s+p_j)}$
- Hence characteristic equation is equiv. to $\prod_{j=1}^n (s + p_j) + K \prod_{i=1}^M (s + z_i) = 0$
- Where does the locus start?
- Where are poles for $K = 0$?
- They are the poles of $P(s)$. Mark each with an \times

$$\prod_{j=1}^n (s + p_j) + K \prod_{i=1}^M (s + z_i) = 0$$

- Where do the poles end up?
- Where are poles for $K \rightarrow \infty$?
- Rewrite as $(1/K) \prod_{j=1}^n (s + p_j) + \prod_{i=1}^M (s + z_i) = 0$
- The zeros of $P(s)$. Mark each with a \circ
- Since $M \leq n$ there will often be zeros at ∞ , too

Summary: Root locus starts at poles of $P(s)$ and ends at zeros of $P(s)$

Note: Often $P(s) \propto G_c(s)G(s)$ and $K \propto$ an amplifier gain. In that case, root locus (of the closed loop) starts at the open-loop poles and ends at the open-loop zeros.

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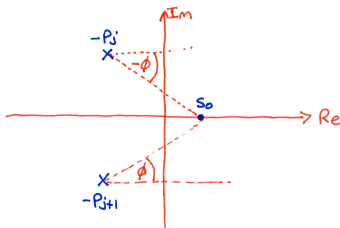
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Example

Parameter
Design"Negative"
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Phase condition:

$$\angle K + \sum_{i=1}^M \angle(s_0 + z_i) - \sum_{j=1}^n \angle(s_0 + p_j) = 180^\circ + \ell 360^\circ$$

Recall that for $K > 0$, $\angle K = 0$.What does this tell us when s_0 is on the real axis?

Any complex conjugate pairs have no impact

Step 2, cont.

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Phase condition for $K > 0$:

$$\sum_{i=1}^M \angle(s_0 + z_i) - \sum_{j=1}^n \angle(s_0 + p_j) = 180^\circ + \ell 360^\circ$$

Let's examine effects of poles of $P(s)$ on the real axis

- For $s_{0,1}$, all angles from poles to $s_{0,1}$ are zero
- For $s_{0,2}$, right pole generates an angle of 180° , others zero
- For $s_{0,3}$, $-\sum_{j=1}^n \angle(s_0 + p_j) = -360^\circ$
- For $s_{0,4}$, $-\sum_{j=1}^n \angle(s_0 + p_j) = -540^\circ$

Something similar for zeros.

Therefore: sections of real axis on the locus must lie to left of odd number of (real-valued) poles and (real-valued) zeros of $P(s)$

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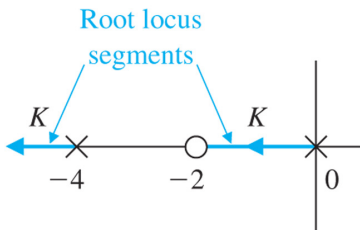
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$$P(s) = \frac{(s+2)}{s(s+4)}$$

Step 1: Poles of $P(s)$: $s = 0, -4$; Zeros of $P(s)$: $s = -2$ **Step 2:** Determine segments on real axis

In this case, this is enough to generate the complete locus of closed-loop poles as K goes from 0 to $+\infty$



Review of Principles of Root Locus

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Parameter Design

“Negative” Root Locus

- We would like to know where the closed-loop poles go as a parameter of the loop (typically a controller design parameter) is changed.
- We would like to gain insight from how the closed-loop poles move in order to guide our design of the controller

