

Linear Matrix Inequality Formulation of Spectral Mask Constraints With Applications to FIR Filter Design

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Abstract—The design of a finite impulse response (FIR) filter often involves a spectral “mask” that the magnitude spectrum must satisfy. The mask specifies upper and lower bounds at each frequency and, hence, yields an infinite number of constraints. In current practice, spectral masks are often approximated by discretization, but in this paper, we will derive a result that allows us to precisely enforce piecewise constant and piecewise trigonometric polynomial masks in a finite and convex manner via linear matrix inequalities. While this result is theoretically satisfying in that it allows us to avoid the heuristic approximations involved in discretization techniques, it is also of practical interest because it generates competitive design algorithms (based on interior point methods) for a diverse class of FIR filtering and narrowband beamforming problems. The examples we provide include the design of standard linear and nonlinear phase FIR filters, robust “chip” waveforms for wireless communications, and narrowband beamformers for linear antenna arrays. Our main result also provides a contribution to system theory, as it is an extension of the well-known Positive-Real and Bounded-Real Lemmas.

Index Terms—Beamforming, FIR digital filter design, optimization, spectral masks.

I. INTRODUCTION

IN the design of finite impulse response (FIR) filters, one often encounters a spectral mask constraint on the magnitude of the frequency response of the filter (e.g., [1]–[4]). That is, for given $L(e^{j\omega})$ and $U(e^{j\omega})$, constrain the (possibly complex) filter coefficients g_k so that

$$L(e^{j\omega}) \leq |G(e^{j\omega})| \leq U(e^{j\omega}) \quad \text{for all } 0 \leq \omega < 2\pi \quad (1)$$

or determine that the constraint cannot be satisfied. Here, $j = \sqrt{-1}$, and $G(e^{j\omega}) = \sum_k g_k e^{-j\omega k}$ is the frequency response of the filter. A spectral mask constraint can be rather awkward to accommodate into general optimization-based filter design techniques for two reasons. First, it is semi-infinite in the sense that there are two inequality constraints for every $\omega \in [0, 2\pi)$.

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Second, the set of feasible filter coefficients is in general nonconvex due to the lower bound on $|G(e^{j\omega})|$. In order to efficiently solve filter design problems employing such constraints, we must find a way in which (1) can be represented in a finite and convex manner.

There are two established approaches [1] to deal with the problem of nonconvexity of (1). The first is to enforce additional constraints on the parameters g_k so that $G(e^{j\omega})$ has “linear phase.” In that case, $|G(e^{j\omega})|$ becomes a linear function of approximately half the g_k ’s (the rest are determined via the linear phase constraint), and hence, (1) can be reduced to two semi-infinite linear (and hence convex) constraints. However, phase linearity may be an excessively restrictive constraint in some applications [5]. The second approach to deal with nonconvexity is to reformulate (1) in terms of the autocorrelation of the filter [5]–[10]. In particular, if $r_m = \sum_k g_k \bar{g}_{k-m}$ represents the autocorrelation of the filter, then $R(e^{j\omega}) = |G(e^{j\omega})|^2$, and hence, (1) is equivalent to

$$L(e^{j\omega})^2 \leq R(e^{j\omega}) \leq U(e^{j\omega})^2 \quad \text{for all } 0 \leq \omega < 2\pi \quad (2)$$

which amounts to two semi-infinite linear constraints on r_m . (Observe that $r_{-m} = \bar{r}_m$, and hence, $R(e^{j\omega})$ is real.) Hence, by reformulating the mask constraint in terms of r_m , $m \geq 0$, we obtain convex constraints. Note that the constraint that $R(e^{j\omega}) \geq L(e^{j\omega})^2 \geq 0$ is sufficient to ensure that a filter g_k can be extracted (although not uniquely) from a designed autocorrelation r_m via spectral factorization [9], [11].

The problem of representing (1) or (2) in a finite manner is more challenging. [For simplicity, we will phrase our discussion in terms of (2).] One standard, but *ad hoc*, approach is to approximate the constraints by discretizing them uniformly in frequency and enforcing the $2N$ linear constraints

$$L(e^{j\omega_i})^2 + \epsilon \leq R(e^{j\omega_i}) \leq U(e^{j\omega_i})^2 - \epsilon \\ \text{for } \omega_i = \frac{2\pi i}{N}, i = 0, 1, \dots, N-1 \quad (3)$$

where N and ϵ are chosen heuristically. For a fixed N , one must choose ϵ to be small enough so that the overconstraining of the problem at frequencies ω_i does not result in significant performance loss, yet one must choose ϵ to be large enough for satisfaction of (3) to guarantee satisfaction of (2) for all $0 \leq \omega < 2\pi$. Unfortunately, as N is increased so that ϵ can be reduced, the resulting formulation can become prone to numerical difficulties. In practice, N and ϵ are usually chosen according to engineering “rules of thumb” that depend on the design problem

at hand. (Other discretization techniques are also available [12], [13].) For certain design problems, algorithms of the exchange type [1], [3], [4] offer an alternative to direct discretization techniques. These methods employ a nonuniform discretization of (2) at each stage of the algorithm, where the sample points are determined by the stationary points of the current estimate of the optimal $R(e^{j\omega})$ and any points of discontinuity in the mask. (In practice, the stationary points are often approximated using fine uniform discretization [4].) At each stage of the algorithm, an optimization problem is solved subject to appropriate equality constraints derived from (2) at those sample points. Although exchange methods often work well for the design of lowpass filters, substantial effort is required to guarantee the algorithm's convergence [4]. Furthermore, the algorithms may require substantial "retailoring" in order to incorporate additional constraints on the filter coefficients (e.g., [14]). Recently, a precise finite representation of (2) that does not require discretization was developed using dual parameterization methods [15]. However, that representation may result in nonconvex design problems.

In this paper, we derive a precise finite representation of a large class of spectral mask constraints that results in convex design problems. This representation provides a theoretically satisfying characterization of the mask constraint that avoids the heuristic approximation of discretization techniques, yet generates practically competitive design algorithms. Our development begins with the derivation of a (finite) linear matrix inequality (LMI) characterization of the set of trigonometric polynomials of a given order whose real part is positive over a given segment of the unit circle (Theorem 3). Although that result is a contribution to system theory in itself (as outlined later), we also show that it allows us to precisely enforce piecewise constant and piecewise trigonometric polynomial spectral masks in a convex and finite manner. As a result, these masks can be incorporated, without approximation, into the diverse class of FIR filter and narrowband beamformer design problems that can be efficiently solved using well-established interior point methods (e.g., [8]–[10]). We will provide examples that show how our main result leads to effective algorithms for peak-constrained weighted least-squares design of linear-phase and nonlinear-phase FIR filters, for the design of robust "chip" waveforms for digital wireless communication systems based on code division multiple access and for the design of narrowband beamformers for linear antenna arrays with uncertain signal and interference directions.

Our main theoretical result (Theorem 3) provides an LMI characterization of the set of trigonometric polynomials whose real part is positive over a segment of the unit circle. When specialized to the case where the segment is the whole circle, this result generates a new LMI formulation of the Positive Real Lemma [16], [17] (and the closely related Kalman-Yakubovich-Popov [KYP] Lemma) for FIR systems. This new formulation states that for r_m , $-M + 1 \leq m \leq M - 1$, with $r_{-m} = \bar{r}_m$, $R(e^{j\omega}) \geq 0$ for all $\omega \in [0, 2\pi]$ if and only if there exists an $M \times M$ positive semidefinite Hermitian matrix \mathbf{X} such that $\text{tr}(\mathbf{X}) = r_0$ and $\sum_{\ell=0}^{M-1-m} [\mathbf{X}]_{\ell+m, \ell} = r_m$, for $1 \leq m \leq M - 1$. (For later notational convenience, we will index the elements of vectors and matrices starting from zero.) However, Theorem 3 generates LMI formulations of more general constraints of the form $R(e^{j\omega}) \geq \text{Re } A(e^{j\omega})$ for all $\omega \in [\alpha, \beta]$ or

for all $\omega \in [0, 2\pi] \setminus (\alpha, \beta)$, where $A(e^{j\omega}) = \sum_{k=0}^{M_A-1} a_k e^{-j\omega k}$ is a trigonometric polynomial, and $\text{Re} \cdot$ denotes the real part. Since these LMI formulations apply to segments of the unit circle and naturally incorporate nonconstant lower bounds, they can be considered to be generalizations of the Positive Real Lemma. Theorem 3 also generates LMI formulations of constraints of the form $R(e^{j\omega}) \leq \text{Re } B(e^{j\omega})$ for all $\omega \in [\alpha, \beta]$ or for all $\omega \in [0, 2\pi] \setminus (\alpha, \beta)$, which can be considered to be generalizations of the Bounded Real Lemma [18].

