EE3CL4: Introduction to Linear Control Systems
Section 4: Stability and Routh-Hurwitz Condition

Tim Davidson
McMaster University
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1 Stability
   Condition in terms of poles
   Condition in terms of denominator coefficients

2 Routh Hurwitz condition
   Basics
   Disk drive example
   Dealing with zeros
      Zeros in first column
      Zero rows

3 Using Routh Hurwitz for design
   Turning control of a tracked vehicle
A system is said to be stable if all bounded inputs $r(t)$ give rise to bounded outputs $y(t)$

Counterexamples

- Albert Collins, Jeff Beck (Yardbirds), Pete Townshend (The Who), Jimi Hendrix, Tom Morello (Rage Against the Machine), Kurt Cobain (Nirvana)
- Tacoma Narrows
Conditions for stability

\[ R(s) \rightarrow G(s) \rightarrow Y(s) \]

\[ y(t) = \int_{-\infty}^{\infty} g(\tau)r(t - \tau) \, d\tau \]

Let \( r(t) \) be such that \( |r(t)| \leq \bar{r} \)

\[ |y(t)| = \left| \int_{-\infty}^{\infty} g(\tau)r(t - \tau) \, d\tau \right| \]
\[ \leq \int_{-\infty}^{\infty} |g(\tau)r(t - \tau)| \, d\tau \]
\[ \leq \bar{r} \int_{-\infty}^{\infty} |g(\tau)| \, d\tau \]

Using this: system \( G(s) \) is stable iff \( \int_{-\infty}^{\infty} |g(\tau)| \, d\tau \) is finite
Condition in terms of poles?

We want \( \int_{-\infty}^{\infty} |g(\tau)| \, d\tau \) to be finite

Can we determine this from \( G(s) \)?

We can write a general rational transfer function in the form

\[
G(s) = \frac{K \prod_i (s + z_i)}{s^N \prod_k (s + \sigma_k) \prod_m (s^2 + 2\alpha_m s + (\alpha_m^2 + \omega_m^2))}
\]

Poles: 0, \(-\sigma_k\), \(-\alpha_m \pm j\omega_m\)

Assuming \( N = 0 \) and no repeated roots, the impulse response is

\[
g(t) = \sum_k A_k e^{-\sigma_k t} + \sum_m B_m e^{-\alpha_m t} \sin(\omega_m t + \theta_m)
\]

Stability requires \( \int |g(t)| \, dt \) to be bounded;
that requires \( \sigma_k > 0, \alpha_m > 0 \)

In fact, system is stable iff poles have negative real parts
Marginal stability

- Consider integrator: $G(s) = 1/s$; simple pole at origin
  - $y(t) = \int_{-\infty}^{\infty} r(t) \, dt$
  - if $r(t) = \cos(t)$, which is bounded, then $y(t) = \sin(t)$. Bounded
  - If $r(t) = u(t)$, which is bounded, then $y(t) = t$. Not bounded

- Consider $G(s) = 1/(s^2 + 1)$, simple poles at $s = \pm j1$
  - Unit step response: $u(t) - \cos(t)$. Bounded
  - What if $r(t)$ is a sinusoid of frequency $1/(2\pi)$ Hz? Not bounded

If $G(s)$ has a pole with positive real part, or a repeated pole on $j\omega$-axis output is always unbounded
Routh-Hurwitz condition

We have seen how to determine stability from the poles. Much easier than having to find impulse response and then determining if \[ \int |g(\tau)| \, d\tau < \infty \]

Can we determine stability without having to determine the poles?

Yes! Routh-Hurwitz condition
Routh-Hurwitz condition

Let $G(s) = \frac{p(s)}{q(s)}$, where

$$q(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$$
$$= a_n (s - r_1)(s - r_2) \ldots (s - r_n)$$

where $r_i$ are the roots of $q(s) = 0$.

