

Tim Davidson

Stability

Condition in terms of poles

Condition in terms of denominator coefficients

Routh Hurwitz condition

Basics

Disk drive example

Dealing with zeros

Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle

EE3CL4: Introduction to Linear Control Systems Section 4: Stability and Routh-Hurwitz Condition

Tim Davidson

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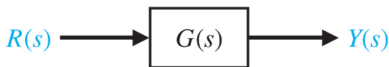
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A system is said to be stable if all bounded inputs $r(t)$ give rise to bounded outputs $y(t)$

Counterexamples

- Albert Collins, Jeff Beck (Yardbirds), Pete Townshend (The Who), Jimi Hendrix, Tom Morello (Rage Against the Machine), Kurt Cobain (Nirvana)
- Tacoma Narrows

Conditions for stability

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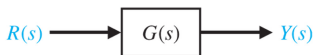
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$$y(t) = \int_{-\infty}^{\infty} g(\tau)r(t - \tau) d\tau$$

Let $r(t)$ be such that $|r(t)| \leq \bar{r}$

$$\begin{aligned} |y(t)| &= \left| \int_{-\infty}^{\infty} g(\tau)r(t - \tau) d\tau \right| \\ &\leq \int_{-\infty}^{\infty} |g(\tau)r(t - \tau)| d\tau \\ &\leq \bar{r} \int_{-\infty}^{\infty} |g(\tau)| d\tau \end{aligned}$$

Using this: system $G(s)$ is stable iff $\int_{-\infty}^{\infty} |g(\tau)| d\tau$ is finite

Condition in terms of poles?

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We want $\int_{-\infty}^{\infty} |g(\tau)| d\tau$ to be finite

Can we determine this from $G(s)$?

We can write a general rational transfer function in the form

$$G(s) = \frac{K \prod_i (s + z_i)}{s^N \prod_k (s + \sigma_k) \prod_m (s^2 + 2\alpha_m s + (\alpha_m^2 + \omega_m^2))}$$

Poles: $0, -\sigma_k, -\alpha_m \pm j\omega_m$

Assuming $N = 0$ and no repeated roots, the impulse response is zero for $t < 0$ and for $t \geq 0$ it is

$$g(t) = \sum_k A_k e^{-\sigma_k t} + \sum_m B_m e^{-\alpha_m t} \sin(\omega_m t + \theta_m)$$

Stability requires $\int_{-\infty}^{\infty} |g(t)| dt$ to be bounded; that requires $\sigma_k > 0, \alpha_m > 0$

In fact, system is stable iff poles have negative real parts

Marginal stability

Stability

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- Consider integrator: $G(s) = 1/s$; simple pole at origin
 - $y(t) = \int_{-\infty}^t r(\lambda) d\lambda$
 - if $r(t) = \cos(t)$, which is bounded, then $y(t) = \sin(t)$. Bounded
 - If $r(t) = u(t)$, which is bounded, then $y(t) = t$. Not bounded
- Consider $G(s) = 1/(s^2 + 1)$, simple poles at $s = \pm j1$
 - Unit step response: $u(t) - \cos(t)$. Bounded
 - What if $r(t)$ is a sinusoid of frequency $1/(2\pi)$ Hz? Not bounded

If $G(s)$ has a pole with positive real part, or a repeated pole on $j\omega$ -axis output is always unbounded

Stability

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Routh-Hurwitz condition

We have seen how to determine stability from the poles.

Much easier than having to find impulse response and then determining if $\int_{-\infty}^{\infty} |g(\tau)| d\tau < \infty$

Can we determine stability without having to determine the poles?

Yes! Routh-Hurwitz condition

Routh-Hurwitz condition

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Let $G(s) = \frac{p(s)}{q(s)}$, where

$$\begin{aligned} q(s) &= a_n s^n + a_{n-1} s^{n-1} + \dots a_1 s + a_0 \\ &= a_n (s - r_1)(s - r_2) \dots (s - r_n) \end{aligned}$$

where r_i are the roots of $q(s) = 0$.

By multiplying out, $q(s) = 0$ can be written as

$$\begin{aligned} q(s) &= a_n s^n - a_n (r_1 + r_2 + \dots + r_n) s^{n-1} \\ &\quad + a_n (r_1 r_2 + r_2 r_3 + \dots) s^{n-2} \\ &\quad - a_n (r_1 r_2 r_3 + r_1 r_2 r_4 + \dots) s^{n-3} \\ &\quad + \dots + (-1)^n a_n (r_1 r_2 r_3 \dots r_n) = 0 \end{aligned}$$

If all r_i are real and in left half plane, what is sign of coeffs of s^k ?
the same!

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Routh-Hurwitz condition

That observation leads to a necessary condition.