Our notational conventions are as follows: Vectors and matrices will be represented by italicized bold lowercase and uppercase letters, respectively. The elements of these structures will be indexed starting from zero and will be denoted by medium weight lowercase letters with appropriate subscripts; e.g., $g_k = [\mathbf{g}]_k$ and $x_{ij} = [\mathbf{X}]_{ij}$. Operators will be represented by upright bold uppercase letters. In order to illuminate the connections between the results for polynomials on the real line and trigonometric polynomials on the unit circle, we define

$$\begin{aligned} \mathbf{u}(t; n) &:= [1 \quad t \quad t^2 \quad \dots \quad t^n]^T \\ \mathbf{v}(\theta; n) &:= [1 \quad e^{j\theta} \quad e^{j2\theta} \quad \dots \quad e^{jn\theta}]^T \end{aligned} \quad (4)$$

where the superscript " T " denotes the transpose (without conjugation). Thus, the components of $\mathbf{u}(t; n)$ form a basis of the (real) function space of polynomials of degree n on the real line, whereas the components of $\mathbf{v}(\theta; n)$ form a basis of the (complex) function space of trigonometric polynomials of degree n on $[0, 2\pi)$. Consequently, an n th-order polynomial $p(t) = \sum_{k=0}^n p_k t^k$ can be written as $p(t) = \mathbf{u}(t; n)^T \mathbf{p}$, where $[\mathbf{p}]_k = p_k$. Similarly, if the sequence g_k denotes the impulse response of a causal FIR filter and if $[\mathbf{g}]_k = g_k$, then the frequency response $G(e^{j\theta}) = \sum_{k=0}^n g_k e^{-jk\theta} = \mathbf{v}(\theta; n)^H \mathbf{g}$, where the superscript " H " denotes the Hermitian transpose. The (complex valued) inner product between two complex matrices \mathbf{X} and \mathbf{Z} is defined as

$$\langle \mathbf{X}, \mathbf{Z} \rangle = \text{tr } \mathbf{X}^H \mathbf{Z}. \quad (5)$$

We will denote by $\mathcal{H}_+^{n \times n}$ the set of $n \times n$ positive semidefinite (complex) Hermitian matrices and by $\mathcal{S}_+^{n \times n}$ the subset of $\mathcal{H}_+^{n \times n}$ consisting of the real symmetric positive semidefinite matrices. For a complex number $x \in \mathbb{C}$, we denote the polar coordinates as $(|x|, \arg x) \in \mathbb{R}_+ \times [0, 2\pi)$, i.e., $x = |x| e^{j \arg x}$, with $\arg x \in [0, 2\pi)$.

II. TRANSFORMATION OF POLYNOMIAL BASIS

In this section, we establish a one-to-one correspondence between polynomials of degree $2n$ on the real line and trigonometric polynomials of degree n on $(0, 2\pi)$, where the coefficients of the trigonometric polynomials may be complex numbers. The main result of this section is stated in Theorem 1, but we first state some intermediate results.

By a result from classical complex analysis [19], the complex exponential function $e^{j\theta}$ can be represented as a (complex) rational function of t over the real line

$$e^{j\theta} = \frac{t + j}{t - j} = \frac{(t + j)^2}{1 + t^2}.$$

This mapping from $t \in (-\infty, \infty)$ to $\theta \in (0, 2\pi)$ is one-to-one. In fact, it is a conformal mapping and is closely related to the “bilinear transform,” which is used to map the left half plane to the unit disc in the standard transformation of analog filter designs into the discrete-time domain [20]. This mapping provides the basis for relating the polynomials over the real line with trigonometric polynomials over the unit circle. The following lemma further relates an arbitrary power of $e^{j\theta}$ to a rational function of t .

Lemma 1: Let $\theta \in (0, 2\pi)$ and $t \in \mathbb{R}$ be related by

$$e^{j\theta} = \frac{(t+j)^2}{1+t^2}.$$

Then, for any positive integer $i \geq 1$, we have

$$e^{ji\theta} = \frac{1}{(1+t^2)^i} \left(\sum_{k=0}^i (-1)^k \binom{2i}{2k} t^{2(i-k)} + j \sum_{k=0}^{i-1} (-1)^k \binom{2i}{2k+1} t^{2(i-k)-1} \right).$$

Proof: This is a simple application of Newton’s binomial formula. **Q.E.D.**

Let us define a lower triangular matrix $\mathbf{G}(n)$ of size $(n+1) \times (n+1)$ whose (i, j) th entry is given by

$$g_{ij}(n) := \begin{cases} 0, & \text{for } 0 \leq i \leq j-1 \\ \binom{n-j}{n-i}, & \text{for } j \leq i \leq n. \end{cases} \quad (6)$$

Notice that the diagonal entries of $\mathbf{G}(n)$ are equal to 1; hence, it is invertible. We will denote the columns of $\mathbf{G}(n)$ by $\mathbf{g}_j(n)$, $j = 0, 1, \dots, n$. In addition, for each $0 \leq k \leq n$, we define a $(n+1) \times (n+1)$ matrix $\mathbf{H}(k; n)$ whose (i, j) th entry is given by

$$h_{ij}(k; n) := \begin{cases} \binom{n-j}{n-i-k}, & \text{for } k \leq j \leq n, j-k \leq i \leq n-k \\ 0, & \text{otherwise.} \end{cases} \quad (7)$$

We will denote the j th column of this matrix by $\mathbf{h}_j(k; n)$. Obviously, we have $\mathbf{H}(0; n) = \mathbf{G}(n)$. We remark that

$$\begin{aligned} \mathbf{u}(x; n)^T \mathbf{h}_j(k; n) &= \sum_{i=j-k}^{n-k} \binom{n-j}{n-i-k} x^i \\ &= x^{j-k} \sum_{\ell=0}^{n-j} \binom{n-j}{n-j-\ell} x^\ell \\ &= x^{j-k} (1+x)^{n-j}. \end{aligned} \quad (8)$$

Based on (8), we can now restate Lemma 1 as follows.

Lemma 2: Let $\theta \in (0, 2\pi)$ and $t \in \mathbb{R}$ be related by

$$e^{j\theta} = \frac{(t+j)^2}{1+t^2}.$$

Then

$$1 = e^{j0\theta} = \frac{\mathbf{u}(t^2; n)^T \mathbf{f}_0}{(1+t^2)^n}$$

and

$$e^{j\ell\theta} = \frac{\mathbf{u}(t^2; n)^T \mathbf{f}_\ell + jt \left(\mathbf{u}(t^2; n-1)^T \tilde{\mathbf{f}}_{\ell-1} \right)}{(1+t^2)^n} \quad \ell = 1, 2, \dots, n$$

where

$$\mathbf{f}_\ell := \sum_{k=0}^{\ell} (-1)^k \binom{2\ell}{2k} \mathbf{h}_\ell(k; n) \quad \ell = 0, 1, \dots, n \quad (9)$$

and

$$\tilde{\mathbf{f}}_\ell := \sum_{k=0}^{\ell} (-1)^k \binom{2(\ell+1)}{2k+1} \mathbf{h}_\ell(k; n-1) \quad \ell = 0, 1, \dots, n-1. \quad (10)$$

Proof: The proof follows directly from Lemma 1 and the definitions of \mathbf{f}_ℓ and $\tilde{\mathbf{f}}_\ell$. **Q.E.D.**

Now, consider the elementary identities

$$\begin{aligned} \binom{n-1}{m} &= \frac{n-m}{n} \binom{n}{m}, \quad \binom{n-1}{m-1} = \frac{m}{n} \binom{n}{m} \\ \binom{n-1}{n-1} &= \binom{n}{n} = 1 \end{aligned}$$

which are valid for any $m = 0, 1, \dots, n-1$. An immediate consequence of the above identities and (7) is that

$$\mathbf{h}_j(k; n) + \mathbf{h}_j(k-1; n) = \mathbf{h}_{j-1}(k-1; n). \quad (11)$$

Alternatively, the above identity can be established from (8). Since $\mathbf{h}_j(0; n) = \mathbf{g}_j(n)$, a simple induction argument shows that

$$\mathbf{h}_j(k; n) \in \text{span} \{ \mathbf{g}_0(n), \mathbf{g}_1(n), \dots, \mathbf{g}_j(n) \}. \quad (12)$$

The following lemma further strengthens the above relation.

