EE3CL4: Introduction to Linear Control Systems
Section 2: System Models

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Outline

1. Modelling physical systems
   - Translational Newtonian Mechanics
   - Rotational Newtonian Mechanics

2. Linearization

3. Laplace transforms

4. Laplace transforms in action

5. Transfer function

6. Step response

7. Transfer function of DC motor

8. Our first control system design

9. Block diagram models
   - Block diagram transformations
Differential equation models

- Most of the systems that we will deal with are dynamic
- Differential equations provide a powerful way to describe dynamic systems
- Will form the basis of our models

- We saw differential equations for inductors and capacitors in 2CI, 2CJ
- What about mechanical systems? both translational and rotational
Translational Spring

\[ F(t) : \text{resultant force in direction } x \]
Recall free body diagrams and "action and reaction"

- Spring. \( k \): spring constant, \( L_r \): relaxed length of spring

\[
F(t) = k \left( [x_2(t) - x_1(t)] - L_r \right)
\]
Translational Damper

\( F(t) \): resultant force in direction \( x \)

- Viscous damper. \( b \): viscous friction coefficient

\[
F(t) = b \left( \frac{dx_2(t)}{dt} - \frac{dx_1(t)}{dt} \right) = b \left( v_2(t) - v_1(t) \right)
\]
$F(t)$: resultant force in direction $x$

- **Mass**: $M$

\[
F(t) = M \frac{d^2 x_m(t)}{dt^2} = M \frac{dv_m(t)}{dt} = Ma_m(t)
\]
Rotational spring

\( T(t) \): resultant torque in direction \( \theta \)

- Rotational spring. \( k \): rotational spring constant, \( \phi_r \): rotation of relaxed spring

\[
T(t) = k(\theta_2(t) - \theta_1(t)) - \phi_r
\]
Rotational damper

\[ T(t): \text{resultant torque in direction } \theta \]

- Rotational viscous damper.
  \[ b: \text{rotational viscous friction coefficient} \]

\[ T(t) = b\left(\frac{d\theta_2(t)}{dt} - \frac{d\theta_1(t)}{dt}\right) = b(\omega_2(t) - \omega_1(t)) \]
Rotational inertia

\( T(t) \): resultant torque in direction \( \theta \)

- Rotational inertia: \( J \)

\[
T(t) = J \frac{d^2 \theta_m(t)}{dt^2} = J \frac{d \omega_m(t)}{dt} = J \alpha_m(t)
\]
Example system (translational)

Horizontal. Origin for $y$: $y = 0$ when spring relaxed

- $F = M \frac{dv(t)}{dt}$
- $v(t) = \frac{dy(t)}{dt}$
- $F(t) = r(t) - b \frac{dy(t)}{dt} - ky(t)$

$$M \frac{d^2 y(t)}{dt^2} + b \frac{dy(t)}{dt} + ky(t) = r(t)$$
Example, continued

\[ M \frac{d^2 y(t)}{dt} + b \frac{dy(t)}{dt} + ky(t) = r(t) \]

Resembles equation for parallel RLC circuit.
Example, continued

- Stretch the spring a little and hold.
- Assume an under-damped system.
- What happens when we let it go?
Taylor’s series

- Nature does not have many linear systems
- However, many systems behave approximately linearly in the neighbourhood of a given point
- Apply first-order Taylor’s Series at a given point
- Obtain a locally linear model

- In this course we will focus on the case of a single linearized differential equation model for the system, in which the coefficients are constants
- e.g., in previous examples mass, viscosity and spring constant did not change with time, position, velocity, temperature, etc
• Assume shaft is light with respect to $M$, and stiff with respect to gravitational forces
• Torque due to gravity: $T(\theta) = MgL \sin(\theta)$
• Apply Taylor’s series around $\theta = 0$:
  $$T(\theta) = MgL \left( \theta - \theta^3/3! + \theta^5/5! - \theta^7/7! + \ldots \right)$$
• For small $\theta$ around $\theta = 0$ we can build an approximate model that is linear
  $$T(\theta) \approx MgL\theta$$
Laplace transform

- Once we have a linearized differential equation (with constant coefficients) we can take Laplace Transforms to obtain the transfer function.