By multiplying out, $q(s) = 0$ can be written as

$$q(s) = a_n s^n - a_n (r_1 + r_2 + \cdots + r_n) s^{n-1}$$
$$+ a_n (r_1 r_2 + r_2 r_3 + \ldots) s^{n-2}$$
$$- a_n (r_1 r_2 r_3 + r_1 r_2 r_4 + \ldots) s^{n-3}$$
$$+ \cdots + (-1)^n a_n (r_1 r_2 r_3 \ldots r_n) = 0$$

If all $r_i$ are real and in left half plane, what is sign of coeffs of $s^k$? the same!
That observation leads to a necessary condition.
Hence, not that useful for design

A more sophisticated analysis leads to the Routh-Hurwitz condition, which is necessary and sufficient
Hence, can be quite useful for design
Consider $G(s) = \frac{p(s)}{q(s)}$. Poles are solutions to $q(s) = 0$; i.e.,

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0 = 0$$

Construct a table of the form

<table>
<thead>
<tr>
<th>$s^n$</th>
<th>$a_n$</th>
<th>$a_{n-2}$</th>
<th>$a_{n-4}$</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^{n-1}$</td>
<td>$a_{n-1}$</td>
<td>$a_{n-3}$</td>
<td>$a_{n-5}$</td>
<td>\ldots</td>
</tr>
<tr>
<td>$s^{n-2}$</td>
<td>$b_{n-1}$</td>
<td>$b_{n-3}$</td>
<td>$b_{n-5}$</td>
<td>\ldots</td>
</tr>
<tr>
<td>$s^{n-3}$</td>
<td>$c_{n-1}$</td>
<td>$c_{n-3}$</td>
<td>$c_{n-5}$</td>
<td>\ldots</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\ldots</td>
</tr>
</tbody>
</table>
| $s^0$ | $h_{n-1}$ |}

where

$$b_{n-1} = \frac{a_{n-1} a_{n-2} - a_n a_{n-3}}{a_{n-1}} = \frac{-1}{a_{n-1}} a_n a_{n-2}$$

$$b_{n-3} = \frac{-1}{a_{n-1}} a_n a_{n-4}$$

$$c_{n-1} = \frac{-1}{b_{n-1}} b_{n-1} b_{n-3}$$
R-H cond: A first look

Now consider the table that we have just constructed

\[
\begin{array}{c|cccc}
  s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\
  s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
  s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \ldots \\
  s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \ldots \\
  s^0 & h_{n-1} & & & \\
\end{array}
\]

Loosely speaking:

- Number of roots in the right half plane is equal to the number of sign changes in the first column of the table
- Stability iff no sign changes in the first column

Now let’s move towards a more sophisticated statement
Stability (Revision)

Let \( G(s) = \frac{p(s)}{q(s)} \), where

\[
q(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0
\]

System is stable iff all poles of \( G(s) \) have negative real parts

Recall, poles are solutions to \( q(s) = 0 \)

Can we find a necessary and sufficient condition that depends only on \( a_k \) so that we don’t have to solve \( q(s) = 0 \)?
Routh-Hurwitz condition

1. Consider \( q(s) \) with \( a_n > 0 \)

\[
a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_1 s + a_0 = 0
\]

2. Construct a table of the form

<table>
<thead>
<tr>
<th>Row ( n )</th>
<th>( a_n )</th>
<th>( a_{n-2} )</th>
<th>( a_{n-4} )</th>
<th>( \ldots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row ( n-1 )</td>
<td>( a_{n-1} )</td>
<td>( a_{n-3} )</td>
<td>( a_{n-5} )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>Row ( n-2 )</td>
<td>( b_{n-1} )</td>
<td>( b_{n-3} )</td>
<td>( b_{n-5} )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>Row ( n-3 )</td>
<td>( c_{n-1} )</td>
<td>( c_{n-3} )</td>
<td>( c_{n-5} )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>Row 0</td>
<td>( h_{n-1} )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Procedure provided on the following slides

3. Count the sign changes in the first column

4. That is the number of roots in the right half plane

Stability (poles in LHP) iff \( a_k > 0 \) and all terms in first col. > 0
Constructing RH table

\[ a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots a_1 s + a_0 = 0 \]