Hence, not that useful for design

A more sophisticated analysis leads to the Routh-Hurwitz condition, which is necessary and sufficient

Hence, can be quite useful for design

R-H cond: A first look

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Consider $G(s) = \frac{p(s)}{q(s)}$. Poles are solutions to $q(s) = 0$; i.e.,

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_1 s + a_0 = 0$$

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Construct a table of the form

Row n	a_n	a_{n-2}	a_{n-4}	\dots
Row $n - 1$	a_{n-1}	a_{n-3}	a_{n-5}	\dots
Row $n - 2$	b_{n-1}	b_{n-3}	b_{n-5}	\dots
Row $n - 3$	c_{n-1}	c_{n-3}	c_{n-5}	\dots
\vdots	\vdots	\vdots	\vdots	\dots
Row 0	h_{n-1}			

where

$$b_{n-1} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$b_{n-3} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} \quad c_{n-1} = \frac{-1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}$$

R-H cond: A first look

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Now consider the table that we have just constructed

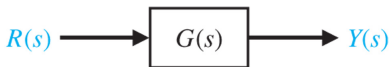
Row n	a_n	a_{n-2}	a_{n-4}	\dots
Row $n - 1$	a_{n-1}	a_{n-3}	a_{n-5}	\dots
Row $n - 2$	b_{n-1}	b_{n-3}	b_{n-5}	\dots
Row $n - 3$	c_{n-1}	c_{n-3}	c_{n-5}	\dots
\vdots	\vdots	\vdots	\vdots	\dots
Row 0	h_{n-1}			

Loosely speaking:

- Number of roots in the right half plane is equal to the number of sign changes in the first column of the table
- Stability iff no sign changes in the first column

Now let's move towards a more sophisticated statement

Stability (Revision)



Let $G(s) = \frac{p(s)}{q(s)}$, where

$$q(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$$

System is stable iff all poles of $G(s)$ have negative real parts

Recall, poles are solutions to $q(s) = 0$

Can we find a necessary and sufficient condition that depends only on the set of coefficients $\{a_k\}_{k=0}^n$ so that we don't have to solve $q(s) = 0$?

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Turning control of a tracked vehicle

Routh-Hurwitz condition

- 1 Consider $q(s)$ with $a_n > 0$

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots a_1 s + a_0 = 0$$

- 2 Construct a table of the form

Row n	a_n	a_{n-2}	a_{n-4}	\dots
Row $n - 1$	a_{n-1}	a_{n-3}	a_{n-5}	\dots
Row $n - 2$	b_{n-1}	b_{n-3}	b_{n-5}	\dots
Row $n - 3$	c_{n-1}	c_{n-3}	c_{n-5}	\dots
\vdots	\vdots	\vdots	\vdots	\dots
Row 0	h_{n-1}			

Procedure provided on the following slides

- 3 Count the sign changes in the first column
- 4 That is the number of roots in the right half plane

Stability (poles in LHP) iff all terms in first col. have same sign

Constructing RH table

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Stability

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Turning control of a tracked vehicle

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_1 s + a_0 = 0$$

Step 2.1: Arrange coefficients of $q(s)$ in first two rows

$$\begin{array}{l|llll} \text{Row } n & a_n & a_{n-2} & a_{n-4} & \dots \\ \text{Row } n-1 & a_{n-1} & a_{n-3} & a_{n-5} & \dots \end{array}$$

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Determinant of a 2×2 matrix:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

Constructing RH table

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Step 2.2: Construct 3rd row using determinants of 2×2 matrices constructed from rows above

$$\begin{array}{l} \text{Row } n \\ \text{Row } n - 1 \\ \text{Row } n - 2 \end{array} \left| \begin{array}{cccc} a_n & a_{n-2} & a_{n-4} & \dots \\ a_{n-1} & a_{n-3} & a_{n-5} & \dots \\ b_{n-1} & & & \end{array} \right.$$

$$b_{n-1} = \frac{-1}{a_{n-1}} \left| \begin{array}{cc} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{array} \right|$$

Constructing RH table

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Step 2.2, cont: Construct 3rd row using determinants of 2×2 matrices constructed from rows above

$$\begin{array}{l|llll}
 \text{Row } n & a_n & a_{n-2} & a_{n-4} & \dots \\
 \text{Row } n-1 & a_{n-1} & a_{n-3} & a_{n-5} & \dots \\
 \text{Row } n-2 & b_{n-1} & b_{n-3} & \dots &
 \end{array}$$

$$b_{n-3} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

Constructing RH table

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Step 2.3: Construct 4th row using determinants of 2×2 matrices constructed from rows above

$$\begin{array}{l|llll}
 \text{Row } n & a_n & a_{n-2} & a_{n-4} & \dots \\
 \text{Row } n-1 & a_{n-1} & a_{n-3} & a_{n-5} & \dots \\
 \text{Row } n-2 & b_{n-1} & b_{n-3} & \dots & \\
 \text{Row } n-3 & c_{n-1} & \dots & &
 \end{array}$$

$$c_{n-1} = \frac{-1}{b_{n-1}} \left| \begin{array}{cc} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{array} \right|$$

Step 2.4: Continue in this pattern.