- We will consider the “one-sided” Laplace transform, for signals that are zero to the left of the origin.

\[ F(s) = \int_{0^-}^{\infty} f(t)e^{-st} \, dt \]

- What does \( \int_{0^-}^{\infty} \) mean? \( \lim_{T \to \infty} \int_{0^-}^{T} \).

- Does this limit exist?

- If \( |f(t)| < Me^{\alpha t} \), then exists for all \( \text{Re}(s) > \alpha \).

  Includes all physically realizable signals

- Note: When multiplying transfer function by Laplace of input, output is only valid for values of \( s \) in intersection of regions of convergence.
Poles and zeros

- In this course, most Laplace transforms will be rational functions, that is, a ratio of two polynomials in \( s \); i.e.,

\[
F(s) = \frac{n_F(s)}{d_F(s)}
\]

where \( n_F(s) \) and \( d_F(s) \) are polynomials

- Definitions:
  - Poles of \( F(s) \) are the roots of \( d_F(s) \)
  - Zeros of \( F(s) \) are the roots of \( n_F(s) \)

- Hence,

\[
F(s) = \frac{K_F \prod_{i=1}^{M} (s + z_i)}{\prod_{j=1}^{n} (s + p_j)} = \left( \frac{K_F \prod_{i=1}^{M} z_i}{\prod_{j=1}^{n} p_j} \right) \frac{\prod_{i=1}^{M} (s/z_i + 1)}{\prod_{j=1}^{n} (s/p_j + 1)}
\]

where \(-z_i\) are the zeros and \(-p_j\) are the poles
Visualizing poles and zeros

- Consider the simple Laplace transform $F(s) = \frac{s(s+3)}{s^2+2s+5}$.
- Zeros: 0, $-3$; poles: $-1 + j2$, $-1 - j2$
- Pole-zero plot (left) and magnitude of $F(s)$ (right)
Visualizing poles and zeros

- \( F(s) = \frac{s(s+3)}{s^2+2s+5} \); zeros: 0, \(-3\); poles: \(-1 + j2, -1 - j2\)
- \(|F(s)|\) from above (left) and prev. view of \(|F(s)|\) (right)
Laplace transform pairs

- Simple ones can be computed analytically; often available in tables; see Tab. 2.3 in 12th ed. of text

- For more complicated ones, one can typically obtain the inverse Laplace transform by
  - identifying poles
  - constructing partial fraction expansion
  - using of properties and some simple pairs to invert each component of partial fraction expansion
Recall that complex poles come in conjugate pairs.
Key properties

Linearity

\[ \frac{df(t)}{dt} \longleftrightarrow sF(s) - f(0^-) \]

\[ \int_{-\infty}^{t} f(x) \, dx \longleftrightarrow \frac{F(s)}{s} + \frac{1}{s} \int_{-\infty}^{0^-} f(x) \, dx \]
Final value theorem

Can we avoid having to do an inverse Laplace transform? Sometimes.

Consider the case when we only want to find the final value of $f(t)$, namely $\lim_{t \to \infty} f(t)$.

- If $F(s)$ has all its poles in the left half plane, except, perhaps, for a single pole at the origin, then

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Common application: Steady state value of step response

What if there are poles in RHP, or on the $j\omega$-axis and not at the origin?
Mass-spring-damper system

- Horizontal (no gravity)
- Set origin of $y$ where spring is “relaxed”
- $F = M \frac{dv(t)}{dt}$
- $v(t) = \frac{dy(t)}{dt}$
- $F(t) = r(t) - b \frac{dy(t)}{dt} - ky(t)$

\[ M \frac{d^2 y(t)}{dt^2} + b \frac{dy(t)}{dt} + ky(t) = r(t) \]
MSD system

\[ M \frac{d^2 y(t)}{dt} + b \frac{dy(t)}{dt} + ky(t) = r(t) \]

Consider \( t \geq 0 \) and take Laplace transform

\[ M \left( s^2 Y(s) - sy(0^-) - \left. \frac{dy(t)}{dt} \right|_{t=0^-} \right) + b(sY(s) - y(0^-)) + kY(s) = R(s) \]

Hence

\[ Y(s) = \frac{1/M}{s^2 + (b/M)s + k/M} R(s) \]

\[ + \frac{(s + b/M)}{s^2 + (b/M)s + k/M} y(0^-) \]

\[ + \frac{1}{s^2 + (b/M)s + k/M} \left. \frac{dy(t)}{dt} \right|_{t=0^-} \]
Response to static init. cond.