Step 2.1: Arrange coefficients of \( q(s) \) in first two rows

<table>
<thead>
<tr>
<th>Row ( n )</th>
<th>( a_n )</th>
<th>( a_{n-2} )</th>
<th>( a_{n-4} )</th>
<th>( \ldots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row ( n - 1 )</td>
<td>( a_{n-1} )</td>
<td>( a_{n-3} )</td>
<td>( a_{n-5} )</td>
<td>( \ldots )</td>
</tr>
</tbody>
</table>
Determinant of a $2 \times 2$ matrix:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$
Constructing RH table

Step 2.2: Construct 3rd row using determinants of $2 \times 2$ matrices constructed from rows above

<table>
<thead>
<tr>
<th>Row $n$</th>
<th>$a_n$</th>
<th>$a_{n-2}$</th>
<th>$a_{n-4}$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row $n-1$</td>
<td>$a_{n-1}$</td>
<td>$a_{n-3}$</td>
<td>$a_{n-5}$</td>
<td>...</td>
</tr>
<tr>
<td>Row $n-2$</td>
<td>$b_{n-1}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$$b_{n-1} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$
Constructing RH table

Step 2.2, cont: Construct 3rd row using determinants of 2 × 2 matrices constructed from rows above

<table>
<thead>
<tr>
<th>Row $n$</th>
<th>$a_n$</th>
<th>$a_{n-2}$</th>
<th>$a_{n-4}$</th>
<th>…</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row $n-1$</td>
<td>$a_{n-1}$</td>
<td>$a_{n-3}$</td>
<td>$a_{n-5}$</td>
<td>…</td>
</tr>
<tr>
<td>Row $n-2$</td>
<td>$b_{n-1}$</td>
<td>$b_{n-3}$</td>
<td>…</td>
<td>…</td>
</tr>
</tbody>
</table>

$$b_{n-3} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$
Constructing RH table

Step 2.3: Construct 4th row using determinants of $2 \times 2$ matrices constructed from rows above

<table>
<thead>
<tr>
<th>Row $n$</th>
<th>$a_n$</th>
<th>$a_{n-2}$</th>
<th>$a_{n-4}$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row $n-1$</td>
<td>$a_{n-1}$</td>
<td>$a_{n-3}$</td>
<td>$a_{n-5}$</td>
<td>...</td>
</tr>
<tr>
<td>Row $n-2$</td>
<td>$b_{n-1}$</td>
<td>$b_{n-3}$</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>Row $n-3$</td>
<td>$c_{n-1}$</td>
<td>...</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$c_{n-1} = \frac{-1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}$

Step 2.4: Continue in this pattern.

Caveat: Requires all elements of first column to be non-zero
Will come back to that. Let’s see some examples, first
RH table, second-order system

\[ q(s) = a_2 s^2 + a_1 s + a_0 \]

| Row 2 | \[ a_2 \quad a_0 \] \\
| Row 1 | \[ a_1 \quad 0 \] \\
| Row 0 | \[ b_1 \] \\

\[ b_1 = \frac{-1}{a_1} \begin{vmatrix} a_2 & a_0 \\ a_1 & 0 \end{vmatrix} = a_0 \]

Therefore, second order system is stable iff all three denominator coefficients have the same sign.
RH table, third order system

\[ q(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0 \]

<table>
<thead>
<tr>
<th>Row 3</th>
<th>( a_3 )</th>
<th>( a_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 2</td>
<td>( a_2 )</td>
<td>( a_0 )</td>
</tr>
<tr>
<td>Row 1</td>
<td>( b_1 )</td>
<td>0</td>
</tr>
<tr>
<td>Row 0</td>
<td>( c_1 )</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ b_1 = \frac{-1}{a_2} \left| \begin{array}{cc} a_3 & a_1 \\ a_2 & a_0 \end{array} \right| \quad c_1 = \frac{-1}{b_1} \left| \begin{array}{cc} a_2 & a_0 \\ b_1 & 0 \end{array} \right| = a_0 \]

Therefore, if \( a_3 > 0 \), necessary and sufficient condition for third-order system to be stable is that \( a_2 > 0 \), \( b_1 > 0 \) and \( a_0 > 0 \).