Sketching the Root Locus

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Table 7.2 Seven Steps for Sketching a Root Locus

| Step | Related Equation or Rule |
|--|--|
| 1. Prepare the root locus sketch. | |
| (a) Write the characteristic equation so that the parameter of interest, K , appears as a multiplier. | $1 + KP(s) = 0.$ |
| (b) Factor $P(s)$ in terms of n poles and M zeros. | $1 + K \frac{\prod_{i=1}^M (s + z_i)}{\prod_{j=1}^n (s + p_j)} = 0.$ |
| (c) Locate the open-loop poles and zeros of $P(s)$ in the s -plane with selected symbols. | $\times = \text{poles}, \circ = \text{zeros}$ Locus begins at a pole and ends at a zero. |
| (d) Determine the number of separate loci, SL . | $SL = n$ when $n \geq M$; $n =$ number of finite poles, $M =$ number of finite zeros. |
| (e) The root loci are symmetrical with respect to the horizontal real axis. | |
| 2. Locate the segments of the real axis that are root loci. | Locus lies to the left of an odd number of poles and zeros. |
| 3. The loci proceed to the zeros at infinity along asymptotes centered at σ_A and with angles ϕ_A . | $\sigma_A = \frac{\sum (-p_j) - \sum (-z_i)}{n - M}.$ $\phi_A = \frac{2k + 1}{n - M} 180^\circ, k = 0, 1, 2, \dots, (n - M - 1).$ Use Routh–Hurwitz criterion (see Section 6.2). |
| 4. Determine the points at which the locus crosses the imaginary axis (if it does so). | |
| 5. Determine the breakaway point on the real axis (if any). | a) Set $K = p(s).$ b) Determine roots of $dp(s)/ds = 0$ or use graphical method to find maximum of $p(s).$ $\angle P(s) = 180^\circ + k360^\circ$ at $s = -p_j$ or $-z_i.$ |
| 6. Determine the angle of locus departure from complex poles and the angle of locus arrival at complex zeros, using the phase criterion. | |
| 7. Complete the root locus sketch. | |

Step 1

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- Write the denominator of the closed-loop transfer function in the form

$$1 + KP(s) = 0,$$

$$\text{where } P(s) = \frac{\prod_{i=1}^M (s+z_i)}{\prod_{j=1}^n (s+p_j)}$$

K contains the parameter of interest

- We will focus on the case in which $K \geq 0$
- We will discuss the “negative” root locus case later
- Root loci start at poles of $P(s)$ and end at zeros of $P(s)$, including the zeros of $P(s)$ at infinity
- Mark the poles of $P(s)$ with an \times
- Mark the (finite) zeros of $P(s)$ with a \circ

Step 2

Using the phase condition, we showed that for $K > 0$,

- any part of the root locus on the real axis lies to the left of an odd number of (real-valued) poles and (real-valued) zeros of $P(s)$

- $P(s) = \frac{\prod_{i=1}^M (s+z_i)}{\prod_{j=1}^n (s+p_j)}$
- closed-loop characteristic equation

$$(1/K) \prod_{j=1}^n (s + p_j) + \prod_{i=1}^M (s + z_i) = 0$$

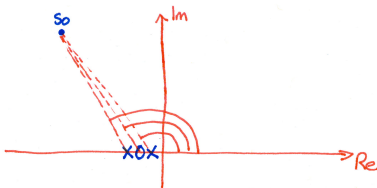
- As $K \rightarrow +\infty$, there are M finite values of s that satisfy the equation
- How many zeros at infinity? Recall that $P(s) = \frac{s^M + \dots}{s^n + \dots}$. Therefore, $n - M$ zeros at infinity
- How do the loci approach the zeros at infinity?

Step 3, Approaching infinity

- Consider a point s_0 on the root locus far from the poles of $P(s)$ and the finite zeros of $P(s)$
- Phase condition (for positive K):

$$\sum_{i=1}^M \angle(s_0 + z_i) - \sum_{j=1}^n \angle(s_0 + p_j) = 180^\circ + \ell 360^\circ$$

- Since the point s_0 is far away from all $-z_i$ and $-p_j$, all angles are approximately the same, say ϕ



- Hence, phase cond. is approx: $(M - n)\phi = 180^\circ + \ell 360^\circ$
- Re-arranging, and using multiples of 360° ,

$$\phi = (2\ell + 1) \frac{180^\circ}{(n - M)} \quad \text{for } \ell = 0, 1, \dots, (n - M - 1)$$