Spring stretched to a point $y_0$, held, then let go at time $t = 0$

Hence, $r(t) = 0$ and $\left. \frac{dy(t)}{dt} \right|_{t=0^-} = 0$

Hence,

$$Y(s) = \frac{(s + b/M)}{s^2 + (b/M)s + k/M} y_0$$

What can we learn about this response without having to invert $Y(s)$
Standard form

\[
Y(s) = \frac{(s + b/M)}{s^2 + (b/M)s + k/M} y_0
\]

\[
= \frac{(s + 2\zeta \omega_n)}{s^2 + 2\zeta \omega_n s + \omega_n^2} y_0
\]

where \( \omega_n = \sqrt{k/M} \) and \( \zeta = \frac{b}{2\sqrt{kM}} \)

Poles: \( s_1, s_2 = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \)

- \( \zeta > 1 \) (equiv. \( b > 2\sqrt{kM} \)): distinct real roots, overdamped
- \( \zeta = 1 \) (equiv. \( b = 2\sqrt{kM} \)): equal real roots, critically damped
- \( \zeta < 1 \) (equiv. \( b < 2\sqrt{kM} \)): complex conj. roots, underdamped
Overdamped case

- \( s_1, s_2 = -ζω_n \pm ω_n\sqrt{ζ^2 - 1} \)
- Overdamped response: \( ζ > 1 \) (equiv. \( b > 2\sqrt{kM} \))
- \( y(t) = c_1 e^{s_1t} + c_2 e^{s_2t} \)
- \( y(0) = y_0 \implies c_1 + c_2 = y_0 \)
- \( \frac{dy(t)}{dt} \bigg|_{t=0} = 0 \implies s_1 c_1 + s_2 c_2 = 0 \)

- What does this look like when strongly overdamped
  - \( s_2 \) is large and negative, \( s_1 \) is small and negative
  - Hence \( e^{s_2t} \) decays much faster than \( e^{s_1t} \)
  - Also, \( c_2 = -c_1 s_1 / s_2 \). Hence, small
  - Hence \( y(t) \approx c_1 e^{s_1t} \)
  - Looks like a first order system!
Critically damped case

- \( s_1 = s_2 = -\omega_n \)
- \( y(t) = c_1 e^{-\omega_n t} + c_2 te^{-\omega_n t} \)
- \( y(0) = y_0 \implies c_1 = y_0 \)
- \( \left. \frac{dy(t)}{dt} \right|_{t=0} = 0 \implies -c_1 \omega_n + c_2 = 0 \)
Underdamped case

- $s_1, s_2 = -\zeta \omega_n \pm j\omega_n \sqrt{1 - \zeta^2}$
- Therefore, $|s_i| = \omega_n$: poles lies on a circle
- Angle to negative real axis is $\cos^{-1}(\zeta)$. 

![Diagram showing complex plane with poles and angles](image-url)
Underdamped case

- Define $\sigma = \zeta \omega_n$, $\omega_d = \omega_n \sqrt{1 - \zeta^2}$. Response is:

\[
y(t) = c_1 e^{-\sigma t} \cos(\omega_d t) + c_2 e^{-\sigma t} \sin(\omega_d t)
= Ae^{-\sigma t} \cos(\omega_d t + \phi)
\]

- Homework: Relate $A$ and $\phi$ to $c_1$ and $c_2$.
- Homework: Write the initial conditions $y(0) = y_0$ and

\[
\left. \frac{dy(t)}{dt} \right|_{t=0} = 0
\]

in terms of $c_1$ and $c_2$, and in terms of $A$ and $\phi$. 
Numerical examples

- \( Y(s) = \frac{(s+2\zeta \omega_n)}{s^2+2\zeta \omega_n s+\omega_n^2} y_0 \), where \( \omega_n = \sqrt{\frac{k}{M}} \), \( \zeta = \frac{b}{2\sqrt{km}} \)

- Poles: \( s_1, s_2 = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \)