Caveat: Requires all elements of first column to be non-zero
Will come back to that. Let's see some examples, first

RH table, second-order system

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$$q(s) = a_2 s^2 + a_1 s + a_0$$

$$\begin{array}{l|ll} \text{Row 2} & a_2 & a_0 \\ \text{Row 1} & a_1 & 0 \\ \text{Row 0} & b_1 & \end{array}$$

$$b_1 = \frac{-1}{a_1} \left| \begin{array}{ll} a_2 & a_0 \\ a_1 & 0 \end{array} \right| = a_0$$

Therefore, second order system is stable iff all three denominator coefficients have the same sign

RH table, third order system

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Stability

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$$q(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

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Zero rows

$$\begin{array}{l|ll} \text{Row 3} & a_3 & a_1 \\ \text{Row 2} & a_2 & a_0 \\ \text{Row 1} & b_1 & 0 \\ \text{Row 0} & c_1 & 0 \end{array}$$

Using Routh Hurwitz for design

Turning control of a tracked vehicle

$$b_1 = \frac{-1}{a_2} \begin{vmatrix} a_3 & a_1 \\ a_2 & a_0 \end{vmatrix} \quad c_1 = \frac{-1}{b_1} \begin{vmatrix} a_2 & a_0 \\ b_1 & 0 \end{vmatrix} = a_0$$

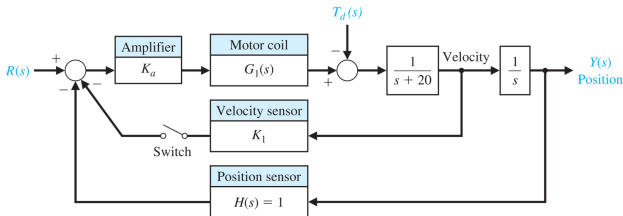
Therefore, if $a_3 > 0$, necessary and sufficient condition for third-order system to be stable is that $a_2 > 0$, $b_1 > 0$ and $a_0 > 0$.

$b_1 > 0$ is equiv. to $a_2 a_1 > a_0 a_3$, and this implies $a_1 > 0$.

Disk drive read control

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Add velocity feedback (switch closed)



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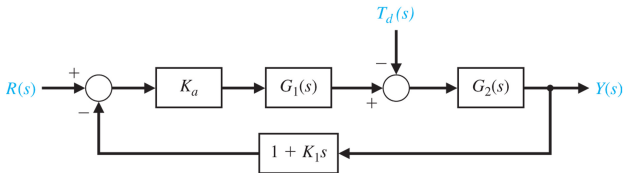
Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle

Using block diagram manipulation



$$G_1(s) = \frac{5000}{s + 1000}$$

$$G_2(s) = \frac{1}{s(s + 20)}$$

Closed loop

Stability

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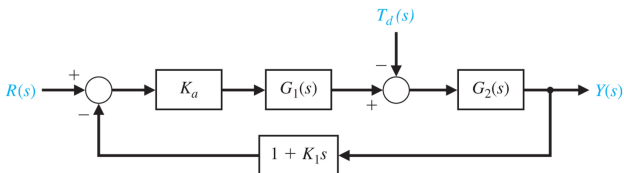
Dealing with zeros

Zeros in first column

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Using Routh Hurwitz for design

Turning control of a tracked vehicle



$$T(s) = \frac{Y(s)}{R(s)} = \frac{K_a G_1(s) G_2(s)}{1 + K_a G_1(s) G_2(s) (1 + K_1 s)}$$

Hence, char. eqn:

$$s^3 + 1020s^2 + (20000 + 5000K_a K_1)s + 5000K_a = 0$$

Stabilizing values of K_1 and K_a

$$s^3 + 1020s^2 + (20000 + 5000K_aK_1)s + 5000K_a = 0$$

Routh table

Row 3	1	20000 + 5000 K_aK_1
Row 2	1020	5000 K_a
Row 1	b_1	
Row 0	5000 K_a	

$$b_1 = \frac{1020(20000 + 5000K_aK_1) - 5000K_a}{1020}$$

For stability we require $b_1 > 0$ and $K_a > 0$. That is,

$$K_a > 0 \quad \text{and} \quad K_1 > \frac{K_a - 4080}{1020K_a}$$

Note that for $0 < K_a < 4080$, any positive K_1 will stabilize the loop, and some negative ones will, too.

For example, $K_a = 100$ and $K_1 = 0.05$.