Step 3, Centroid

- From where do these rays emanate?
- Recall $P(s) = \frac{\prod_{i=1}^M (s+z_i)}{\prod_{j=1}^n (s+p_j)}$
- For large s , effects of finite zeros almost cancelled out by that of M of the finite poles
- Therefore, as s gets large, the roots follow a similar path to those of $\tilde{P}(s) = 1/(s - \sigma_A)^{n-M}$.
- By equating first couple of terms of Taylor's expansion,

$$\begin{aligned} \sigma_A &= \frac{\sum \text{poles of } P(s) - \sum \text{zeros of } P(s)}{n - M} \\ &= \frac{\sum_{j=1}^n (-p_j) - \sum_{i=1}^M (-z_i)}{n - M} \end{aligned}$$

Example

Sketch the root locus of the char. eqn: $1 + K \frac{s+1}{s(s+2)(s+4)^2} = 0$

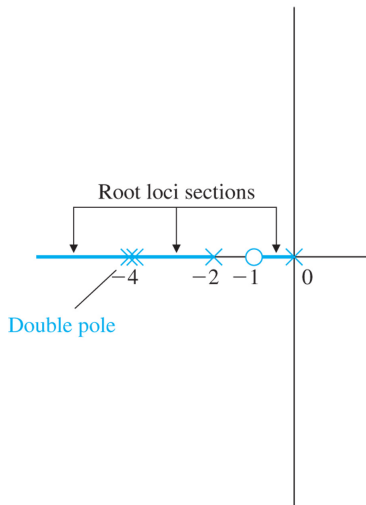
Step 1: Poles of $P(s)$: $s = 0, -2, -4, -4$;

Zeros of $P(s)$: $s = -1$

Step 2: Intervals on real axis:

- Order poles and zeros of $P(s)$: $-4, -4, -2, -1, 0$
- Examine from the right for intervals that are to the left of an odd number of poles and zeros
- $[-1, 0], [-4, -2], [-\infty, -4]$

Partial root locus after Step 2



Preliminary examples

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Parameter Design

"Negative" Root Locus

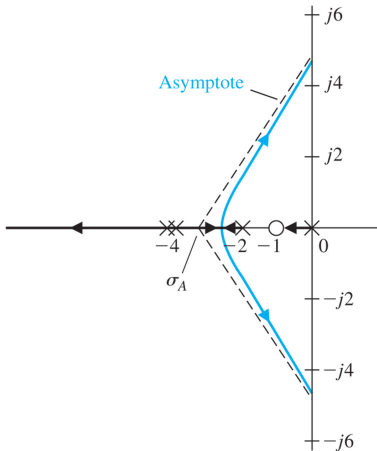
Example

Step 3: Asymptotes:

- Angles: $n - M = 4 - 1 = 3$.
Hence, angles are 60, 180, 300
Note that we already knew 180!
- Centroid: $\sigma_A = -3$

Example

Hence the complete root locus



What is the largest gain for which system is stable?

- Find values of K for which closed-loop poles lie on imaginary axis.
- Also find the positions of these closed-loop poles
- How can we do this?

- Routh-Hurwitz table

- Gains of interest correspond to zero rows, but remember not all zero rows correspond to closed-loop poles on $j\omega$ -axis

- Find the closed-loop pole positions by factorizing the auxiliary polynomial
(polynomial with coeffs in row above zero row)

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Example

Parameter
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Table 7.2 Seven Steps for Sketching a Root Locus