- \( \zeta > 1 \): overdamped; \( \zeta < 1 \): underdamped

- Consider the case of \( M = 1, k = 1 \). Hence, \( \omega_n = 1 \),

- \( b = 3 \rightarrow 0 \). Hence, \( \zeta = 1.5 \rightarrow 0 \)

- Initial conds: \( y_0 = 1 \), \( \frac{dy(t)}{dt}\bigg|_{t=0} = 0 \)
Poles and transient response, $b = 3$
Poles and transient response, \( b = 2.75 \)
Poles and transient response, 

\[ b = 2.5 \]
Poles and transient response, \( b = 2.25 \)
Poles and transient response,

\[ b = 2 \]
Poles and transient response, $b = 1.95$
Poles and transient response, \( b = 1.75 \)
Poles and transient response,

\[ b = 1.5 \]
Poles and transient response, \( b = 1.25 \)
Poles and transient response, $b = 1$
Poles and transient response, $b = 0.75$

![Diagram showing poles and transient response with $b = 0.75$.]
Poles and transient response,

\[ b = 0.5 \]
Poles and transient response, \( b = 0.25 \)

![Graph showing poles and transient response](image)

- **Re(s)**
- **Im(s)**
- **y(t)**

**Block diagram models**
**Block dia. transform.**

**Our first control system design**

**Laplace in action**

**Transfer function**

**Step response**

**Transfer fn of DC motor**
Poles and transient response, $b = 0$
Transfer function

**Definition:** Laplace transform of output over Laplace transform of input when initial conditions are zero

- Most of the transfer functions in this course will be ratios of polynomials in $s$.
- Hence, poles and zeros of transfer functions have natural definitions

Example: Recall the mass-spring-damper system,
Transfer function, MSD system

For the mass-spring-damper system,

\[
Y(s) = \frac{1/M}{s^2 + (b/M)s + k/M} R(s) + \frac{(s + b/M)}{s^2 + (b/M)s + k/M} y(0^-) + \frac{1}{s^2 + (b/M)s + k/M} \left. \frac{dy(t)}{dt} \right|_{t=0^-}
\]

Therefore, transfer function is:

\[
\frac{1/M}{s^2 + (b/M)s + k/M} = \frac{1}{Ms^2 + bs + k}
\]
Step response

- Recall that $u(t) \leftrightarrow \frac{1}{s}$
- Therefore, for transfer function $G(s)$, the step response is:

$$\mathcal{L}^{-1}\left\{ \frac{G(s)}{s} \right\}$$

- For the mass-spring-damper system, step response is

$$\mathcal{L}^{-1}\left\{ \frac{1}{s(Ms^2 + bs + k)} \right\}$$

- What is the final position for a step input? Recall final value theorem. Final position is $1/k$.
- What about the complete step response?
Step response

- Step response: $\mathcal{L}^{-1}\left\{ \frac{G(s)}{s} \right\}$

- Hence poles of Laplace transform of step response are poles of $G(s)$, plus an additional pole at $s = 0$.

- For the mass-spring-damper system, using partial fractions, step response is:

  $$\mathcal{L}^{-1}\left\{ \frac{1}{s(Ms^2 + bs + k)} \right\}$$

  $$= \mathcal{L}^{-1}\left\{ \frac{1}{s} \right\} - \frac{1}{k} \mathcal{L}^{-1}\left\{ \frac{Ms + b}{Ms^2 + bs + k} \right\}$$

  $$= \frac{1}{k} u(t) - \frac{1}{k} \mathcal{L}^{-1}\left\{ \frac{Ms + b}{Ms^2 + bs + k} \right\}$$

- Consider again the case of $M = k = 1, b = 3 \rightarrow 0$. $\omega_n = 1, \zeta = 1.5 \rightarrow 0$. 
Poles and step response, $b = 3$
Poles and step resp., $b = 2.75$
Poles and step resp., $b = 2.5$

![Graph showing poles and step response with $b = 2.5$.]
Poles and step resp., $b = 2.25$

The image shows a graph with the real part of $s$ on the horizontal axis and the imaginary part of $s$ on the vertical axis. The graph is labeled with $b=2.25$. The step response graph is also shown, with time $t$ on the horizontal axis and the response $y(t)$ on the vertical axis, indicating a positive response trajectory that approaches a steady state.