That pair gives a 2% settling time of 260ms

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Using Routh Hurwitz for design

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Reminder: Construction procedure

$$\begin{array}{l|llll}
 \text{Row } k + 2 & p_1 & p_3 & p_5 & \dots \\
 \text{Row } k + 1 & q_1 & q_3 & q_5 & \dots \\
 \text{Row } k & r_1 & r_3 & r_5 & \dots
 \end{array}$$

To compute r_3 , multiply

- $\frac{-1}{\text{first element of previous row}} = \frac{-1}{q_1}$ by
- determinant of 2×2 matrix formed in the following way:
 - The first column contains the first elements of the two rows above the element to be calculated
 - The second column contains the elements of the two rows above that lie one column to the right of the element to be calculated
- Therefore

$$r_3 = \frac{-1}{q_1} \begin{vmatrix} p_1 & p_5 \\ q_1 & q_5 \end{vmatrix} = \frac{-1}{q_1} (p_1 q_5 - q_1 p_5)$$

RH table, dealing with zeros

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Zeros in first column

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Turning control of a tracked vehicle

- The Routh-Hurwitz table encounters trouble when there is a zero in the first column
- The next row involves $(-1/0)$ times a determinant
- When some other elements in that row are not zero, we can proceed by replacing the zero by a small positive number ϵ , and then taking the limit as $\epsilon \rightarrow 0$ after the table has been constructed.
- When a whole row is zero, we need to be a bit more sophisticated (later)

RH table, zero first element in non-zero row

As an example, consider

$$q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

Routh table

Row 5	1	2	11
Row 4	2	4	10
Row 3	$0 \leftarrow \epsilon$	6	0
Row 2	c_1	10	0
Row 1	d_1	0	0
Row 0	10	0	0

$$c_1 = \frac{4\epsilon - 12}{\epsilon} = \frac{-12}{\epsilon} \quad d_1 = \frac{6c_1 - 10\epsilon}{c_1} \rightarrow 6$$

Two sign changes, hence unstable with two RHP poles

Zero row

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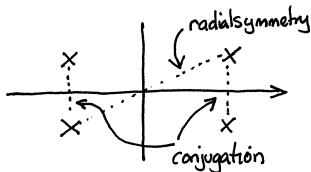
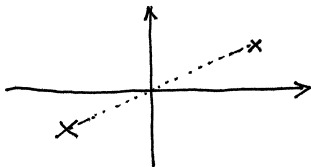
Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle

- It is possible that the Routh Hurwitz procedure can produce a zero row
- While this complicates the procedure, it yields useful information for design
- Zero rows occur when polynomial has roots that are radially symmetric.
- Since the roots must also occur in conjugate pairs, this means that there is at least one pair of roots that is symmetric in the imaginary axis.
(All roots are symmetric in the real axis.)



Zero row

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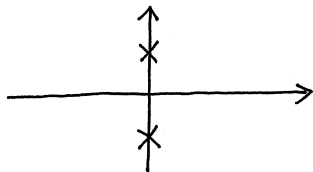
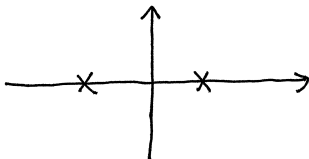
Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle

- Common examples include:
 - equal and opposite roots on the real axis,
 - a pair of complex conjugate roots on the imaginary axis.
- The latter is more common, and more useful in design



- So how can we deal with the zero row?

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Dealing with a zero row

- Routh Hurwitz procedure provides an “auxiliary polynomial”, $a(s)$, that contains the roots of interest as factors
- The auxiliary polynomial is a factor of the original polynomial; i.e., $q(s) = a(s)b(s)$; $b(s)$ can be found by polyn. division
- Given the symmetry of the roots, the auxiliary polynomial is of even order (or will have all its roots at the origin)
- The coefficients of the auxiliary polynomial appear in the row above the zero row
- Let k denote the row number of the row above the zero row, and let $c_{k,1}$, $c_{k,2}$, $c_{k,3}$ denote the coefficients in that row. The auxiliary polynomial is constructed as

$$a(s) = c_{k,1}s^k + c_{k,2}s^{k-2} + c_{k,3}s^{k-4} + \dots$$

- Finally, we replace the zero row by the coefficients of the derivative of the auxiliary polynomial; i.e., the zero row is replaced by $kc_{k,1}$, $(k-2)c_{k,2}$, $(k-4)c_{k,3}$

Zero row example

- $q(s) = s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$

- Construct table

Row 5	1	24	-25
Row 4	2	48	-50
Row 3	0	0	

- Auxiliary polynomial: $a(s) = 2s^4 + 48s^2 - 50$

This is actually a factor of $q(s)$.

Using quadratic formula, roots of $a(s)$ are $s^2 = 1, -25$

Hence roots of $a(s)$ are $s = \pm 1, \pm j5$

- $\frac{da(s)}{ds} = 8s^3 + 96s.$

Replace zero row by these coefficients

Row 5	1	24	-25
Row 4	2	48	-50
Row 3	8	96	

Zero row example, cont

Tim Davidson

Stability

Condition in terms of poles

Condition in terms of denominator coefficients

Routh Hurwitz condition

Basics

Disk drive example

Dealing with zeros

Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle

- Now complete the table in the usual way

Row 5	1	24	-25
Row 4	2	48	-50
Row 3	8	96	
Row 2	24	-50	
Row 1	112.7	0	
Row 0	-50		

- One sign change in first column.
Indicates one root in right half plane.
- Recall $a(s)$ is a factor of $q(s)$.
Indeed, by polyn division $q(s) = (s + 2)a(s)$
We have seen that roots of $a(s)$ are ± 1 and $\pm j5$.
- Hence $q(s)$ does indeed have one root with a positive real part.