Lemma 3: It holds that

$$(-1)^k \mathbf{h}_j(k; n) - \mathbf{g}_j(n) \in \text{span} \{ \mathbf{g}_0(n), \mathbf{g}_1(n), \dots, \mathbf{g}_{j-1}(n) \}. \quad (13)$$

Proof: For a given j , let $\mathbf{w}(k) = (-1)^k \mathbf{h}_j(k; n) - \mathbf{g}_j(n)$. We will prove the lemma by induction on k . For $k = 0$, we have $\mathbf{w}(0) = \mathbf{h}_j(0; n) - \mathbf{g}_j(n) = \mathbf{g}_j(n) - \mathbf{g}_j(n) = 0$; therefore, (13) holds trivially. Consider now a $k \in \{1, 2, \dots, j\}$, and make the hypothesis that $\mathbf{w}(k-1) \in \text{span} \{ \mathbf{g}_0(n), \mathbf{g}_1(n), \dots, \mathbf{g}_{j-1}(n) \}$. We have

$$\begin{aligned} \mathbf{w}(k) &= (-1)^k \mathbf{h}_j(k; n) - \mathbf{g}_j(n) \\ &= (-1)^k (\mathbf{h}_j(k; n) + \mathbf{h}_j(k-1; n)) \\ &\quad + (-1)^{k-1} \mathbf{h}_j(k-1; n) - \mathbf{g}_j(n) \\ &= (-1)^k \mathbf{h}_{j-1}(k-1; n) + (-1)^{k-1} \mathbf{h}_j(k-1; n) - \mathbf{g}_j(n) \\ &= (-1)^k \mathbf{h}_{j-1}(k-1; n) + \mathbf{w}(k-1). \end{aligned}$$

By (12), we know that

$$\mathbf{h}_{j-1}(k-1; n) \in \text{span} \{ \mathbf{g}_0(n), \mathbf{g}_1(n), \dots, \mathbf{g}_{j-1}(n) \}.$$

Hence, if $\mathbf{w}(k-1) \in \text{span} \{ \mathbf{g}_0(n), \mathbf{g}_1(n), \dots, \mathbf{g}_{j-1}(n) \}$, then $\mathbf{w}(k) \in \text{span} \{ \mathbf{g}_0(n), \mathbf{g}_1(n), \dots, \mathbf{g}_{j-1}(n) \}$. Since $\mathbf{w}(0)$ satisfies (13), a simple induction argument completes the proof. **Q.E.D.**

Now, we can substitute (13) into (9) and (10) to obtain the following relations:

$$\begin{aligned} \mathbf{f}_j - \left(\sum_{k=0}^j \binom{2j}{2k} \right) \mathbf{g}_j(n) \\ \in \text{span} \{ \mathbf{g}_0(n), \mathbf{g}_1(n), \dots, \mathbf{g}_{j-1}(n) \} \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathbf{f}}_j - \left(\sum_{k=0}^j \binom{2(j+1)}{2k+1} \right) \mathbf{g}_j(n-1) \\ \in \text{span} \{ \mathbf{g}_0(n-1), \mathbf{g}_1(n-1), \dots, \mathbf{g}_{j-1}(n-1) \}. \end{aligned}$$

As a result, we have that

$$\text{span} \{ \mathbf{f}_0, \mathbf{f}_1, \dots, \mathbf{f}_j \} = \text{span} \{ \mathbf{g}_0(n), \mathbf{g}_1(n), \dots, \mathbf{g}_j(n) \}$$

for $j = 0, 1, \dots, n$

and

$$\text{span} \{ \tilde{\mathbf{f}}_0, \tilde{\mathbf{f}}_1, \dots, \tilde{\mathbf{f}}_j \} = \text{span} \{ \mathbf{g}_0(n-1), \mathbf{g}_1(n-1), \dots, \mathbf{g}_j(n-1) \}$$

for $j = 0, 1, \dots, n-1$.

This implies that the matrices

$$\begin{aligned} \mathbf{F} &= [\mathbf{f}_0, \mathbf{f}_1, \dots, \mathbf{f}_n] \text{ and} \\ \tilde{\mathbf{F}} &= [\tilde{\mathbf{f}}_0, \tilde{\mathbf{f}}_1, \dots, \tilde{\mathbf{f}}_{n-1}] \end{aligned}$$

must be invertible. In light of Lemma 2, this establishes a one-to-one correspondence between polynomials of degree $2n$ on the real line and trigonometric polynomials of degree n on $(0, 2\pi)$. We summarize the result in the following theorem.

Theorem 1: The mapping $\eta(t) := \arg((t+j)^2/(1+t^2))$ is a bijection between \mathbb{R} and $(0, 2\pi)$. In particular, the inverse function $\eta^{-1}(\theta)$ for $\theta \in (0, 2\pi)$ is given by

$$\eta^{-1}(\theta) = \frac{\sin \theta}{1 - \cos \theta}.$$

Furthermore, for any vector $\mathbf{p} \in \mathbb{R}^{2n+1}$, there exists a vector $\mathbf{c} \in \mathbb{R}^{2n+1}$ such that

$$\frac{\sum_{i=0}^{2n} p_i t^i}{(1+t^2)^n} = \text{Re} \left(c_0 + \sum_{k=1}^n (c_{2k} - j c_{2k-1}) e^{jk\eta(t)} \right)$$

for all $t \in \mathbb{R}$.

Conversely, for any vector $\mathbf{c} \in \mathbb{R}^{2n+1}$, there exists a vector $\mathbf{p} \in \mathbb{R}^{2n+1}$ such that

$$\frac{\sum_{i=0}^{2n} p_i (\eta^{-1}(\theta))^i}{(1+(\eta^{-1}(\theta))^2)^n} = \text{Re} \left(c_0 + \sum_{k=1}^n (c_{2k} - j c_{2k-1}) e^{jk\theta} \right)$$

for all $\theta \in (0, 2\pi)$.

Proof: The bijectivity of $\eta(t)$ follows from simple calculus. The remaining part of the theorem is due to the invertibility of \mathbf{F} and $\tilde{\mathbf{F}}$. **Q.E.D.**

III. CHARACTERIZATION OF NON-NEGATIVE POLYNOMIALS ON A SEGMENT

In this section, we characterize the set of trigonometric polynomials that are non-negative over a segment of the unit circle. The main result will be stated in Theorem 2, but first, we state some preliminary results. We begin with a review of a well-known characterization [21], [22] of non-negative polynomials over a line segment in \mathbb{R} . We refer to Powers and Reznick [23] for a recent survey on characterizations for polynomials that are non-negative on an interval.

Proposition 1 (Markov-Lukacs): Let $\mathbf{p} \in \mathbb{R}^{2n+1}$ and $a, b \in \mathbb{R}, a < b$. Then

$$\mathbf{u}(t; 2n)^T \mathbf{p} \geq 0 \text{ for all } t \in [a, b]$$

if and only if there exist $\mathbf{q} \in \mathbb{R}^{n+1}$ and $\mathbf{r} \in \mathbb{R}^n$ such that

$$\mathbf{u}(t; 2n)^T \mathbf{p} = (\mathbf{u}(t; n)^T \mathbf{q})^2 + (t-a)(b-t) (\mathbf{u}(t; n-1)^T \mathbf{r})^2.$$

Moreover, it holds that

$$\mathbf{u}(t; 2n)^T \mathbf{p} \geq 0 \text{ for all } t \in [a, \infty)$$

if and only if there exist $\mathbf{q} \in \mathbb{R}^{n+1}$ and $\mathbf{r} \in \mathbb{R}^n$ such that

$$\mathbf{u}(t; 2n)^T \mathbf{p} = (\mathbf{u}(t; n)^T \mathbf{q})^2 + (t-a) (\mathbf{u}(t; n-1)^T \mathbf{r})^2.$$

The following corollary of Proposition 1 provides a characterization for polynomials that are non-negative over the complement of a finite symmetric interval in \mathbb{R} .