| Step | Related Equation or Rule |
|--|--|
| 1. Prepare the root locus sketch. | |
| (a) Write the characteristic equation so that the parameter of interest, K , appears as a multiplier. | $1 + KP(s) = 0.$ |
| (b) Factor $P(s)$ in terms of n poles and M zeros. | $1 + K \frac{\prod_{i=1}^M (s + z_i)}{\prod_{j=1}^n (s + p_j)} = 0.$ |
| (c) Locate the open-loop poles and zeros of $P(s)$ in the s -plane with selected symbols. | $\times = \text{poles}, \circ = \text{zeros}$ Locus begins at a pole and ends at a zero. |
| (d) Determine the number of separate loci, SL . | $SL = n$ when $n \geq M$; $n =$ number of finite poles, $M =$ number of finite zeros. |
| (e) The root loci are symmetrical with respect to the horizontal real axis. | |
| 2. Locate the segments of the real axis that are root loci. | Locus lies to the left of an odd number of poles and zeros. |
| 3. The loci proceed to the zeros at infinity along asymptotes centered at σ_A and with angles ϕ_A . | $\sigma_A = \frac{\sum(-p_j) - \sum(-z_i)}{n - M}.$ $\phi_A = \frac{2k + 1}{n - M} 180^\circ, k = 0, 1, 2, \dots, (n - M - 1).$ |
| 4. Determine the points at which the locus crosses the imaginary axis (if it does so). | Use Routh–Hurwitz criterion (see Section 6.2). |
| 5. Determine the breakaway point on the real axis (if any). | a) Set $K = p(s).$ b) Determine roots of $dp(s)/ds = 0$ or use graphical method to find maximum of $p(s).$ $\angle P(s) = 180^\circ + k360^\circ$ at $s = -p_j$ or $-z_i.$ |
| 6. Determine the angle of locus departure from complex poles and the angle of locus arrival at complex zeros, using the phase criterion. | |
| 7. Complete the root locus sketch. | |

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Example

Parameter
Design"Negative"
Root Locus

- 1 Write the denominator of the closed loop as

$$1 + KP(s), \text{ with } P(s) = \frac{\prod_{i=1}^M (s+z_i)}{\prod_{j=1}^n (s+p_j)}$$

Put an \times at the $-p_j$'s; put a \circ at the $-z_i$'s

Loci start at the \times 's and end at the \circ 's or at infinity

- 2 Parts of loci on real axis: to the left of an odd number of (real-valued) poles and (real-valued) zeros of $P(s)$
- 3 $n - M$ asymptotes as K gets large: Angles

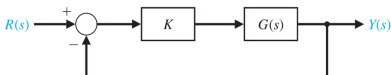
$$\phi = (2\ell + 1) \frac{180^\circ}{(n - M)} \quad \text{for } \ell = 0, 1, \dots, (n - M - 1)$$

Centroid:

$$\sigma_A = \frac{\sum_{j=1}^n (-p_j) - \sum_{i=1}^M (-z_i)}{n - M}$$

- 4 Roots on $j\omega$ -axis and corresponding K 's from zero rows and auxiliary polynomial of Routh-Hurwitz procedure

Using root locus for design



- For this loop, $P(s)$ in root locus procedure is $G(s)$
- What can we do if the root locus is not to our liking.
- Can we use the insight that we have developed to design a compensator $C(s) = \frac{\prod (s + \tilde{z}_i)}{\prod (s + \tilde{p}_j)}$ that we insert between the amplifier and $G(s)$ so that the root locus with $P(s) \propto C(s)G(s)$ is more to our liking?
- Note that in the compensated system
 - the zeros of $P(s)$ are the $-z_i$'s from $G(s)$ and the $-\tilde{z}_i$'s from $C(s)$
 - the poles of $P(s)$ are the $-p_j$'s from $G(s)$ and the $-\tilde{p}_j$'s from $C(s)$
- Let's attempt this for a VTOL aircraft

Compensator design for VTOL aircraft

- In this experiment we will work with a model for the vertical control system for a VTOL aircraft, such as the Harrier jump jet
- The transfer function of the process/plant can be approximated by $G(s) = \frac{1}{s(s-1)}$
- Do you notice anything interesting about this model?
- Tasks:
 - Sketch the root locus of a proportional controller
 - Highlight some features of that root locus

Compensator design for VTOL aircraft

- Is there a point on the root locus where the closed-loop poles are in a position that will correspond to satisfactory performance?
- That is, is there any satisfactory proportional controller?
- If not, use insight from the root locus sketching procedure to choose a compensator so that the closed-loop has a satisfactory root locus.
- That is, replace $G_C(s) = K$ by $G_C(s) = KC(s)$
- Start with something simple. Try $C(s) = \frac{1}{s+a}$, with a being a design parameter
- If that doesn't work, try $C(s) = \frac{s+b}{s+a}$, with both a and b being design parameters