Turning control of a tracked vehicle

Stability

Condition in terms of poles

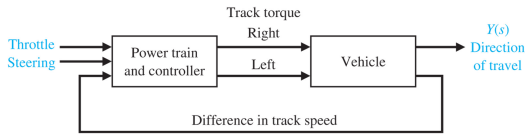
Condition in terms of denominator coefficients

Routh Hurwitz condition

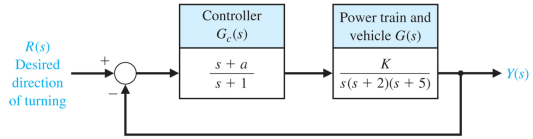
- Basics
- Disk drive example
- Dealing with zeros
- Zeros in first column
- Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle



(a)



(b)

Select K and a so that

- the closed-loop is stable, and
- the steady-state error due to a ramp is at most 24% of the magnitude of the command

Deal with stability first

Tim Davidson

Stability

Condition in terms of poles

Condition in terms of denominator coefficients

Routh Hurwitz condition

Basics

Disk drive example

Dealing with zeros

Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle

$$\text{Transfer function: } T(s) = \frac{G_c(s)G(s)}{1+G_c(s)G(s)}$$

$$\text{Char. equation: } s^4 + 8s^3 + 17s^2 + (K + 10)s + Ka = 0$$

Routh table

Row 4	1	17	Ka
Row 3	8	K + 10	
Row 2	b_3	Ka	
Row 1	c_3		
Row 0	Ka		

$$b_3 = \frac{126 - K}{8} \quad c_3 = \frac{b_3(K + 10) - 8Ka}{b_3}$$

For stability we require $b_3 > 0$, $c_3 > 0$ and $Ka > 0$

Stability region

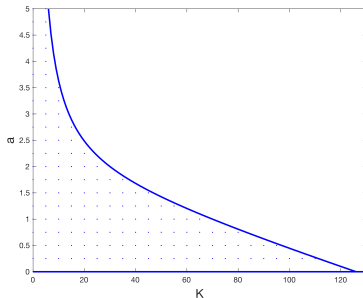
For positive K , these constraints can be rewritten as

$$K < 126$$

$$a > 0$$

$$a < \frac{(K + 10)(126 - K)}{64K}$$

Region of stable parameters is between blue curves



Stability

Condition in terms of poles

Condition in terms of denominator coefficients

Routh Hurwitz condition

Basics

Disk drive example

Dealing with zeros

Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle

Steady-state error to ramp

Stability

Condition in terms of poles

Condition in terms of denominator coefficients

Routh Hurwitz condition

Basics

Disk drive example

Dealing with zeros

Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle

- For a ramp input $r(t) = At$, we have $e_{ss} = \frac{A}{K_V}$, where

$$K_V = \lim_{s \rightarrow 0} sG_c(s)G(s) = \frac{Ka}{10}$$

- Therefore, $e_{ss} = \frac{A}{K_V} = \frac{10A}{Ka}$
- To obtain $e_{ss} < 0.24A \simeq \frac{A}{4.167}$, we need $K_V > 4.167$
- That means we need $Ka > 41.67$.
- Any (K, a) pair in stable region with $a > \frac{41.67}{K}$ will satisfy design constraints

Tim Davidson

Stability

Condition in terms of poles

Condition in terms of denominator coefficients

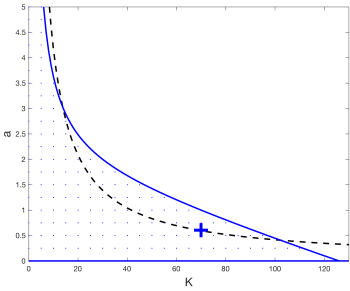
Routh Hurwitz condition

- Basics
- Disk drive example
- Dealing with zeros
- Zeros in first column
- Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle

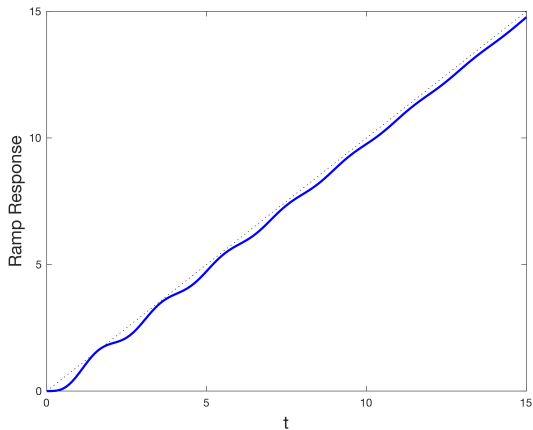
Set of parameters with desired performance



For positive K ,

- stability region is between the blue solid curves
- desired steady-state error region is above the black dashed curve and between the blue solid curves
- Design example: $K = 70, a = 0.6$, marked by + sign

Ramp response



- Steady-state error criterion satisfied ($K_v = 4.200 > 4.167$)
- Transient time is quite long

Stability

Condition in terms of poles

Condition in terms of denominator coefficients

Routh Hurwitz condition

Basics

Disk drive example

Dealing with zeros

Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle

What about settling times?