**Mathematical Perspectives**

The graphs represent the behavior of a system with poles at $s = \pm 2.25 + 0i$. The step response indicates a system with a finite overshoot and settling time.

**System Dynamics**

The system dynamics are described by the Laplace transform, where $b$ plays a crucial role in determining the stability and response characteristics. The diagram illustrates the impact of $b$ on the system's transient behavior, highlighting the importance of $b=2.25$ in achieving desired performance attributes.
Poles and step resp., $b = 2$
Poles and step resp., $b = 1.95$

```
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<tr>
<th>Im(s)</th>
<th>Re(s)</th>
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</tr>
<tr>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>0</td>
<td>-3</td>
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<table>
<thead>
<tr>
<th>y(t)</th>
<th>t</th>
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</thead>
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</tr>
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<td>10</td>
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<tr>
<td>0.4</td>
<td>15</td>
</tr>
<tr>
<td>0.2</td>
<td>0</td>
</tr>
</tbody>
</table>
```

$P = 1.95$
Poles and step resp., $b = 1.75$

$$|m(s)|$$

$$y(t)$$

$$b=1.75$$

$$b=1.75$$

$$Re(s)$$

$$t$$
Poles and step resp., \( b = 1.5 \)
Poles and step resp., $b = 1.25$
Poles and step resp., $b = 1$

![Graph showing poles and step response with $b = 1$.]
Poles and step resp., $b = 0.75$

![Diagram showing poles and step response](image-url)
Poles and step resp., $b = 0.5$

![Diagram showing poles and step response. The real part of the poles is represented on the left graph, and the time response is shown on the right graph. The parameter $b = 0.5$ is noted.]
Poles and step resp., $b = 0.25$

![Graph showing poles and step response with $b = 0.25$.]
Poles and step resp., $b = 0$
We will consider linearized model for each component

Flux in the air gap: \( \phi(t) = K_f i_f(t) \) (Magnetic cct, 2CJ4)

Torque: \( T_m(t) = K_1 \phi(t) i_a(t) = K_1 K_f i_f(t) i_a(t) \).

Is that linear?

Only if one of \( i_f(t) \) or \( i_a(t) \) is constant

We will consider "armature control": \( i_f(t) \) constant
Armature controlled DC motor

- \(i_f(t)\) will be constant (to set up magnetic field), \(i_f(t) = I_f\)
- Torque: \(T_m(t) = K_1 K_f I_f i_a(t) = K_m i_a(t)\)
- Will control motor using armature voltage \(V_a(t)\)
- What is the transfer function from \(V_a(s)\) to angular position \(\theta(s)\)?
- Origin?
Towards transfer function

- \( T_m(t) = K_m i_a(t) \iff T_m(s) = K_m i_a(s) \)
- KVL: \( V_a(s) = (R_a + sL_a) i_a(s) + V_b(s) \)
- \( V_b(s) \) is back-emf voltage, due to Faraday’s Law
- \( V_b(s) = K_b \omega(s) \), where \( \omega(s) = s\theta(s) \) is rot. velocity
- Remember: transfer function implies zero init. conds
Towards transfer function

- Torque on load: \( T_L(s) = T_m(s) - T_d(s) \)
- \( T_d(s) \): disturbance. Often small, unknown
- Load torque and load angle (Newton plus friction):
  \[
  T_L(s) = Js^2 \theta(s) + bs\theta(s)
  \]
- Now put it all together
Towards transfer function

- \( T_m(s) = K_m I_a(s) = K_m \left( \frac{V_a(s) - V_b(s)}{R_a + sL_a} \right) \)
- \( V_b(s) = K_b \omega(s) \)
- \( T_L(s) = T_m(s) - T_d(s) \)
- \( T_L(s) = Js^2 \theta(s) + bs\theta(s) = Js\omega(s) + b\omega(s) \)
- Hence \( \omega(s) = \frac{T_L(s)}{Js + b} \)
- \( \theta(s) = \omega(s)/s \)
Modelling physical systems
Linearization
Laplace transforms
Laplace in action
Transfer function
Step response
Transfer fn of DC motor
Our first control system design
Block diagram models
Block dia. transform.