General Procedure

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| (c) Locate the open-loop poles and zeros of $P(s)$ in the s -plane with selected symbols. | $\times = \text{poles}, \circ = \text{zeros}$ Locus begins at a pole and ends at a zero. |
| (d) Determine the number of separate loci, SL . | $SL = n$ when $n \geq M$; $n =$ number of finite poles, $M =$ number of finite zeros. |
| (e) The root loci are symmetrical with respect to the horizontal real axis. | |
| 2. Locate the segments of the real axis that are root loci. | Locus lies to the left of an odd number of poles and zeros. |
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| 4. Determine the points at which the locus crosses the imaginary axis (if it does so). | Use Routh–Hurwitz criterion (see Section 6.2). |
| 5. Determine the breakaway point on the real axis (if any). | a) Set $K = p(s).$ b) Determine roots of $dp(s)/ds = 0$ or use graphical method to find maximum of $p(s).$ $\angle P(s) = 180^\circ + k360^\circ$ at $s = -p_j$ or $-z_i.$ |
| 6. Determine the angle of locus departure from complex poles and the angle of locus arrival at complex zeros, using the phase criterion. | |
| 7. Complete the root locus sketch. | |

Steps 1 to 4

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Example

Parameter
Design"Negative"
Root Locus

- 1 Write the denominator of the closed loop as

$$1 + KP(s), \text{ with } P(s) = \frac{\prod_{i=1}^M (s+z_i)}{\prod_{j=1}^n (s+p_j)}$$

Put an \times at the $-p_j$'s; put a \circ at the $-z_i$'s

Loci start at the \times 's and end at the \circ 's or at infinity

- 2 Parts of loci on real axis: to the left of an odd number of (real-valued) poles and (real-valued) zeros of $P(s)$
- 3 $n - M$ asymptotes as K gets large: Angles

$$\phi = (2\ell + 1) \frac{180^\circ}{(n - M)} \quad \text{for } \ell = 0, 1, \dots, (n - M - 1)$$

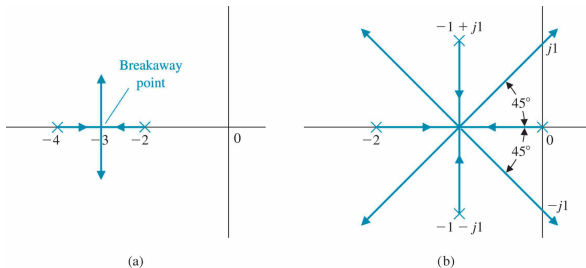
Centroid:

$$\sigma_A = \frac{\sum_{j=1}^n (-p_j) - \sum_{i=1}^M (-z_i)}{n - M}$$

- 4 Roots on $j\omega$ -axis and corresponding K 's from zero rows and auxiliary polynomial of Routh-Hurwitz procedure

Step 5 (approximated)

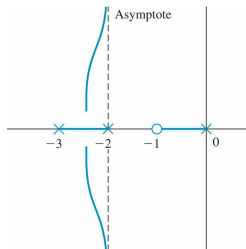
- Since complex poles appear in conjugate pairs, the root locus can leave the real axis only in even multiplicities; often just a pair
- Due to phase criterion, angles of break away are evenly spaced; when a pair, they depart at $\pm 90^\circ$; Examples:


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- Same insight applies when poles return to real axis
- You can calculate the point of departure/return, but often enough to use insight to approximate that point.

