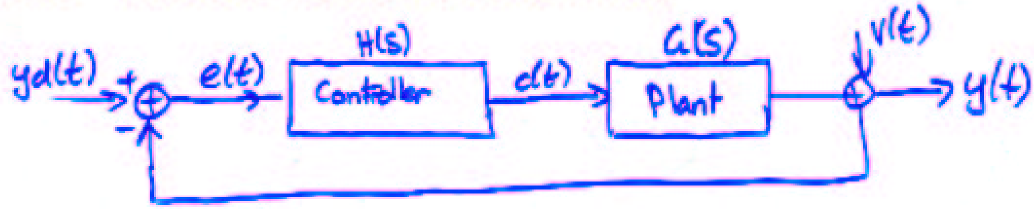
UNITY FEEDBACK SYSTEM (REVISITED)



Using straight forward analysis

$$E(s) = Y_d(s) - Y(s)$$

$$Y(s) = G(s)H(s)E(s) + V(s)$$

$$\Rightarrow \quad Y(s) = \frac{G(s)H(s)Y_d(s)}{1+G(s)H(s)} + \frac{1}{1+G(s)H(s)}V(s)$$

- There are many possible control objectives, but a common one is to make $y(t) \approx y_d(t)$
- That is make $e(t) = y_d(t) - y(t)$ small.
- Systems which achieve this goal are said to "track" the input

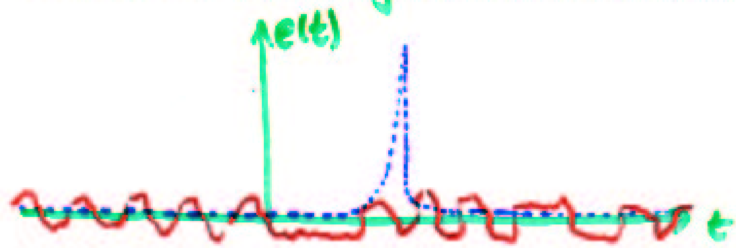
STEADY-STATE ERROR.

What do we mean by $y(t) \approx y_d(t)$?

Perhaps $e(t) = y_d(t) - y(t)$ is small

What do we mean by small ?

Both of the following $e(t)$ are small in some sense !



- One measure that is often used is the steady-state error, $\lim_{t \to \infty} e(t)$, for certain special signals
- How can we find this out?

$$E(s) = Y_d(s) - Y(s) = [1 - T(s)]\, Y_d(s)$$

$$= \frac{1}{1 + G(s)H(s)}\, Y_d(s)$$

- Using the final value theorem. (assuming system is stable)

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

$$= \lim_{s \to 0} \frac{s\,Y_d(s)}{1 + G(s)H(s)}$$

Hence E_{ss} depends on.

* $L(s) = G(s)H(s)$ the open loop transfer function
* $Y_d(s)$ the "command" signal

- Since many controllers introduce integration (usually to "average out" noise and disturbances) it is convenient to factorize this out + write

$$G(s)H(s) = \frac{P(s)}{s^{k} Q_1(s)}$$

where neither $P(s)$ or $Q_1(s)$ have zeros at $s=0$. (these are polynomials).

- Using this factorization we will say that $G(s)H(s)$ is of

$$\begin{array}{ll} \text{type } 0 & \text{if } k=0 \\ \text{type } 1 & \text{if } k=1 \\ \text{type } 2 & \text{if } k=2,\ \text{etc} \end{array}$$

- We will now show how the integral action affects the steady state error for step, ramp and parabolic inputs

SIDEBAR ON INTEGRAL ACTION

$$\xrightarrow{x(t)} \boxed{H(s) = \frac{1}{s}} \longrightarrow y(t)$$

what is $y(t)$.

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau.$$

$$h(\tau) = \mathcal{L}^{-1}\left\{ \frac{1}{s} \right\}$$ inverse Laplace transform.

If the system is causal,

$$h(\tau) = u(\tau)$$ [unit step].

$$\Rightarrow \quad y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{0}^{\infty} x(t-\tau)\, d\tau$$

Let $\mu = t-\tau$
$d\mu = -d\tau$

$$= -\int_{t}^{t-\infty} x(\mu)\, d\mu$$

$$= \int_{-\infty}^{t} x(\mu)\, d\mu.$$

STEP INPUT

$$y_d(t) = u(t) \Rightarrow Y_d(s) = \frac{1}{s}$$

$$E(s) = \frac{Y_d(s)}{1+G(s)H(s)}$$

$$\epsilon_{ss} = \lim_{s\to 0} sE(s).$$

$$\Rightarrow \epsilon_{ss} = \lim_{s\to 0} \frac{1}{1+G(s)H(s)}$$

$$= \frac{1}{1+K_p}.$$

where
$$K_p = \lim_{s\to 0} G(s)H(s)$$
$$= \lim_{s\to 0} \frac{P(s)}{s^k Q_1(s)}$$

is called position error constant.

If $k \geq 1$, $K_p \to \infty \Rightarrow \epsilon_{ss} = 0$.

If $k = 0$, K_p is finite, and hence there is steady state error for a step command in type 0 systems.