• \( T_m(s) = K_m I_a(s) = K_m \left( \frac{V_a(s) - V_b(s)}{R_a + sL_a} \right) \)
• \( V_b(s) = K_b \omega(s) \)
• \( T_L(s) = T_m(s) - T_d(s) \)
• \( T_L(s) = J s^2 \theta(s) + b s \theta(s) = J s \omega(s) + b \omega(s) \)
• Hence \( \omega(s) = \frac{T_L(s)}{J s + b} \)
• \( \theta(s) = \omega(s) / s \)
Transfer function

- Set $T_d(s) = 0$ and solve (you MUST do this yourself)

$$G(s) = \frac{\theta(s)}{V_a(s)} = \frac{K_m}{s\left[(R_a + sL_a)(Js + b) + K_bK_m\right]}$$

$$= \frac{K_m}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

- Third order :(
Second-order approximation

\[ G(s) = \frac{\theta(s)}{V_a(s)} = \frac{K_m}{s[(R_a + sL_a)(Js + b) + K_bK_m]} \]

- Sometimes armature time constant, \( \tau_a = L_a/R_a \), is negligible
- Hence (you MUST derive this yourself)

\[ G(s) \approx \frac{K_m}{s[R_a(Js + b) + K_bK_m]} = \frac{K_m/(R_a b + K_bK_m)}{s(\tau_1 s + 1)} \]

where \( \tau_1 = R_aJ/(R_a b + K_bK_m) \)
Model for a disk drive read system

- Uses a permanent magnet DC motor
- Can be modelled using arm. contr. model with $K_b = 0$
- Hence, motor transfer function:

$$G(s) = \frac{\theta(s)}{V_a(s)} = \frac{K_m}{s(R_a + sL_a)(Js + b)}$$

- Assume for now that the arm is stiff
Typical values

\[
G(s) = \frac{\theta(s)}{V_a(s)} = \frac{K_m}{s(R_a + sL_a)(Js + b)}
\]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Typical Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inertia of arm and read head</td>
<td>(J)</td>
<td>1 N m s(^2)/rad</td>
</tr>
<tr>
<td>Friction</td>
<td>(b)</td>
<td>20 N m s/rad</td>
</tr>
<tr>
<td>Amplifier</td>
<td>(K_a)</td>
<td>10–1000</td>
</tr>
<tr>
<td>Armature resistance</td>
<td>(R)</td>
<td>1 (\Omega)</td>
</tr>
<tr>
<td>Motor constant</td>
<td>(K_m)</td>
<td>5 N m/A</td>
</tr>
<tr>
<td>Armature inductance</td>
<td>(L)</td>
<td>1 mH</td>
</tr>
</tbody>
</table>

\[
G(s) = \frac{5000}{s(s + 20)(s + 1000)}
\]
Time constants

- Initial model

\[ G(s) = \frac{5000}{s(s + 20)(s + 1000)} \]

- Motor time constant = \(1/20 = 50\text{ms}\)
- Armature time constant = \(1/1000 = 1\text{ms}\)
- Hence

\[ G(s) \approx \hat{G}(s) = \frac{5}{s(s + 20)} \]
A simple feedback controller

Now that we have a model, how to control?

Simple idea: Apply voltage to motor that is proportional to error between where we are and where we want to be.

Here, \( V(s) = V_a(s) \) and \( Y(s) = \theta(s) \).
What is the transfer function from command to position? Derive this yourself

\[ \frac{Y(s)}{R(s)} = \frac{K_a G(s)}{1 + K_a G(s)} \]

Using second-order approx. \( G(s) \approx \hat{G}(s) = \frac{5}{s(s+20)} \),

\[ Y(s) \approx \frac{5K_a}{s^2 + 20s + 5K_a} R(s) \]

For \( 0 < K_a < 20 \): overdamped; for \( K_a > 20 \): underdamped
Response to $r(t) = 0.1u(t)$; $K_a = 10$

Slow. Slower than IBM’s first drive from late 1950’s. Disks in the 1970’s had 25ms seek times; now < 10ms. Perhaps increase $K_a$? That would result in a “bigger” input to the motor for a given error.
Response to \( r(t) = 0.1u(t); \) \( K_a = 10, 15 \)