\( b_1 > 0 \) is equiv. to \( a_2 a_1 > a_0 a_3 \), and this implies \( a_1 > 0 \).
Disk drive read control

Add velocity feedback (switch closed)

Using block diagram manipulation

\[ G_1(s) = \frac{5000}{s + 1000} \quad G_2(s) = \frac{1}{s(s + 20)} \]
Stability
Condition in terms of poles
Condition in terms of denominator coefficients
Routh Hurwitz condition
Basics
Disk drive example
Dealing with zeros
Zeros in first column
Zero rows
Using Routh Hurwitz for design
Turning control of a tracked vehicle

\[ T(s) = \frac{Y(s)}{R(s)} = \frac{K_a G_1(s) G_2(s)}{1 + K_a G_1(s) G_2(s)(1 + K_1 s)} \]

Hence, char. eqn:
\[ s^3 + 1020s^2 + (20000 + 5000K_a K_1) s + 5000K_a = 0 \]
Stabilizing values of $K_1$ and $K_a$

$$s^3 + 1020s^2 + (20000 + 5000K_aK_1)s + 5000K_a = 0$$

Routh table

<table>
<thead>
<tr>
<th>Row 3</th>
<th>1</th>
<th>20000 + 5000K_aK_1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 2</td>
<td>1020</td>
<td>5000K_a</td>
</tr>
<tr>
<td>Row 1</td>
<td>$b_1$</td>
<td>$b_1$</td>
</tr>
<tr>
<td>Row 0</td>
<td>5000K_a</td>
<td>5000K_a</td>
</tr>
</tbody>
</table>

$$b_1 = \frac{1020(20000 + 5000K_aK_1) - 5000K_a}{1020}$$

For stability we require $b_1 > 0$ and $K_a > 0$

For example, $K_a = 100$ and $K_1 = 0.05$.
That pair gives a 2% settling time of 260ms
**Reminder: Construction procedure**

<table>
<thead>
<tr>
<th>Row $k + 2$</th>
<th>$p_1$</th>
<th>$p_3$</th>
<th>$p_5$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row $k + 1$</td>
<td>$q_1$</td>
<td>$q_3$</td>
<td>$q_5$</td>
<td>...</td>
</tr>
<tr>
<td>Row $k$</td>
<td>$r_1$</td>
<td>$r_3$</td>
<td>$r_5$</td>
<td>...</td>
</tr>
</tbody>
</table>

To compute $r_3$, multiply

- $\frac{-1}{\text{first element of previous row}} = \frac{-1}{q_1}$ by $q_1$
- determinant of $2 \times 2$ matrix formed in the following way:
  - The first column contains the first elements of the two rows above the element to be calculated
  - The second column contains the elements of the two rows above that lie one column to the right of the element to be calculated

Therefore

$$r_3 = \frac{-1}{q_1} \begin{vmatrix} p_1 & p_5 \\ q_1 & q_5 \end{vmatrix} = \frac{-1}{q_1} (p_1 q_5 - q_1 p_5)$$
RH table, dealing with zeros

- The Routh-Hurwitz table encounters trouble when there is a zero in the first column.
- The next row involves \((-1/0)\) times a determinant.
- When some other elements in that row are not zero, we can proceed by replacing the zero by a small positive number \(\epsilon\), and then taking the limit as \(\epsilon \to 0\) after the table has been constructed.
- When a whole row is zero, we need to be a bit more sophisticated (later).
RH table, zero first element in non-zero row

As an example, consider

\[ q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10 \]

Routh table

<table>
<thead>
<tr>
<th>Row 5</th>
<th>1</th>
<th>2</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 4</td>
<td>2</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>Row 3</td>
<td>0</td>
<td>ε</td>
<td>6</td>
</tr>
<tr>
<td>Row 2</td>
<td>(c_1)</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>Row 1</td>
<td>(d_1)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Row 0</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ c_1 = \frac{4\epsilon - 12}{\epsilon} = \frac{-12}{\epsilon} \quad d_1 = \frac{6c_1 - 10\epsilon}{c_1} \rightarrow 6 \]

Two sign changes, hence unstable with two RHP poles
Zero row

- It is possible that the Routh Hurwitz procedure can produce a zero row

- While this complicates the procedure, it yields useful information for design

- Zero rows occur when polynomial has roots that are radially symmetric.