Corollary 1: Let $\mathbf{p} \in \mathbb{R}^{2n+1}$, and let $a > 0$ be a given positive number. Then

$$\mathbf{u}(t; 2n)^T \mathbf{p} \geq 0 \text{ for all } t \notin (-a, a)$$

if and only if there exist $\mathbf{q} \in \mathbb{R}^{n+1}$ and $\mathbf{r} \in \mathbb{R}^n$ such that

$$\mathbf{u}(t; 2n)^T \mathbf{p} = (\mathbf{u}(t; n)^T \mathbf{q})^2 + (t^2 - a^2) (\mathbf{u}(t; n-1)^T \mathbf{r})^2.$$

Proof: Let $\tilde{p}_i := p_{2n-i}$ for $i = 0, 1, \dots, 2n$. Then, for $t \neq 0$, we have

$$\mathbf{u}(t; 2n)^T \mathbf{p} = t^{2n} \left(\mathbf{u} \left(\frac{1}{t}; 2n \right)^T \tilde{\mathbf{p}} \right).$$

Since $t^2 \geq a^2 > 0$ for all $t \notin (-a, a)$, it follows that

$$\mathbf{u}(t; 2n)^T \mathbf{p} \geq 0 \text{ for all } t \notin (-a, a)$$

if and only if

$$\mathbf{u}(s; 2n)^T \tilde{\mathbf{p}} \geq 0 \text{ for all } s \in \left[-\frac{1}{a}, \frac{1}{a} \right].$$

By Proposition 1, the above relation holds if and only if there exist $\tilde{\mathbf{q}} \in \mathbb{R}^{n+1}$ and $\tilde{\mathbf{r}} \in \mathbb{R}^n$ such that

$$\mathbf{u}(s; 2n)^T \tilde{\mathbf{p}} = (\mathbf{u}(s; n)^T \tilde{\mathbf{q}})^2 + \frac{1-a^2 s^2}{a^2} (\mathbf{u}(s; n-1)^T \tilde{\mathbf{r}})^2.$$

Letting

$$q_i := \tilde{q}_{n-i} \text{ and } r_i := \frac{\tilde{r}_{n-1-i}}{a}, \quad \text{for } i = 0, 1, \dots, n$$

proves the result. **Q.E.D.**

The next corollary further extends Corollary 1 to the case where the interval is nonsymmetric.

Corollary 2: Let $\mathbf{p} \in \mathbb{R}^{2n+1}$ and $a, b \in \mathbb{R}$, $a < b$. Then

$$\mathbf{u}(t; 2n)^T \mathbf{p} \geq 0 \text{ for all } t \notin (a, b) \quad (14)$$

if and only if there exist $\mathbf{q} \in \mathbb{R}^{n+1}$ and $\mathbf{r} \in \mathbb{R}^n$ such that

$$\mathbf{u}(t; 2n)^T \mathbf{p} = (\mathbf{u}(t; n)^T \mathbf{q})^2 + (t-a)(t-b) (\mathbf{u}(t; n-1)^T \mathbf{r})^2. \quad (15)$$

Proof: It is obvious that (15) implies (14). Suppose now that (14) holds. Let us define

$$s := 2t - (a+b), \quad \tilde{a} := b - a.$$

It is clear (e.g., from Newton's binomial formula) that there exists $\tilde{\mathbf{p}} \in \mathbb{R}^{n+1}$ such that

$$\mathbf{u}(t; 2n)^T \mathbf{p} = \mathbf{u}(2t - (a+b); 2n)^T \tilde{\mathbf{p}} = \mathbf{u}(s; 2n)^T \tilde{\mathbf{p}}.$$

Since $(s^2 - \tilde{a}^2) = 4(t-a)(t-b)$, Corollary 1 implies that (15) holds for some $\mathbf{q} \in \mathbb{R}^{n+1}$ and $\mathbf{r} \in \mathbb{R}^n$. **Q.E.D.**

The next lemma determines a simple trigonometric polynomial that is non-negative over a given segment of the unit circle but is nonpositive over its complement. In particular, given $\alpha, \beta \in [0, 2\pi)$, we define a vector $\mathbf{d}(\alpha, \beta) \in \mathbb{R} \times \mathbb{C}$ as

$$\mathbf{d}(\alpha, \beta) := \begin{cases} \begin{bmatrix} \cos \alpha + \cos \beta - \cos(\beta - \alpha) - 1 \\ (1 - e^{j\alpha})(e^{j\beta} - 1) \end{bmatrix}, & \text{if } \alpha > 0 \\ \begin{bmatrix} -\sin \beta \\ j(1 - e^{j\beta}) \end{bmatrix}, & \text{if } \alpha = 0. \end{cases} \quad (16)$$

We remark that

$$\mathbf{d}(0, \beta) = \lim_{\alpha \downarrow 0} \frac{1}{\sin \alpha} \mathbf{d}(\alpha, \beta) \quad (17)$$

and

$$\mathbf{d}(\alpha, 2\pi - \alpha) = 2(1 - \cos \alpha) \begin{bmatrix} \cos \alpha \\ -1 \end{bmatrix}. \quad (18)$$

Lemma 4: Let $0 \leq \alpha < \beta < 2\pi$, and let $\mathbf{d}(\alpha, \beta)$ be defined as in (16). Then, the trigonometric polynomial $\text{Re } \mathbf{v}(\theta; 1)^H \mathbf{d}(\alpha, \beta)$ satisfies the following properties:

$$\begin{cases} \text{Re } \mathbf{v}(\theta; 1)^H \mathbf{d}(\alpha, \beta) > 0, & \text{for all } \theta \in (\alpha, \beta) \\ \text{Re } \mathbf{v}(\theta; 1)^H \mathbf{d}(\alpha, \beta) < 0, & \text{for all } \theta \in [0, 2\pi) \setminus [\alpha, \beta]. \end{cases}$$

Finally, we also need the following well-known representation result for trigonometric polynomials that are non-negative over the entire unit circle.

Proposition 2 (Riesz-Féjer): Let $\mathbf{p} \in \mathbb{R} \times \mathbb{C}^n$. Then

$$\text{Re } \mathbf{v}(\theta; n)^H \mathbf{p} \geq 0 \text{ for all } \theta \in [0, 2\pi)$$

if and only if there exists $\mathbf{q} \in \mathbb{R} \times \mathbb{C}^n$ such that

$$\text{Re } \mathbf{v}(\theta; n)^H \mathbf{p} = |\mathbf{v}(\theta; n)^H \mathbf{q}|^2.$$

We are now ready to establish the main result of this section.

Theorem 2: Let $\mathbf{p} \in \mathbb{R} \times \mathbb{C}^n$, $0 \leq \alpha < \beta < 2\pi$, and let $\mathbf{d}(\alpha, \beta)$ be given by (16). Then

$$\text{Re } \mathbf{v}(\theta; n)^H \mathbf{p} \geq 0 \text{ for all } \theta \in [\alpha, \beta] \quad (19)$$

if and only if there exist $\mathbf{q} \in \mathbb{R} \times \mathbb{C}^n$ and $\mathbf{r} \in \mathbb{R} \times \mathbb{C}^{n-1}$ such that

$$\text{Re } \mathbf{v}(\theta; n)^H \mathbf{p} = |\mathbf{v}(\theta; n)^H \mathbf{q}|^2 + (\text{Re } \mathbf{v}(\theta; 1)^H \mathbf{d}(\alpha, \beta)) |\mathbf{v}(\theta; n-1)^H \mathbf{r}|^2. \quad (20)$$

Moreover

$$\text{Re } \mathbf{v}(\theta; n)^H \mathbf{p} \geq 0 \text{ for all } \theta \in [0, 2\pi) \setminus (\alpha, \beta) \quad (21)$$

if and only if there exist $\mathbf{q} \in \mathbb{R} \times \mathbb{C}^n$ and $\mathbf{r} \in \mathbb{R} \times \mathbb{C}^{n-1}$ such that

$$\text{Re } \mathbf{v}(\theta; n)^H \mathbf{p} = |\mathbf{v}(\theta; n)^H \mathbf{q}|^2 - (\text{Re } \mathbf{v}(\theta; 1)^H \mathbf{d}(\alpha, \beta)) |\mathbf{v}(\theta; n-1)^H \mathbf{r}|^2. \quad (22)$$

Proof: In light of Lemma 4, it is obvious that (20) implies (19), and similarly, (22) implies (21). We now establish the converse relations. Fix some $\mathbf{p} \in \mathbb{R} \times \mathbb{C}^n$ such that (19) holds. We introduce a vector $\tilde{\mathbf{p}} \in \mathbb{R}^{2n+1}$ as

$$\begin{cases} \tilde{p}_{2i} = \text{Re } p_i, & \text{for } i = 0, 1, \dots, n \\ \tilde{p}_{2i-1} = \text{Im } p_i, & \text{for } i = 1, 2, \dots, n. \end{cases}$$