Stability

Condition in terms of poles

Condition in terms of denominator coefficients

Routh Hurwitz condition

Basics

Disk drive example

Dealing with zeros

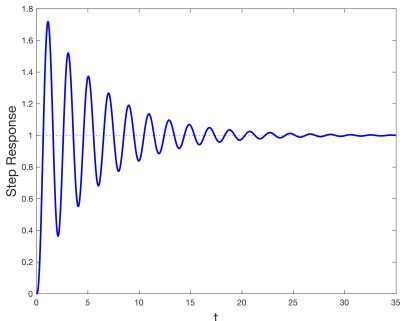
Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle

- That system takes a long time to settle.
- What does the step response look like?



- Could we have predicted the long settling time and large overshoot?
- We know that the settling time of *each component* of the response is related to the real parts of the poles

Closed-loop poles and zeros

Stability

Condition in terms of poles

Condition in terms of denominator coefficients

Routh Hurwitz condition

Basics

Disk drive example

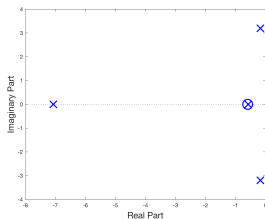
Dealing with zeros

Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle



- One “fast” real pole
- One “reasonably slow” real pole that is close to a zero
- Complex conjugate pair with small real parts (≈ 0.17), and large angles with negative real axis ($\approx 87^\circ$). Conjugate pair dominates
- 2% settling time of response to conjugate pair is 4 time constants, $4\left(\frac{1}{0.17}\right) \approx 23$
- Damping ratio of conjugate pair $\zeta = \cos(87^\circ) \approx 0.053$. Corresponds to overshoot of $\approx 85\% = 0.85$
- Actual performance of the fourth-order system with one zero is quite close to this guidance from second-order system with no finite zeros

Stability

Condition in terms of poles

Condition in terms of denominator coefficients

Routh Hurwitz condition

Basics

Disk drive example

Dealing with zeros

Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle

Managing settling times

- It is disappointing that using Routh-Hurwitz for design gives us control over stability, but does not allow us to manage settling times
- Particularly disappointing, given “shape” of region to left of a certain real number is same as that to left of origin
- Can we do anything?
- Recall closed-loop denominator polynomial is:

$$q(s) = a_n \prod_{j=1}^n (s - (-p_{cl,j}))$$

where $-p_{cl,j}$ are the closed-loop poles

- If we want $q(s)$ to have closed-loop poles to left of $-\sigma$, then we need

$$\tilde{q}(s) = a_n \prod_{j=1}^n (s - (-p_{cl,j} + \sigma))$$

to be stable

Managing settling times, II

Tim Davidson

Stability

Condition in terms of poles

Condition in terms of denominator coefficients

Routh Hurwitz condition

Basics

Disk drive example

Dealing with zeros

Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle

- That means we want

$$\begin{aligned}\tilde{q}(s) &= q(s - \sigma) \\ &= a_n(s - \sigma)^n + a_{n-1}(s - \sigma)^{n-1} + \\ &\quad \dots + a_1(s - \sigma) + a_0\end{aligned}$$

to be stable

- Using binomial theorem, $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$, we can write

$$\tilde{q}(s) = \tilde{a}_n s^n + \tilde{a}_{n-1} s^{n-1} + \dots + \tilde{a}_1 s + \tilde{a}_0$$

where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$

- Now you can apply Routh-Hurwitz procedure to $\tilde{q}(s)$
- What happens if there is a pair of closed-loop poles with real parts equal to $-\sigma$? A zero row in the table

Application to tracked vehicle

Tim Davidson

Stability

Condition in terms of poles

Condition in terms of denominator coefficients

Routh Hurwitz condition

Basics

Disk drive example

Dealing with zeros

Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle

- Settling time of component due to complex-conjugate pair in current design is around 23 seconds
- Can we reduce this? Let's try to get it down to around 16 seconds.
- That means that we want poles to the left of $-\sigma = -0.25$
- $\tilde{q}(s) \simeq s^4 + 7s^3 + 11.375s^2 + (K + 2.9375)s + \tilde{a}_0(K, a)$, where $\tilde{a}_0(K, a) = Ka - 0.25K - 1.3086$
- Routh table

Row 4	1	11.375	$\tilde{a}_0(K, a)$
Row 3	7	$K + 2.9375$	
Row 2	\tilde{b}_3	$\tilde{a}_0(K, a)$	
Row 1	\tilde{c}_3		
Row 0	$\tilde{a}_0(K, a)$		

$$\tilde{b}_3 = \frac{76.6876 - K}{7}, \quad \tilde{c}_3 = \frac{\tilde{b}_3(K + 2.9375) - 7\tilde{a}_0(K, a)}{\tilde{b}_3}$$

- Hence, we require $\tilde{b}_3 > 0$, $\tilde{c}_3 > 0$ and $\tilde{a}_0(K, a) > 0$