- Since the roots must also occur in conjugate pairs, this means that there is at least one pair of roots that is symmetric in the imaginary axis. (All roots are symmetric in the real axis.)
Zero row

- Common examples include:
  - equal and opposite roots on the real axis,
  - a pair of complex conjugate roots on the imaginary axis.

- The latter is more common, and more useful in design

- So how can we deal with the zero row?
Dealing with a zero row

- Routh Hurwitz procedure provides an “auxiliary polynomial”, $a(s)$, that contains the roots of interest as factors.

- The auxiliary polynomial is a factor of the original polynomial; i.e., $q(s) = a(s)b(s)$; $b(s)$ can be found by polyn. division.

- Given the symmetry of the roots, the auxiliary polynomial is of even order (or will have all its roots at the origin).

- The coefficients of the auxiliary polynomial appear in the row above the zero row.

- Let $k$ denote the row number of the row above the zero row, and let $c_{k,1}$, $c_{k,2}$, $c_{k,3}$ denote the coefficients in that row. The auxiliary polynomial is constructed as

$$a(s) = c_{k,1}s^k + c_{k,2}s^{k-2} + c_{k,3}s^{k-4} + \ldots$$

- Finally, we replace the zero row by the coefficients of the derivative of the auxiliary polynomial; i.e., the zero row is replaced by $kc_{k,1}$, $(k-2)c_{k,2}$, $(k-4)c_{k,3}$.
**Zero row example**

- \( q(s) = s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0 \)

- Construct table

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>24</th>
<th>-25</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 5</td>
<td>1</td>
<td>24</td>
<td>-25</td>
</tr>
<tr>
<td>Row 4</td>
<td>2</td>
<td>48</td>
<td>-50</td>
</tr>
<tr>
<td>Row 3</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

- Auxiliary polynomial: \( a(s) = 2s^4 + 48s^2 - 50 \)
  - This is actually a factor of \( q(s) \).
  - Using quadratic formula, roots of \( a(s) \) are \( s^2 = 1, -25 \)
  - Hence roots of \( a(s) \) are \( s = \pm 1, \pm j5 \)

- \( \frac{da(s)}{ds} = 8s^3 + 96s \)
  - Replace zero row by these coefficients

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>24</th>
<th>-25</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 5</td>
<td>1</td>
<td>24</td>
<td>-25</td>
</tr>
<tr>
<td>Row 4</td>
<td>2</td>
<td>48</td>
<td>-50</td>
</tr>
<tr>
<td>Row 3</td>
<td>8</td>
<td>96</td>
<td></td>
</tr>
</tbody>
</table>
Zero row example, cont

- Now complete the table in the usual way

<table>
<thead>
<tr>
<th>Row 1</th>
<th>112.7</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 2</td>
<td>24</td>
<td>−50</td>
</tr>
<tr>
<td>Row 3</td>
<td>8</td>
<td>96</td>
</tr>
<tr>
<td>Row 4</td>
<td>2</td>
<td>48</td>
</tr>
<tr>
<td>Row 5</td>
<td>1</td>
<td>24</td>
</tr>
</tbody>
</table>

- One sign change in first column.
  Indicates one root in right half plane.

- Recall $a(s)$ is a factor of $q(s)$.
  Indeed, by polyn division $q(s) = (s + 2)a(s)$
  We have seen that roots of $a(s)$ are $\pm 1$ and $\pm j5$.