Using Theorem 1 and its notations, we have

$$\text{Re } \mathbf{v}(\theta; n)^H \mathbf{p} = \frac{\mathbf{u}(\eta^{-1}(\theta); 2n)^T \tilde{\mathbf{p}}}{(1 + (\eta^{-1}(\theta))^2)^2} \text{ for all } \theta \in (0, 2\pi). \quad (23)$$

Recall from Theorem 1 that $\eta(t) := \arg((t+j)^2/(1+t^2))$ is a bijection between \mathbb{R} and $(0, 2\pi)$. For convenience, we let $\eta(\infty) := 0$ to obtain a bijection between $\mathbb{R} \cup \{\infty\}$ and $[0, 2\pi)$. Let $a := \eta^{-1}(\alpha)$ and $b := \eta^{-1}(\beta)$. Since η is a bijection between $\mathbb{R} \cup \{\infty\}$ and $[0, 2\pi)$ and η is a decreasing function, it follows that

$$\eta([b, \infty)) = [0, \beta], \quad [b, \infty) = \eta^{-1}([0, \beta]).$$

and for $\alpha > 0$ that

$$\eta([b, a]) = [\alpha, \beta], \quad [b, a] = \eta^{-1}([\alpha, \beta]).$$

Since (19) holds, it follows from (23) that $\mathbf{u}(t; 2n)^T \tilde{\mathbf{p}} \geq 0$ for all $t \in [a, b]$. As a result, we may apply Proposition 1 to conclude that for given α and β , there exist $\tilde{\mathbf{q}} \in \mathbb{R}^{n+1}$ and $\tilde{\mathbf{r}} \in \mathbb{R}^n$ such that

$$\mathbf{u}(t; 2n)^T \tilde{\mathbf{p}} = (\mathbf{u}(t; n)^T \tilde{\mathbf{q}})^2 + (t-b)(a-t) (\mathbf{u}(t; n-1)^T \tilde{\mathbf{r}})^2$$

if $\alpha > 0$, or

$$\mathbf{u}(t; 2n)^T \tilde{\mathbf{p}} = (\mathbf{u}(t; n)^T \tilde{\mathbf{q}})^2 + (t-b) (\mathbf{u}(t; n-1)^T \tilde{\mathbf{r}})^2$$

if $\alpha = 0$. Due to Theorem 1 and Proposition 2, there must exist a $\mathbf{q} \in \mathbb{R} \times \mathbb{C}^n$ and a $\hat{\mathbf{r}} \in \mathbb{R} \times \mathbb{C}^{n-1}$ such that

$$\frac{(\mathbf{u}(t; n)^T \tilde{\mathbf{q}})^2}{(1+t^2)^n} = |\mathbf{v}(\eta(t); n)^H \mathbf{q}|^2$$

and

$$\frac{(\mathbf{u}(t; n-1)^T \tilde{\mathbf{r}})^2}{(1+t^2)^{n-1}} = |\mathbf{v}(\eta(t); n-1)^H \hat{\mathbf{r}}|^2.$$

Furthermore, we know from Theorem 1 that for given α and β , there must exist some $\mathbf{c} \in \mathbb{R} \times \mathbb{C}$ such that

$$\frac{(t-b)(a-t)}{1+t^2} = \operatorname{Re} \mathbf{v}(\eta(t); 1)^H \mathbf{c} \quad (24)$$

if $\alpha > 0$, or

$$\frac{t-b}{1+t^2} = \operatorname{Re} \mathbf{v}(\eta(t); 1)^H \mathbf{c} \quad (25)$$

if $\alpha = 0$. We have now shown that (19) implies the existence of vectors \mathbf{q} , $\hat{\mathbf{r}}$ and \mathbf{c} such that

$$\operatorname{Re} \mathbf{v}(\theta; n)^H \mathbf{p} = |\mathbf{v}(\theta; n)^H \mathbf{q}|^2 + (\operatorname{Re} \mathbf{v}(\theta; 1)^H \mathbf{c}) |\mathbf{v}(\theta; n-1)^H \hat{\mathbf{r}}|^2.$$

With an analogous argument, using Corollary 2, Theorem 1, and Proposition 2, we can show that if (21) holds, then

$$\operatorname{Re} \mathbf{v}(\theta; n)^H \mathbf{p} = |\mathbf{v}(\theta; n)^H \mathbf{q}|^2 - (\operatorname{Re} \mathbf{v}(\theta; 1)^H \mathbf{c}) |\mathbf{v}(\theta; n-1)^H \hat{\mathbf{r}}|^2.$$

It remains to show that \mathbf{c} is a positive multiple of $\mathbf{d}(\alpha, \beta)$.

Notice that

$$\begin{aligned} \operatorname{Re} \mathbf{v}(\eta(t); 1)^H \mathbf{c} &= \operatorname{Re} \mathbf{e}^H \mathbf{v}(\eta(t); 1) = \operatorname{Re} (c_0 + \bar{c}_1 e^{j\eta(t)}) \\ &= \operatorname{Re} \left(c_0 + \bar{c}_1 \frac{(j+t)^2}{(1+t^2)} \right) \\ &= \frac{1}{(1+t^2)} \\ &\quad \cdot (c_0 - \operatorname{Re} c_1 + 2t \operatorname{Im} c_1 + (c_0 + \operatorname{Re} c_1) t^2). \end{aligned}$$

Comparing this with (24), we obtain for $\alpha > 0$ that

$$\begin{aligned} \frac{(t-b)(a-t)}{1+t^2} &= \frac{1}{(1+t^2)} \\ &\quad \cdot (c_0 - \operatorname{Re} c_1 + 2t \operatorname{Im} c_1 + (c_0 + \operatorname{Re} c_1) t^2). \end{aligned}$$

This implies that

$$c_0 = -\frac{1+b}{2}, \quad c_1 = \frac{ab-1+(a+b)j}{2}.$$

For $\alpha = 0$, we have to obtain \mathbf{c} from

$$\frac{t-b}{1+t^2} = \frac{1}{(1+t^2)} (c_0 - \operatorname{Re} c_1 + 2t \operatorname{Im} c_1 + (c_0 + \operatorname{Re} c_1) t^2)$$

yielding $c_0 = -b/2$ and $c_1 = (b+j)/2$. It is easily verified that

$$\mathbf{d}(\alpha, \beta) = \begin{cases} 2(1-\cos \alpha)(1-\cos \beta) \mathbf{c}, & \text{for } \alpha > 0 \\ 2(1-\cos \beta) \mathbf{c}, & \text{for } \alpha = 0. \end{cases}$$

We complete the proof by setting $\mathbf{r} = 2(1-\cos \alpha)(1-\cos \beta) \hat{\mathbf{r}}$ for $\alpha > 0$ and $\mathbf{r} = 2(1-\cos \beta) \hat{\mathbf{r}}$ for $\alpha = 0$. **Q.E.D.**

Theorem 2 provides a complete analytical characterization of the set, say Ω , of trigonometric polynomials that are non-negative over a segment of the unit circle. In particular, it shows that any trigonometric polynomial in Ω can be written as a (non-negatively) weighted sum of two squared trigonometric polynomials [see (20)–(22)]. This result will be used in the next section to develop an LMI representation of the polynomials in Ω .

IV. LINEAR MATRIX INEQUALITY FORMULATION

In this section, we will use a result of Nesterov [24] and Theorem 2 of Section III to develop an LMI representation for Ω , which is the set of trigonometric polynomials that are non-negative over a segment of the unit circle.

In [24], Nesterov showed how to obtain an LMI representation for the cone of functions representable as a (weighted) sum of squares of functions in a given linear functional space V_n . In our case, the linear functional space V_n under consideration is the space of all trigonometric polynomials of degree at most n . Clearly, the components of $\mathbf{v}(\theta; n)$ form a basis of V_n . Below, we paraphrase the representation result of Nesterov for our functional space V_n .