Step 5 (approx'd), Example

- Root locus of $1 + KP(s)$, with $P(s) = \frac{(s+1)}{s(s+2)(s+3)}$.
- Outcome of Steps 1-3 of root locus sketching procedure (Step 4 is not relevant in this case)



- Step 5 (approx'd): Poles depart real axis at right angles, somewhere between -2 and -3

Step 6

- For isolated real-valued poles of $P(s)$, root locus can only depart to the left or right
- For isolated real-valued zeros of $P(s)$, root locus can only arrive from the left or right
- What about complex-valued poles and zeros of $P(s)$?
- Need to determine angle of departure from (complex) poles and angle of arrival to (complex) zeros of $P(s)$
- These angles can be determined directly from the zeros and poles of $P(s)$ using a geometric analysis and the phase condition
- Angles of departure/arrival preserve conjugate symmetry
- However, we will not focus on such systems in this course

Step 7

- Join the segments that have been drawn with a smooth curve
- Curve should be as simple as possible
- Curve must respect conjugate symmetry of poles and zeros of a system with real inputs and real outputs

General Procedure

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| (d) Determine the number of separate loci, SL . | $SL = n$ when $n \geq M$; $n =$ number of finite poles, $M =$ number of finite zeros. |
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| 5. Determine the breakaway point on the real axis (if any). | a) Set $K = p(s).$ b) Determine roots of $dp(s)/ds = 0$ or use graphical method to find maximum of $p(s).$ $\angle P(s) = 180^\circ + k360^\circ$ at $s = -p_j$ or $-z_i.$ |
| 6. Determine the angle of locus departure from complex poles and the angle of locus arrival at complex zeros, using the phase criterion. | |
| 7. Complete the root locus sketch. | |

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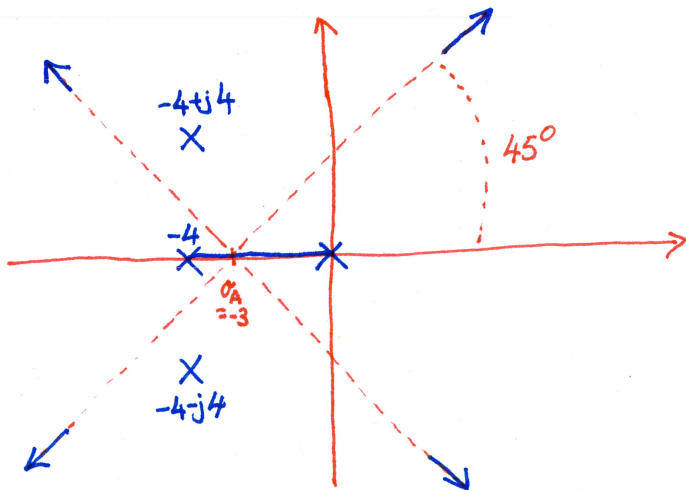
Sketch root locus of $1 + KP(s) = 0$ for $K \geq 0$, where

$$P(s) = \frac{1}{s^4 + 12s^3 + 64s^2 + 128s}$$

- 1 poles of $P(s)$: $0, -4, -4 \pm j4$;
zeros of $P(s)$: no finite zeros;
 $n - M = 4 - 0 \implies 4$ asymptotes
- 2 Segments of real axis: $[-4, 0]$
- 3 Angles of asymptotes: $45^\circ, 135^\circ, 225^\circ, 315^\circ$
Centroid: $(-4 - 4 - 4)/4 = -3$

Example, cont

Partial sketch from Steps 1–3



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Example

Parameter Design

"Negative" Root Locus

Example, cont

- 4 Closed loop denom: $s^4 + 12s^3 + 64s^2 + 128s + K = 0$
Routh table implies stability for $K < 568.89$.
Poles on $j\omega$ axis at $\pm j3.266$
- 5 Real poles will break away from real axis at right angles, somewhere between 0 and 4.
Using full version of Step 5 can show that breakaway point ≈ -1.577