Stability

Condition in terms of poles

Condition in terms of denominator coefficients

Routh Hurwitz condition

Basics

Disk drive example

Dealing with zeros

Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle

Settling time region

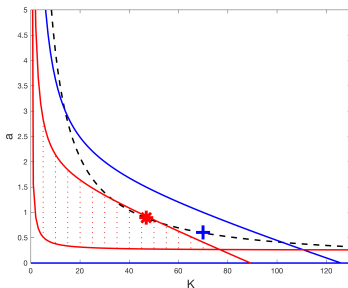
- For positive K , the constraints can be rewritten as

$$K < 76.6876$$

$$a > 0.25 + \frac{1.3086}{K}$$

$$a < \frac{(76.6896 - K)(K + 2.9375) + 12.25K + 64.12}{49K}$$

- Region between the red lines (inside blue region)
- For steady state error we need $a > \frac{41.67}{K}$; above dashed line



- Not much choice this time. Example: $K = 47$, $a = 0.9$, marked by *

Closed-loop poles and zeros

Tim Davidson

Stability

Condition in terms of poles

Condition in terms of denominator coefficients

Routh Hurwitz condition

Basics

Disk drive example

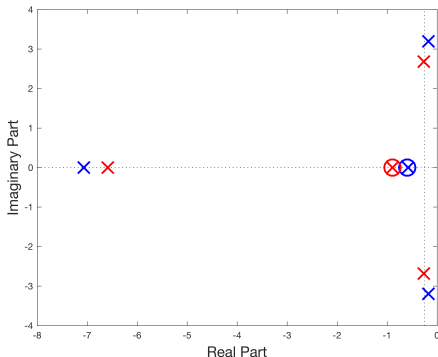
Dealing with zeros

Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle



Since complex conjugate pair of poles still dominates, this suggests new design (red) will have

- Somewhat reduced settling time (poles are to left of 0.25)
- Slightly reduced overshoot in step response (angle slightly reduced)

Stability

Condition in terms of poles

Condition in terms of denominator coefficients

Routh Hurwitz condition

Basics

Disk drive example

Dealing with zeros

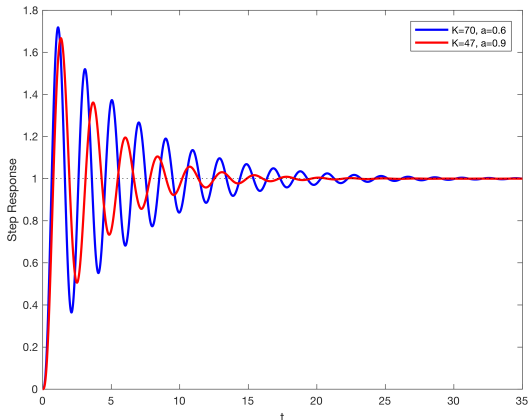
Zeros in first column

Zero rows

Using Routh Hurwitz for design

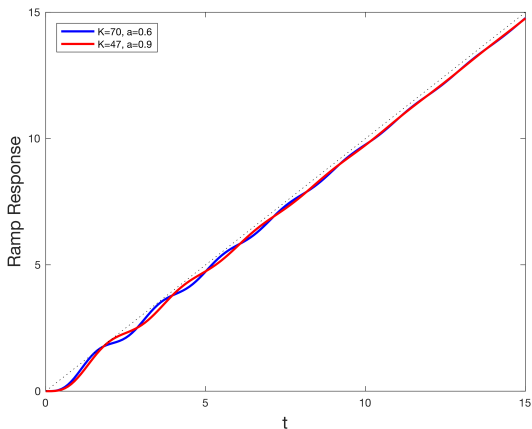
Turning control of a tracked vehicle

Step response



- Settling time has come down; not far from 16 seconds
- Overshoot has been reduced a little bit

Ramp response



- Steady-state error criterion still satisfied ($K_v = 4.230 > 4.167$)
- Transient time has been reduced

Stability

Condition in terms of poles

Condition in terms of denominator coefficients

Routh Hurwitz condition

Basics

Disk drive example

Dealing with zeros

Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle

Ramp response, zoomed in

Tim Davidson

Stability

Condition in terms of poles

Condition in terms of denominator coefficients

Routh Hurwitz condition

Basics

Disk drive example

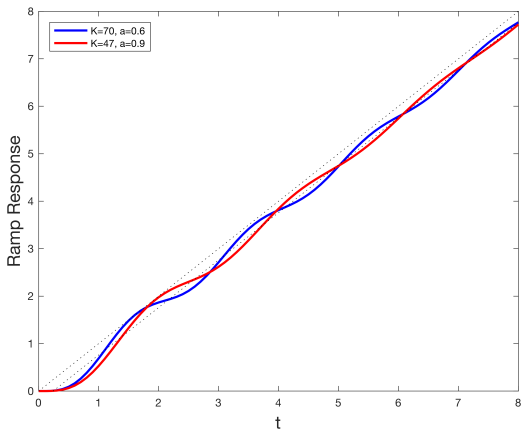
Dealing with zeros

Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle



Discussion

Tim Davidson

Stability

Condition in terms of poles

Condition in terms of denominator coefficients

Routh Hurwitz condition

Basics

Disk drive example

Dealing with zeros

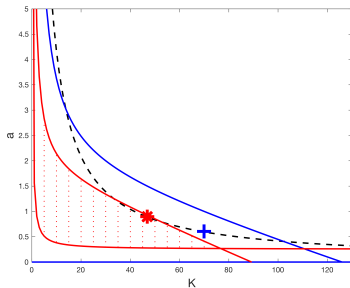
Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle

- Transient times have been reduced, but only a little bit
- Unfortunately, not much room to do any better



- Current compensator is $G_c(s) = \frac{s+a}{s+1}$ and gain is K
- To get significantly better performance, we need to be able to adjust the pole of the compensator, as well as the zero and the gain
- For example, consider $G_c(s) = \frac{(s+a)}{(s+b)} = \frac{s+1.9}{s+6}$ with $K = 133$.
- Note $K_v = \lim_{s \rightarrow 0} sG_c(s)G(s) \simeq 4.212 > 4.167$, so ramp steady-state error criterion still satisfied

Closed-loop poles and zeros

Tim Davidson

Stability

Condition in terms of poles

Condition in terms of denominator coefficients

Routh Hurwitz condition

Basics

Disk drive example

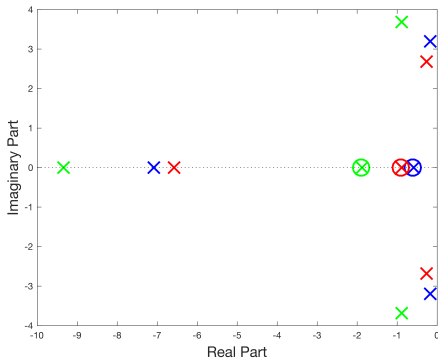
Dealing with zeros

Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle



Since complex conjugate pair of poles still dominates, this suggests new design (green) will have

- Significantly reduced settling time (dominant poles further to left)
- Somewhat reduced overshoot in step response (smaller angle)

Stability

Condition in terms of poles

Condition in terms of denominator coefficients

Routh Hurwitz condition

Basics

Disk drive example

Dealing with zeros

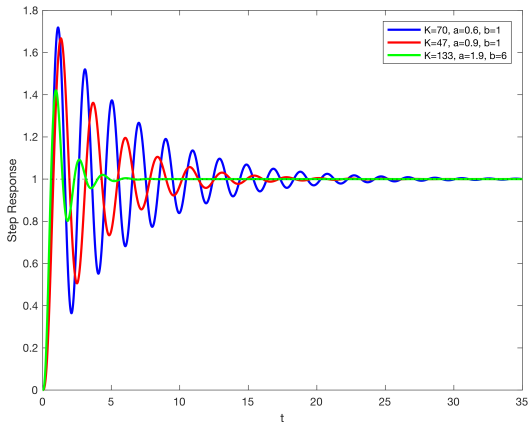
Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle

Step response



- Insight from pole and zero positions reasonably accurate

Ramp response, zoomed in

Tim Davidson

Stability

Condition in terms of poles

Condition in terms of denominator coefficients

Routh Hurwitz condition

Basics

Disk drive example

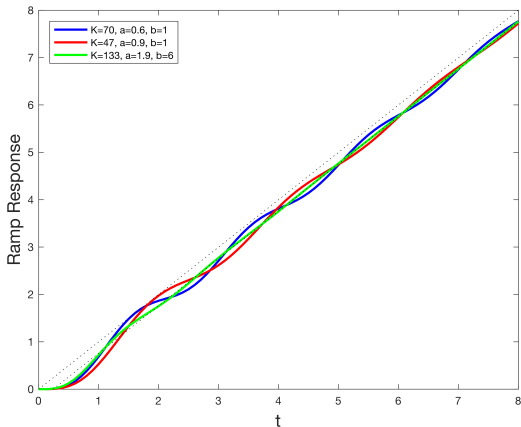
Dealing with zeros

Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle



- Steady-state error criterion still satisfied for new design ($K_v = 4.212 > 4.167$)
- Transient time reduced

Stability

Condition in terms of poles

Condition in terms of denominator coefficients

Routh Hurwitz condition

Basics

Disk drive example

Dealing with zeros

Zeros in first column

Zero rows

Using Routh Hurwitz for design

Turning control of a tracked vehicle

Discussion, cont.

- New design does indeed do better in our chosen metrics
- However, design problem now has three parameters, not two; zero pos'n ($-a$), gain (K), and also pole pos'n ($-b$)
- Hence desired parameter region is a volume, not an area
- Routh-Hurwitz procedure and resulting equations for boundaries of desired parameter space get even more complicated
- Is there an easier way to gain insight into how to choose the pole and zero positions, and to choose the gain?
- One option is the root locus procedure that we will develop in the next section