Proposition 3: Let $\Delta \subseteq [0, 2\pi)$ be a given subset. Let there be m given trigonometric polynomials that are non-negative over Δ :

$$\operatorname{Re} \mathbf{v}(\theta; n_k)^H \mathbf{w}_k \geq 0 \text{ for all } \theta \in \Delta$$

where for each $k = 1, 2, \dots, m$, $n_k \in \{0, 1, \dots, n\}$, and $\mathbf{w}_k \in \mathbb{R} \times \mathbb{C}^{n_k}$. Let $\mathbf{L}_k(\cdot) : \mathbb{C}^{n+1} \mapsto \mathbb{C}^{(n+1) \times (n+1)}$ be a linear operator such that

$$\begin{aligned} \mathbf{L}_k(\mathbf{v}(\theta; n)) + \mathbf{L}_k(\mathbf{v}(\theta; n))^H &= 2 \left(\operatorname{Re} \mathbf{v}(\theta; n_k)^H \mathbf{w}_k \right) \\ &\quad \cdot \mathbf{v}(\theta; n - n_k) \mathbf{v}(\theta; n - n_k)^H \\ &\quad \forall \theta \in [0, 2\pi). \end{aligned}$$

Let $\ell \geq 1$ and $N \geq n+1$. Consider the cone

$$\begin{aligned} \mathcal{K} = \left\{ \mathbf{p} \in \mathbb{R} \times \mathbb{C}^n \mid \operatorname{Re} \mathbf{v}(\theta; n)^H \mathbf{p} \right. \\ \left. = \sum_{k=0}^{\ell} \left(\operatorname{Re} \mathbf{v}(\theta; n_k)^H \mathbf{w}_k \right) \sum_{m=0}^N \left| \mathbf{v}(\theta; n - n_k)^H \mathbf{q}_{m,k} \right|^2 \right. \\ \left. \text{for some } \mathbf{q}_{m,k} \in \mathbb{R} \times \mathbb{C}^{n-n_k} \right\}. \end{aligned} \quad (26)$$

Then, we have the following alternative LMI description of \mathcal{K} :

$$\begin{aligned} \mathcal{K} = \left\{ \mathbf{p} \in \mathbb{R} \times \mathbb{C}^n \mid \mathbf{p} + \xi \mathbf{e}_0 = \sum_{k=1}^m \mathbf{L}_k^*(\mathbf{X}_k) \right. \\ \left. \text{for some } \mathbf{X}_k \in \mathcal{H}_+^{(n+1-n_k) \times (n+1-n_k)}, \xi \in \mathbb{R} \right\} \end{aligned}$$

where \mathbf{e}_0 is the first column of the $(n+1) \times (n+1)$ identity matrix, and $\mathbf{L}_k^* : \mathbb{C}^{(n+1) \times (n+1)} \mapsto \mathbb{C}^{n+1}$ is the adjoint linear operator of \mathbf{L}_k .

Consider now the set of (the real part of) trigonometric polynomials of order n , i.e., functions $\text{Re } \mathbf{v}(\theta; n)^H \mathbf{p}$ in $\theta \in [0, 2\pi)$, with coefficients $\mathbf{p} \in \mathbb{R} \times \mathbb{C}^n$. Given $0 \leq \alpha < \beta < 2\pi$, we define the sets

$$\mathcal{K}(\alpha, \beta) := \left\{ \mathbf{p} \in \mathbb{R} \times \mathbb{C}^n \mid \text{Re } \mathbf{v}(\theta; n)^H \mathbf{p} \geq 0 \right. \\ \left. \text{for all } \theta \in [\alpha, \beta] \right\} \quad (27)$$

and

$$\bar{\mathcal{K}}(\alpha, \beta) := \left\{ \mathbf{p} \in \mathbb{R} \times \mathbb{C}^n \mid \text{Re } \mathbf{v}(\theta; n)^H \mathbf{p} \geq 0 \right. \\ \left. \text{for all } \theta \in [0, 2\pi) \setminus (\alpha, \beta) \right\}. \quad (28)$$

Thus, $\mathcal{K}(\alpha, \beta)$ and $\bar{\mathcal{K}}(\alpha, \beta)$ describe the sets of (the real parts of) trigonometric polynomials that are non-negative over a segment of the unit circle. We also let

$$\mathcal{K}(0, 2\pi) := \left\{ \mathbf{p} \in \mathbb{R} \times \mathbb{C}^n \mid \text{Re } \mathbf{v}(\theta; n)^H \mathbf{p} \geq 0 \right. \\ \left. \text{for all } \theta \in [0, 2\pi) \right\} \quad (29)$$

describe the trigonometric polynomials that are non-negative on the entire unit circle. We may interpret $\mathcal{K}(\alpha, \beta)$ and $\bar{\mathcal{K}}(\alpha, \beta)$ as convex cones in \mathbb{R}^{2n+1} . Since (the real part of) a trigonometric polynomial is non-negative on a closed segment if and only if it is non-negative on the corresponding open segment, these cones are invariant to the opening or closure of either end of the given segment. This fact simplifies the application of the LMI descriptions of these cones that we now develop. By Theorem 2, these cones can be equivalently described as

$$\mathcal{K}(\alpha, \beta) = \left\{ \mathbf{p} \in \mathbb{R} \times \mathbb{C}^n \mid \text{Re } \mathbf{v}(\theta; n)^H \mathbf{p} = |\mathbf{v}(\theta; n)^H \mathbf{q}|^2 \right. \\ \left. + (\text{Re } \mathbf{v}(\theta; 1)^H \mathbf{d}(\alpha, \beta)) |\mathbf{v}(\theta; n-1)^H \mathbf{r}|^2 \right. \\ \left. \text{for some } \mathbf{q} \in \mathbb{R} \times \mathbb{C}^n \text{ and } \mathbf{r} \in \mathbb{R} \times \mathbb{C}^{n-1} \right\} \quad (30)$$

and

$$\bar{\mathcal{K}}(\alpha, \beta) = \left\{ \mathbf{p} \in \mathbb{R} \times \mathbb{C}^n \mid \text{Re } \mathbf{v}(\theta; n)^H \mathbf{p} = |\mathbf{v}(\theta; n)^H \mathbf{q}|^2 \right. \\ \left. - (\text{Re } \mathbf{v}(\theta; 1)^H \mathbf{d}(\alpha, \beta)) |\mathbf{v}(\theta; n-1)^H \mathbf{r}|^2 \right. \\ \left. \text{for some } \mathbf{q} \in \mathbb{R} \times \mathbb{C}^n \text{ and } \mathbf{r} \in \mathbb{R} \times \mathbb{C}^{n-1} \right\} \quad (31)$$

where $\mathbf{d}(\alpha, \beta)$ is given by (16). Notice that both $\mathcal{K}(\alpha, \beta)$ and $\bar{\mathcal{K}}(\alpha, \beta)$ are in the form of (26) since each element of $\mathcal{K}(\alpha, \beta)$ or $\bar{\mathcal{K}}(\alpha, \beta)$ can be written as a (non-negatively) weighted sum of squares. Therefore, Proposition 3 implies that $\mathcal{K}(\alpha, \beta)$ and $\bar{\mathcal{K}}(\alpha, \beta)$ both possess an LMI description. To derive an explicit

form of these LMI representations, we need to make precise the linear operators \mathbf{L}_k . This is what we do next.

We define the unit lower triangular $(n+1) \times (n+1)$ Toeplitz matrices $\mathbf{T}_{0,n}, \mathbf{T}_{1,n}, \dots, \mathbf{T}_{n,n}$ as

$$[\mathbf{T}_{k,n}]_{ij} = \begin{cases} 1, & \text{if } i = k + j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{with } i, j \in \{0, 1, \dots, n\}. \quad (32)$$

Thus, $\mathbf{T}_{0,n} = \mathbf{I}$ and $\langle \mathbf{T}_{k,n}, \mathbf{X} \rangle = \sum_{\ell=0}^{n-k} \mathbf{X}_{\ell+k, \ell}$ for all $\mathbf{X} \in \mathbb{C}^{(n+1) \times (n+1)}$. That is, $\langle \mathbf{T}_{k,n}, \mathbf{X} \rangle$ is the sum of the elements on the k th lower off diagonal of \mathbf{X} . Let us define a linear operator $\mathbf{L} : \mathbb{C}^{n+1} \mapsto \mathbb{C}^{(n+1) \times (n+1)}$ as

$$\mathbf{L}(\mathbf{y}) = y_0 \mathbf{T}_{0,n} + 2 \sum_{i=1}^n y_i \mathbf{T}_{i,n}. \quad (33)$$

It can be checked that $\mathbf{L}(\mathbf{y})$ is lower triangular and

$$\mathbf{L}(\mathbf{v}(\theta; n)) + \mathbf{L}(\mathbf{v}(\theta; n))^H = 2\mathbf{v}(\theta; n)\mathbf{v}(\theta; n)^H \\ \text{for all } \theta \in [0, 2\pi). \quad (34)$$