Example, cont

Preliminary examples

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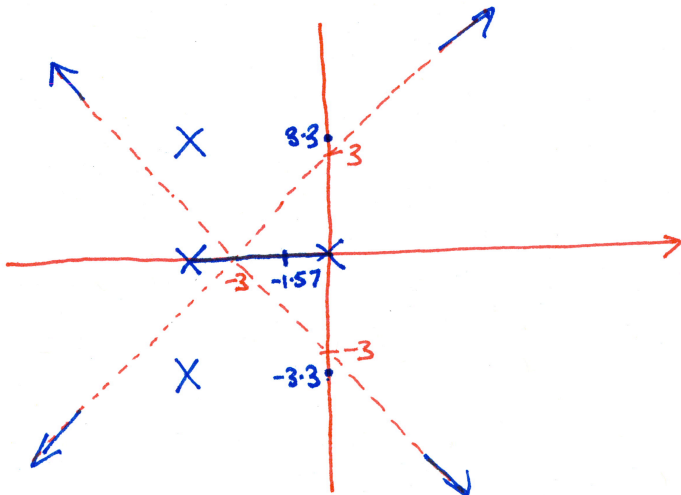
Step 7

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“Negative” Root Locus

Partial sketch from Steps 1–5



Example, cont

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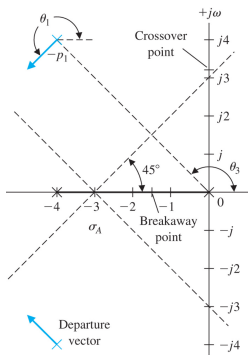
Step 6

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Example

Parameter Design

"Negative" Root Locus



- 6 Angle of departure from pole of $P(s)$ at $-4 + j4$:

Using the full version of Step 6 we can show that:

$$\theta_1 = -225^\circ$$

Example, cont

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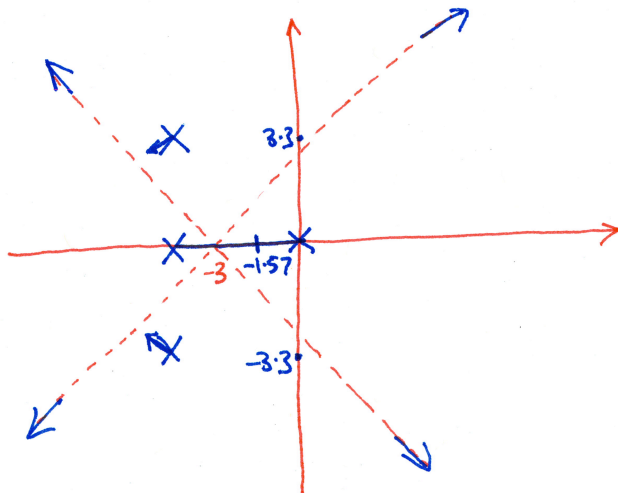
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Example

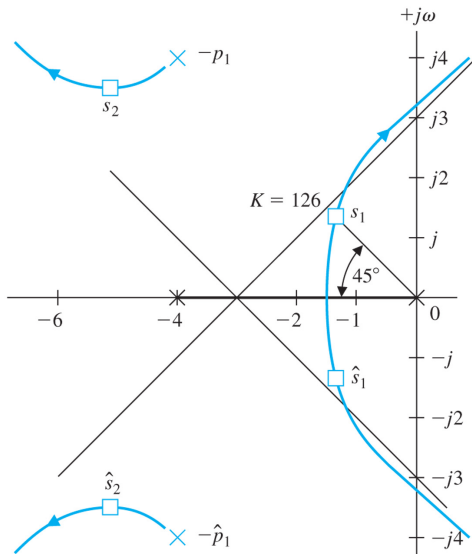
Parameter
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Partial sketch from Steps 1–6



Actual Root Locus



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Example

Parameter Design

"Negative" Root Locus

Parameter Design

- In the examples so far, $1 + KP(s)$ has been the denominator of a closed loop with
 - negative feedback
 - proportional control, with positive gain,
 - possibly, some compensation
 - That is, $G_c(s) = K_c C(s)$
 - Therefore, $P(s) \propto C(s)G(s)$ and $K = K_c K_G$
 - (Often, $K_c = K_{amp}$. However, it may include additional factors)
- However, same principles can also be applied to some other design parameters
- The key step is to rewrite the characteristic polynomial of the closed loop in the form $1 + \alpha P(s)$, where α is the (non-negative) parameter to be designed
- This is not always possible, but when it is possible it can be very useful