RAMP INPUT

$$y_d(t) = t\,u(t) \Rightarrow Y_d(s) = \frac{1}{s^2}$$

$$\Rightarrow \epsilon_{ss} = \lim_{s\to 0} \frac{1}{s + sG(s)H(s)}$$

$$= 1/K_v$$

where
$$K_v = \lim_{s\to 0} sG(s)H(s) = \lim_{s\to 0} \frac{P(s)}{s^{k-1} Q_1(s)}$$

is called the velocity error constant

If $k \geq 2$, $K_v \to \infty$ and hence $e_{ss} = 0$

If $k = 1$, K_v is finite and hence e_{ss} is finite

If $k = 0$, $K_v = 0$ and hence $e_{ss} \to \infty$

Parabolic input

$$y_d(t) = \frac{t^2}{2} u(t) \quad ; \quad Y_d(s) = \frac{1}{s^3}$$

$$e_{ss} = \frac{1}{K_a}$$

$$K_a = \lim_{s \to 0} s \frac{P(s)}{s^{k-2} Q_1(s)}$$

is called the acceleration error constant

If $k \geq 3$, $K_a \to \infty$, $e_{ss} = 0$

If $k = 2$, K_a finite, e_{ss} finite

If $k = 0, 1$, $K_a = 0$, $e_{ss} \to \infty$

APPLICATIONS.

Regulation : If all we need to do is keep a physical variable at a constant level, then all we need is a type 0 loop function $G(s)H(s)$

e.g. : moisture content in paper manufacture

: chemical composition in an industrial reactor.

: temperature, in a building or in the body.

Tracking / Servomechanisms

- if we want the output to follow a more complicated command signal, then wee need $G(s)H(s)$ to have a larger "type number"

eg.
- control of a robot,
- control of aircraft flight surfaces
- control of read head in a disk drive

Control Design Problem

If $G(s) = \frac{1}{s+2}$, design $H(s)$ so that.

i) $\varepsilon_{ss,step} = 0$

ii) $\varepsilon_{ss,ramp} < \frac{1}{50}$

I First guess, try $H(s) = K$.

$$\Rightarrow G(s)\,H(s) = \frac{K}{s+2}$$

This is a type zero system $\Rightarrow \varepsilon_{ss,step} = \frac{1}{1 + K/2}$

II To achieve i) we must make $G(s)H(s)$ at least a type 1 system. Lets try $H(s) = \frac{K}{s}$

$$G(s)H(s) = \frac{K}{s(s+2)}$$

$$\Rightarrow \varepsilon_{ss,step} = 0.$$

$$\varepsilon_{ss,ramp} = \frac{2}{K}$$

$\Rightarrow K > 100$ will do.

However, as you will find out in the lab, if K is too large, the transient performance can be bad.

In practice we would probably choose a combination of proportional and integral control.

e.g. $H(s) = k_p + \frac{k_I}{s}$

Homework: Choose k_p and k_I such that.

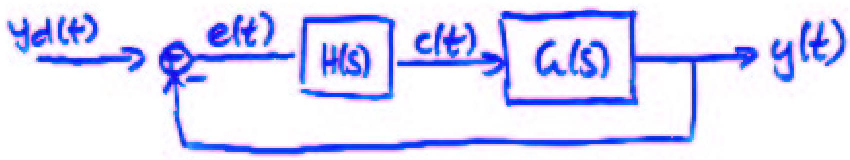
i) $\varepsilon_{ss,step} = 0$

ii) $\varepsilon_{ss,ramp} \leq \frac{1}{50}$

iii) The closed loop poles, ie. the poles of $\frac{1}{1+G(s)H(s)}$ = zeros of $s^k Q_1(s) + P(s)$ have identical real and imaginary parts.

NB: The last property controls some transient performance criteria

Revision: Steady-state errors.



$$E(s) = Y_d(s) - Y(s) = \frac{1}{1+G(s)H(s)} \cdot Y_d(s)$$

- Steady-state error $= \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$

$$= \lim_{s\to 0} \frac{sY_d(s)}{1+G(s)H(s)}$$

- For a step function.

$$\mathcal{E}_{ss,step} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{1}{1+G(s)H(s)} = \frac{1}{1+K_p}$$

- For a ramp,

$$\mathcal{E}_{ss,ramp} = \lim_{s\to 0} \frac{1}{s+sG(s)H(s)} = \frac{1}{K_v}$$

$$K_p = \lim_{s\to 0} G(s)H(s)$$

$$K_v = \lim_{s\to 0} sG(s)H(s)$$

Limits are easier to calculate if we do the following factorization of $G(s)H(s)$

$$G(s)H(s) = \frac{P(s)}{s^k Q_1(s)}$$, $P(s), Q(s)$ are polynomials with no roots at zero

If $k=0$, $k_p = \frac{P(0)}{Q_1(0)} \Rightarrow \varepsilon_{ss,step} = \frac{1}{1 + P(0)/Q_1(0)}$

If $k \geq 1$, $K_p \to \infty \Rightarrow \varepsilon_{ss,step} = 0$

If $k=0$, $K_v = \frac{sP(0)}{Q_1(0)} \to 0 \Rightarrow \varepsilon_{ss,ramp} \to \infty$

If $k=1$ $K_v = \frac{P(0)}{Q_1(0)} \Rightarrow \varepsilon_{ss,ramp} = \frac{Q_1(0)}{P(0)}$

If $k \geq 2$ $K_v \to \infty \Rightarrow \varepsilon_{ss,ramp} = 0$