Changing \( K_a \) changes the position of the closed-loop poles
Hence, step response changes
Response to \( r(t) = 0.1u(t); \) 
\[ K_a = 10, 15, 20 \]

Changing \( K_a \) changes the position of the closed-loop poles
Hence, step response changes (now critically damped)
Response to \( r(t) = 0.1u(t); \)
\[ K_a = 10, 15, 20, 40 \]

Changing \( K_a \) changes the position of the closed-loop poles
Hence, step response changes (now underdamped)
Response to $r(t) = 0.1u(t)$; $K_a = 10, 15, 20, 40, 60$

Changing $K_a$ changes the position of the closed-loop poles. Hence, step response changes (now more underdamped).
Response to $r(t) = 0.1u(t)$;
$K_a = 10, 15, 20, 40, 60, 80$

What is happening to the settling time of the underdamped cases?
Only just beats IBM’s first drive
What else could we do with the controller? Prediction?
Bock diagram models

- As we have just seen, a convenient way to represent a transfer function is via a block diagram

In this case, \( U(s) = G_c(s)R(s) \) and \( Y(s) = G(s)U(s) \)

Hence, \( Y(s) = G(s)G_c(s)R(s) \)

Consistent with the engineering procedure of breaking things up into little bits, studying the little bits, and then put them together
Simple example

Y_1(s) = G_{11}(s)R_1(s) + G_{12}(s)R_2(s)

Y_2(s) = G_{21}(s)R_1(s) + G_{22}(s)R_2(s)
Example: Loop transfer function

- \( E_a(s) = R(s) - B(s) = R(s) - H(s)Y(s) \)
- \( Y(s) = G(s)U(s) = G(s)G_a(s)Z(s) \)
- \( Y(s) = G(s)G_a(s)G_c(s)E_a(s) \)
- \( Y(s) = G(s)G_a(s)G_c(s) \left( R(s) - H(s)Y(s) \right) \)

\[
\frac{Y(s)}{R(s)} = \frac{G(s)G_a(s)G_c(s)}{1 + G(s)G_a(s)G_c(s)H(s)}
\]

- Each transfer function is a ratio of polynomials in \( s \)
- What is \( E_a(s)/R(s) \)?
Block diagram transformations

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Original Diagram</th>
<th>Equivalent Diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Combining blocks in cascade</td>
<td>$X_1 \rightarrow G_1(s) \rightarrow X_2 \rightarrow G_2(s) \rightarrow X_3$</td>
<td>$X_1 \rightarrow G_1G_2 \rightarrow X_1$ or $X_1 \rightarrow G_2G_1 \rightarrow X_3$</td>
</tr>
<tr>
<td>2. Moving a summing point behind a block</td>
<td>$X_1 \rightarrow + \rightarrow G \rightarrow X_3$</td>
<td>$X_1 \rightarrow G \rightarrow + \rightarrow X_3$</td>
</tr>
<tr>
<td>3. Moving a pickoff point ahead of a block</td>
<td>$X_1 \rightarrow G \rightarrow X_2$</td>
<td>$X_1 \rightarrow G \rightarrow X_2$</td>
</tr>
<tr>
<td>4. Moving a pickoff point behind a block</td>
<td>$X_1 \rightarrow G \rightarrow X_2$</td>
<td>$X_1 \rightarrow G \rightarrow X_2$</td>
</tr>
<tr>
<td>5. Moving a summing point ahead of a block</td>
<td>$X_1 \rightarrow G \rightarrow + \rightarrow X_3$</td>
<td>$X_1 \rightarrow G \rightarrow + \rightarrow X_3$</td>
</tr>
<tr>
<td>6. Eliminating a feedback loop</td>
<td>$X_1 \rightarrow + \rightarrow G \rightarrow X_2$</td>
<td>$X_1 \rightarrow \frac{G}{1 + GH} \rightarrow X_3$</td>
</tr>
</tbody>
</table>
Using block diagram transformations

![Block Diagram](image-url)
Using block diagram transformations

(a) [Diagram with block diagram of control system]

(b) [Diagram with block diagram of control system, showing transfer function]

(c) [Diagram with block diagram of control system, showing another transfer function]