To determine the adjoint operator \mathbf{L}^* , we note that

$$\langle \mathbf{L}(\mathbf{y}), \mathbf{X} \rangle = \bar{y}_0 \langle \mathbf{T}_{0,n}, \mathbf{X} \rangle + 2 \sum_{i=1}^n \bar{y}_i \langle \mathbf{T}_{i,n}, \mathbf{X} \rangle \\ = \mathbf{y}^H \mathbf{L}^*(\mathbf{X}), \quad \forall \mathbf{X} \in \mathbb{C}^{(n+1) \times (n+1)}$$

where the adjoint $\mathbf{q} = \mathbf{L}^*(\mathbf{X}) \in \mathbb{C}^{n+1}$ is given by

$$q_0 = \langle \mathbf{T}_{0,n}, \mathbf{X} \rangle, \quad q_i = 2 \langle \mathbf{T}_{i,n}, \mathbf{X} \rangle, \quad \text{for } i = 1, 2, \dots, n. \quad (35)$$

In addition, we need to define a family of operators $\mathbf{\Lambda}(\mathbf{y}; \alpha, \beta) : \mathbb{C}^{n+1} \mapsto \mathbb{C}^{(n+1) \times (n+1)}$ that are linear in $\mathbf{y} \in \mathbb{C}^{n+1}$ and parameterized by $\alpha, \beta \in [0, 2\pi]$. Let $\mathbf{d}(\alpha, \beta) = [d_0(\alpha, \beta), d_1(\alpha, \beta)]^T$ be given by (16). Then

$$\mathbf{\Lambda}(\mathbf{y}; \alpha, \beta) := d_0(\alpha, \beta) \left(y_0 \mathbf{T}_{0,n-1} + 2 \sum_{k=1}^{n-1} y_k \mathbf{T}_{k,n-1} \right) \\ + \overline{d_1(\alpha, \beta)} \left(\sum_{k=1}^n y_k \mathbf{T}_{k-1,n-1} \right) \\ + d_1(\alpha, \beta) \left(\sum_{k=0}^{n-2} y_k \mathbf{T}_{k+1,n-1} \right).$$

It can be checked that

$$\mathbf{\Lambda}(\mathbf{v}(\theta, n)) + \mathbf{\Lambda}(\mathbf{v}(\theta, n))^H = 2(\text{Re } \mathbf{v}(\theta; 1)^H \mathbf{d}(\alpha, \beta)) \mathbf{v}(\theta; n-1) \\ \cdot \mathbf{v}(\theta; n-1)^H, \quad \forall \theta \in [0, 2\pi).$$

To determine the adjoint operator of $\mathbf{\Lambda}$, we fix any $\mathbf{y} \in \mathbb{C}^{n+1}$ and any $\mathbf{X} \in \mathbb{C}^{n \times n}$ and consider

$$\langle \mathbf{\Lambda}(\mathbf{y}; \alpha, \beta), \mathbf{X} \rangle = d_0(\alpha, \beta) \left(\bar{y}_0 \langle \mathbf{T}_{0,n-1}, \mathbf{X} \rangle \right. \\ \left. + 2 \sum_{k=1}^{n-1} \bar{y}_k \langle \mathbf{T}_{k,n-1}, \mathbf{X} \rangle \right) \\ + d_1(\alpha, \beta) \left(\sum_{k=1}^n \bar{y}_k \langle \mathbf{T}_{k-1,n-1}, \mathbf{X} \rangle \right) \\ + \overline{d_1(\alpha, \beta)} \left(\sum_{k=0}^{n-2} \bar{y}_k \langle \mathbf{T}_{k+1,n-1}, \mathbf{X} \rangle \right) \\ = \mathbf{y}^H \mathbf{\Lambda}^*(\mathbf{X}; \alpha, \beta).$$

Thus, the adjoint $\mathbf{q} = \mathbf{\Lambda}^*(\mathbf{X}; \alpha, \beta)$ is given by (36), shown at the bottom of the page. Combining this with (30), (31) and (35) and invoking Proposition 3, we obtain the following key result.

Theorem 3: Let $0 \leq \alpha < \beta < 2\pi$. Suppose $\mathcal{K}(\alpha, \beta)$ and $\bar{\mathcal{K}}(\alpha, \beta)$ are given by (27) and (28). Let \mathbf{L}^* and $\mathbf{\Lambda}^*$ be the adjoint operators defined by (35) and (36). Then, the cones $\mathcal{K}(\alpha, \beta)$ and $\bar{\mathcal{K}}(\alpha, \beta)$ admit the following LMI description:

$$\mathcal{K}(\alpha, \beta) = \left\{ \begin{array}{l} \mathbf{p} | \mathbf{p} + \xi j \mathbf{e}_0 = \mathbf{L}^*(\mathbf{X}) + \mathbf{\Lambda}^*(\mathbf{Z}; \alpha, \beta) \\ \text{for some } \mathbf{X} \in \mathcal{H}_+^{(n+1) \times (n+1)} \\ \mathbf{Z} \in \mathcal{H}_+^{n \times n}, \xi \in \mathbb{R} \end{array} \right\} \quad (37)$$

and

$$\bar{\mathcal{K}}(\alpha, \beta) = \left\{ \begin{array}{l} \mathbf{p} | \mathbf{p} + \xi j \mathbf{e}_0 = \mathbf{L}^*(\mathbf{X}) - \mathbf{\Lambda}^*(\mathbf{Z}; \alpha, \beta) \\ \text{for some } \mathbf{X} \in \mathcal{H}_+^{(n+1) \times (n+1)} \\ \mathbf{Z} \in \mathcal{H}_+^{n \times n}, \xi \in \mathbb{R} \end{array} \right\}. \quad (38)$$

Moreover

$$\mathcal{K}(0, 2\pi) = \left\{ \mathbf{L}^*(\mathbf{X}) | \mathbf{X} \in \mathcal{H}_+^{(n+1) \times (n+1)} \right\}. \quad (39)$$

Theorem 3 provides an equivalent LMI description for a trigonometric polynomial that is non-negative over a given segment $[\alpha, \beta]$ (or its complement) of the unit circle. (Observe that $\mathcal{K}(\alpha, 2\pi) = \bar{\mathcal{K}}(0, \alpha)$.) As mentioned in Section I, this LMI formulation is of practical interest because it generates a precise finite representation of the spectral mask constraints that are often encountered in the design of digital filters. Furthermore, the LMI formulation of the mask results in filter design problems that can be efficiently solved via well-established interior point methods [25]. We will give some detailed examples in Section V.

Equation (39) is a new formulation of the Positive Real Lemma [16], [17] (and the closely related Kalman-Yakubovich-Popov [KYP] Lemma) for FIR systems (see also [24] and [26, Sec. 3.2]). The new formulation is the dual of the standard formulation and states that for r_m , $-M + 1 \leq m \leq M - 1$, with $r_{-m} = \bar{r}_m$, $R(e^{j\theta}) \geq 0$ for all $\theta \in [0, 2\pi)$ if and only if there exists an $\mathbf{X} \in \mathcal{H}_+^{M \times M}$ such that $\text{tr}(\mathbf{X}) = r_0$ and $\sum_{\ell=0}^{M-1-m} [\mathbf{X}]_{\ell+m, \ell} = r_m$ for $1 \leq m \leq M - 1$. Thus, Theorem 3 can be seen as an extension of Positive Real Lemma for FIR systems.

Now, let us consider the special case of real trigonometric polynomials of the form $\sum_{k=0}^n p_k \cos(k\theta)$ with coefficients $\mathbf{p} \in \mathbb{R}^{n+1}$ and segments of the form $[\alpha, 2\pi - \alpha]$. Since $\cos(k\theta)$ and the segment are symmetric with respect to $\theta = \pi$, we need

only consider the subsegment $[\alpha, \pi]$, and the LMI descriptions in Theorem 3 can be simplified to

$$\begin{aligned} \mathcal{K}_{\text{real}}(\alpha) &= \left\{ \mathbf{p} \in \mathbb{R}^{n+1} \mid \sum_{k=0}^n p_k \cos(k\theta) \geq 0, \text{ for all } \theta \in [\alpha, \pi] \right\} \\ &= \left\{ \mathbf{p} \mid \mathbf{p} = \mathbf{L}^*(\mathbf{X}) + \mathbf{\Lambda}^*(\mathbf{Z}; \alpha, 2\pi - \alpha) \right. \\ &\quad \left. \text{for some } \mathbf{X} \in \mathcal{S}_+^{(n+1) \times (n+1)}, \mathbf{Z} \in \mathcal{S}_+^{n \times n} \right\} \quad (40) \end{aligned}$$