- Hence $q(s)$ does indeed have one root with a positive real part.
Select $K$ and $a$ so that

- the closed-loop is stable, and
- the steady-state error due to a ramp is at most 24% of the magnitude of the command
Deal with stability first

Transfer function: \( T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} \)

Char. equation: \( s^4 + 8s^3 + 17s^2 + (K + 10)s + Ka = 0 \)

Routh table

<table>
<thead>
<tr>
<th>Row 4</th>
<th>1</th>
<th>17</th>
<th>Ka</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 3</td>
<td>8</td>
<td>K+10</td>
<td></td>
</tr>
<tr>
<td>Row 2</td>
<td>( b_3 )</td>
<td>Ka</td>
<td></td>
</tr>
<tr>
<td>Row 1</td>
<td>( c_3 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Row 0</td>
<td>Ka</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ b_3 = \frac{126 - K}{8} \quad c_3 = \frac{b_3(K + 10) - 8Ka}{b_3} \]

For stability we require \( b_3 > 0, c_3 > 0 \) and \( Ka > 0 \)
Stability region

For positive $K$, these constraints can be rewritten as

$$K < 126$$

$$a > 0$$

$$a < \frac{(K + 10)(126 - K)}{64K}$$

Region of stable parameters is between blue curves
Steady-state error to ramp

- For a ramp input \( r(t) = At \), we have \( e_{ss} = \frac{A}{K_v} \), where

\[
K_v = \lim_{s \to 0} sG_c(s)G(s) = \frac{Ka}{10}
\]

- Therefore, \( e_{ss} = \frac{A}{K_v} = \frac{10A}{Ka} \)

- To obtain \( e_{ss} < 0.24A \approx \frac{A}{4.167} \), we need \( K_v > 4.167 \)

- That means we need \( Ka > 41.67 \).

- Any \((K, a)\) pair in stable region with \( a > \frac{41.67}{K} \) will satisfy design constraints
Set of parameters with desired performance

For positive $K$,

- stability region is between the blue solid curves
- desired steady-state error region is above the black dashed curve and between the blue solid curves
- Design example: $K = 70$, $a = 0.6$, marked by $+$ sign
Ramp response

- Steady-state error criterion satisfied ($K_v = 4.200 > 4.167$)
- Transient time is quite long
What about settling times?

- That system takes a long time to settle.
- What does the step response look like?

![Step Response Graph]

- Could we have predicted the long settling time and large overshoot?
- We know that the settling time of each component of the response is related to the real parts of the poles.
Closed-loop poles and zeros

- One “fast” real pole
- One “reasonably slow” real pole that is close to a zero
- Complex conjugate pair with small real parts ($\approx 0.17$), and large angles with negative real axis ($\approx 87^\circ$). Conjugate pair dominates
- $2\%$ settling time of response to conjugate pair is $4$ time constants, $4(\frac{1}{0.17}) \approx 23$
- Damping ratio of conjugate pair $\zeta = \cos(87^\circ) \approx 0.053$. Corresponds to overshoot of $\approx 85\% = 0.85$
- Actual performance of the fourth-order system with one zero is quite close to this guidance from second-order system with no finite zeros
Managing settling times

- It is disappointing that using Routh-Hurwitz for design gives us control over stability, but does not allow us to manage settling times.
- Particularly disappointing, given “shape” of region to left of a certain real number is same as that to left of origin.
- Can we do anything?
- Recall closed-loop denominator polynomial is:

\[ q(s) = a_n \prod_{j=1}^{n} (s - (-p_{cl,j})) \]

where \(-p_{cl,j}\) are the closed-loop poles.
- If we want \(q(s)\) to have closed-loop poles to left of \(-\sigma\), then we need

\[ \tilde{q}(s) = a_n \prod_{j=1}^{n} (s - (-p_{cl,j} + \sigma)) \]

to be stable.
Managing settling times, II

- That means we want

\[ \tilde{q}(s) = q(s - \sigma) = a_n(s - \sigma)^n + a_{n-1}(s - \sigma)^{n-1} + \cdots + a_1(s - \sigma) + a_0 \]

to be stable

- Using binomial theorem, \((x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\), we can write

\[ \tilde{q}(s) = \tilde{a}_n s^n + \tilde{a}_{n-1} s^{n-1} + \cdots + \tilde{a}_1 s + \tilde{a}_0 \]

where \( \binom{n}{k} = \frac{n!}{(n-k)!k!} \)

- Now you can apply Routh-Hurwitz procedure to \( \tilde{q}(s) \)

- What happens if there is a pair of closed-loop poles with real parts equal to \(-\sigma\)? A zero row in the table
Application to tracked vehicle