Parameter Design: Example

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Example

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Root Locus

- Suppose that the characteristic equation of the closed loop is:

$$s^3 + (3 + \alpha)s^2 + 3s + 6 = 0$$

- Suppose we are interested in root locus for $\alpha > 0$
- Rewrite as $s^3 + 3s^2 + 3s + 6 + \alpha s^2 = 0$. Hence,

$$1 + \alpha \frac{s^2}{s^3 + 3s^2 + 3s + 6} = 0$$

- Now sketch the root locus of $1 + \alpha P(s)$, where

$$P(s) = \frac{s^2}{s^3 + 3s^2 + 3s + 6}$$

“Negative” Root Locus

- Our root locus procedure has been for parameters that change from 0 to $+\infty$
- What if our parameter of interest goes from 0 to $-\infty$?
- The underlying principles remain the same
- For s_0 to be on the root locus, $1 + KP(s_0) = 0$.
- This implies
 - Magnitude condition: $|KP(s_0)| = 1$
 - Phase condition $\angle KP(s_0) = 180^\circ + \ell 360^\circ$
- However, since K is now negative, its phase is 180° . Therefore, some of the interpretations change.
- That said, interpretations can be derived in the same way as they were for the case of positive K , and they are quite familiar

Sketching Negative Root Locus

Tim Davidson

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From the 12th edition of the textbook

Table 7.9 Seven Steps for Sketching a Negative Gain Root Locus (color text denotes changes from root locus steps in Table 7.2)

| Step | Related Equation or Rule |
|---|--|
| 1. Prepare the root locus sketch. | |
| (a) Write the characteristic equation so that the parameter of interest, K , appears as a multiplier. | (a) $1 + KP(s) = 0$ |
| (b) Factor $P(s)$ in terms of n poles and M zeros | (b) $1 + K \frac{\prod_{i=1}^M (s + z_i)}{\prod_{j=1}^n (s + p_j)} = 0$ |
| (c) Locate the open-loop poles and zeros of $P(s)$ in the s -plane with selected symbols. | (c) $\times =$ poles, $\circ =$ zeros |
| (d) Determine the number of separate loci, SL . | (d) Locus begins at a pole and ends at a zero. $SL = n$ when $n \geq M$; $n =$ number of finite poles, $M =$ number of finite zeros. |
| (e) The root loci are symmetrical with respect to the horizontal real axis. | |

Sketching Negative Root Locus

Preliminary examples

Principles

Sketching the Root Locus, Steps 1–4

Steps 1 and 2

Review of Principles

Review of Steps 1, 2

Step 3

Step 4

Compensator design for VTOL aircraft

Sketching the Root Locus, Steps 5–7

Review of Steps 1–4

Step 5 (approx'd)

Step 6

Step 7

Example

Parameter Design

“Negative” Root Locus

2. Locate the segments of the real axis that are root loci.

3. The loci proceed to the zeros at infinity along asymptotes centered at σ_A and with angles ϕ_A .

4. Determine the points at which the locus crosses the imaginary axis (if it does so).

5. Determine the breakaway point on the real axis (if any).

6. Determine the angle of locus departure from complex at or poles and the angle of locus arrival at complex zeros using the phase criterion.

7. Complete the negative gain root locus sketch.

Locus lies to the left of an **even** number of poles and zeros.

$$\sigma_A = \frac{\sum_{j=1}^n (-p_j) - \sum_{i=1}^M (-z_i)}{n - M}$$

$$\phi_A = \frac{2k + 1}{n - M} 360^\circ, k = 0, 1, 2, \dots, (n - M - 1)$$

Use Routh-Hurwitz criterion (see Section 6.2).

a) Set $K = p(s)$

b) Determine roots of $dp(s)/ds = 0$ or use graphical method to find maximum of $p(s)$.

$$\angle P(s) = \pm k 360^\circ \text{ at } s = -p_j \text{ or } -z_i$$