- Settling time of component due to complex-conjugate pair in current design is around 23 seconds
- Can we reduce this? Let’s try to get it down to around 16 seconds.
- That means that we want poles to the left of \(-\sigma = -0.25\)
- \(\tilde{q}(s) \approx s^4 + 7s^3 + 11.375s^2 + (K + 2.9375)s + \tilde{a}_0(K, a)\), where \(\tilde{a}_0(K, a) = Ka - 0.25K - 1.3086\)

- Routh table

<table>
<thead>
<tr>
<th>Row</th>
<th>(\tilde{b}_3)</th>
<th>(\tilde{c}_3)</th>
<th>(\tilde{a}_0(K, a))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 4</td>
<td>1</td>
<td>11.375</td>
<td>(\tilde{a}_0(K, a))</td>
</tr>
<tr>
<td>Row 3</td>
<td>7</td>
<td>(K + 2.9375)</td>
<td>(\tilde{b}_3)</td>
</tr>
<tr>
<td>Row 2</td>
<td>(\tilde{b}_3)</td>
<td>(\tilde{a}_0(K, a))</td>
<td></td>
</tr>
<tr>
<td>Row 1</td>
<td>(\tilde{c}_3)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Row 0</td>
<td>(\tilde{a}_0(K, a))</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[\tilde{b}_3 = \frac{76.6876 - K}{7}, \quad \tilde{c}_3 = \frac{\tilde{b}_3(K + 2.9375) - 7\tilde{a}_0(K, a)}{\tilde{b}_3}\]

- Hence, we require \(\tilde{b}_3 > 0, \tilde{c}_3 > 0\) and \(\tilde{a}_0(K, a) > 0\)
Settling time region

- For positive $K$, the constraints can be rewritten as
  \[ K < 76.6876 \]
  \[ a > 0.25 + \frac{1.3086}{K} \]
  \[ a < \frac{(76.6896 - K)(K + 2.9375) + 12.25K + 64.12}{49K} \]

- Region between the red lines (inside blue region)
- For steady state error we need $a > \frac{41.67}{K}$; above dashed line

- Not much choice this time. Example: $K = 47, a = 0.9$, marked by *
Closed-loop poles and zeros

Since complex conjugate pair of poles still dominates, this suggests new design (red) will have

- Somewhat reduced settling time (poles are to left of 0.25)
- Slightly reduced overshoot in step response (angle slightly reduced)
• Settling time has come down; not far from 16 seconds
• Overshoot has been reduced a little bit
• Steady-state error criterion still satisfied
  \( K_v = 4.230 > 4.167 \)
• Transient time has been reduced
Ramp response, zoomed in
Discussion

- Transient times have been reduced, but only a little bit
- Unfortunately, not much room to do any better

Current compensator is $G_c(s) = \frac{s+a}{s+1}$ and gain is $K$

To get significantly better performance, we need to be able to adjust the pole of the compensator, as well as the zero and the gain

For example, consider $G_c(s) = \frac{(s+a)}{(s+b)} = \frac{s+1.9}{s+6}$ with $K = 133$.

Note $K_v = \lim_{s \to 0} sG_c(s)G(s) \approx 4.212 > 4.167$, so ramp steady-state error criterion still satisfied
Since complex conjugate pair of poles still dominates, this suggests new design (green) will have

- Significantly reduced settling time (dominant poles further to left)
- Somewhat reduced overshoot in step response (smaller angle)
• Insight from pole and zero positions reasonably accurate
Steady-state error criterion still satisfied for new design ($K_v = 4.212 > 4.167$)

- Transient time reduced
Discussion, cont.

- New design does indeed do better in our chosen metrics.
- However, design problem now has three parameters, not two: zero pos’n \((-a)\), gain \((K)\), and also pole pos’n \((-b)\).
- Hence desired parameter region is a volume, not an area.
- Routh-Hurwitz procedure and resulting equations for boundaries of desired parameter space get even more complicated.
- Is there an easier way to gain insight into how to choose the pole and zero positions, and to choose the gain?
- One option is the root locus procedure that we will develop in the next section.