(for non-negativity on $[\alpha, \pi]$) and

$$\begin{aligned} \bar{\mathcal{K}}_{\text{real}}(\alpha) &= \left\{ \mathbf{p} \in \mathbb{R}^{n+1} \mid \sum_{k=0}^n p_k \cos(k\theta) \geq 0, \text{ for all } \theta \in [0, \alpha] \right\} \\ &= \left\{ \mathbf{p} \mid \mathbf{p} = \mathbf{L}^*(\mathbf{X}) - \mathbf{\Lambda}^*(\mathbf{Z}; \alpha, 2\pi - \alpha) \right. \\ &\quad \left. \text{for some } \mathbf{X} \in \mathcal{S}_+^{(n+1) \times (n+1)}, \mathbf{Z} \in \mathcal{S}_+^{n \times n} \right\} \quad (41) \end{aligned}$$

(for non-negativity on $[0, \alpha]$), where $\mathbf{\Lambda}^*(\mathbf{Z}; \alpha, 2\pi - \alpha)$ is given by (36) with $d_0(\alpha, 2\pi - \alpha)$ and $d_1(\alpha, 2\pi - \alpha)$ simplified to be [see (18)]

$$\begin{aligned} d_0(\alpha, 2\pi - \alpha) &= 2 \cos \alpha (1 - \cos \alpha) \quad \text{and} \\ d_1(\alpha, 2\pi - \alpha) &= -2(1 - \cos \alpha). \end{aligned} \quad (42)$$

Notice that \mathbf{X} and \mathbf{Z} in (40) [and (41)] are real symmetric, rather than complex Hermitian. To see why we can restrict to real symmetric matrices, consider a Hermitian positive semidefinite matrix $\mathbf{X} = (\text{Re } \mathbf{X}) + j(\text{Im } \mathbf{X})$ given in the representation (37). Since $\mathbf{X} \in \mathcal{H}_+^{n \times n}$, for real $\mathbf{q} \in \mathbb{R}^n$, we have that

$$0 \leq \mathbf{q}^T \mathbf{X} \mathbf{q} = \mathbf{q}^T (\text{Re } \mathbf{X}) \mathbf{q} + j \mathbf{q}^T (\text{Im } \mathbf{X}) \mathbf{q} = \mathbf{q}^T (\text{Re } \mathbf{X}) \mathbf{q}.$$

Hence, $\text{Re } \mathbf{X} \in \mathcal{S}_+^{n \times n}$. Moreover, $\langle \mathbf{T}_{i,n}, \mathbf{X} \rangle = \langle \mathbf{T}_{i,n}, \text{Re } \mathbf{X} \rangle$. Thus, if the imaginary parts of the coefficients p_i are restricted to zero, then since d_0 and d_1 in (42) are real, we can replace \mathbf{X} and \mathbf{Z} [obtained from (37)] by $\text{Re } \mathbf{X}$ and $\text{Re } \mathbf{Z}$ with both $\text{Re } \mathbf{X}$ and $\text{Re } \mathbf{Z}$ still being positive semidefinite.

V. APPLICATIONS

We now show how the results of Section IV can be applied to the design of FIR filters and to data-independent narrowband beamformers.

A. FIR Filter Design

In optimization-based designs of (real-valued) FIR filters, one often encounters a (relative) spectral mask constraint of the form

$$\zeta \check{L}(e^{j\theta}) \leq |G(e^{j\theta})| \leq \zeta \check{U}(e^{j\theta}) \quad \text{for all } \theta \in [0, \pi] \quad (43)$$

$$\begin{cases} q_0 = d_0(\alpha, \beta) \langle \mathbf{T}_{0,n-1}, \mathbf{X} \rangle + \overline{d_1(\alpha, \beta)} \langle \mathbf{T}_{1,n-1}, \mathbf{X} \rangle \\ q_k = 2d_0(\alpha, \beta) \langle \mathbf{T}_{k,n-1}, \mathbf{X} \rangle + d_1(\alpha, \beta) \langle \mathbf{T}_{k-1,n-1}, \mathbf{X} \rangle + \overline{d_1(\alpha, \beta)} \langle \mathbf{T}_{k+1,n-1}, \mathbf{X} \rangle, \quad \text{for } k = 1, 2, \dots, n-2 \\ q_{n-1} = 2d_0(\alpha, \beta) \langle \mathbf{T}_{n-1,n-1}, \mathbf{X} \rangle + d_1(\alpha, \beta) \langle \mathbf{T}_{n-2,n-1}, \mathbf{X} \rangle \\ q_n = d_1(\alpha, \beta) \langle \mathbf{T}_{n-1,n-1}, \mathbf{X} \rangle. \end{cases} \quad (36)$$

where $\zeta > 0$, and a normalization constraint either on the filter coefficients or on ζ . [We have used the “breve” notation to distinguish (43) from (1).] As discussed in the introduction, the mask constraint can be made convex by constraining $G(e^{j\theta})$ to have “linear phase,” or by reformulating the constraint in terms of the autocorrelation sequence $r_m = \sum_k g_k g_{k-m}$ as $\zeta^2 \check{L}(e^{j\theta})^2 \leq R(e^{j\theta}) \leq \zeta^2 \check{U}(e^{j\theta})^2$. In both of these cases, we can exploit the results of Section IV to precisely transform the piecewise constant and piecewise trigonometric polynomial portions of the mask into pairs of LMIs. The first step is to write $G(e^{j\theta})$ or $R(e^{j\theta})$ in the form $\text{Re } \mathbf{v}(\theta; \cdot)^H \mathbf{p}$. To do so, we define $\mathbf{M} \in \mathbb{R}^{(2M-1) \times M}$ and $\check{\mathbf{I}} \in \mathbb{R}^{M \times M}$ such that

$$\mathbf{M} := \begin{bmatrix} \mathbf{0} & \mathbf{J} \\ 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \quad \text{and} \quad \check{\mathbf{I}} := \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & 2\mathbf{I} \end{bmatrix} \quad (44)$$

where \mathbf{I} is the $(M-1) \times (M-1)$ identity matrix, and \mathbf{J} is the $(M-1) \times (M-1)$ matrix with ones on the anti-diagonal and zeros elsewhere. For a filter of length M , if we define $\check{\mathbf{r}} \in \mathbb{R}^M$ such that $[\check{\mathbf{r}}]_m = r_m$, $0 \leq m \leq M-1$, then

$$\begin{aligned} R(e^{j\theta}) &= e^{j(M-1)\theta} \mathbf{v}(\theta; 2(M-1))^H \mathbf{M} \check{\mathbf{r}} \\ &= \text{Re } \mathbf{v}(\theta; M-1)^H \check{\mathbf{I}} \check{\mathbf{r}}. \end{aligned}$$

Similarly, for a filter of odd length $2M-1$ that is symmetric and centered at the origin, if we define $\check{\mathbf{g}} \in \mathbb{R}^M$ such that $[\check{\mathbf{g}}]_k = g_k$, $0 \leq k \leq M-1$, then $G(e^{j\theta}) = \text{Re } \mathbf{v}(\theta; M-1)^H \check{\mathbf{I}} \check{\mathbf{g}}$. The frequency response of other linear-phase filters can be written in related ways, but for brevity, we will consider only the odd-length symmetric case.

To further simplify our exposition, we will first consider the design of a simple lowpass filter with a piecewise constant mask. A natural extension to a piecewise trigonometric polynomial mask is provided later in this section, and extensions to bandpass and multiband filters are implicit in the design in Section V-B. The simple piecewise constant lowpass spectral mask can be written in the form of (43), where

$$\begin{aligned} \check{L}(e^{j\theta}) &= \begin{cases} \check{L}_p & 0 \leq \theta \leq 2\pi f_p \\ \check{L}_t & 2\pi f_p < \theta \leq 2\pi f_s \\ \check{L}_s & 2\pi f_s < \theta \leq \pi \end{cases} \quad \text{and} \\ \check{U}(e^{j\theta}) &= \begin{cases} \check{U}_p & 0 \leq \theta < 2\pi f_s \\ \check{U}_s & 2\pi f_s \leq \theta \leq \pi \end{cases} \end{aligned} \quad (45)$$

with f_p and f_s denoting the normalized frequencies of the passband and stopband edges, respectively, $0 \leq f_p < f_s \leq 1/2$ and $0 < \check{U}_s \leq \check{U}_p$, $0 \leq \check{L}_p < \check{U}_p$, and $\check{L}_s \leq \check{L}_p$. In the case of linear-phase filters, we set $\check{L}_t = -\check{U}_p$ and $\check{L}_s = -\check{U}_s$, whereas for autocorrelation designs, we set $\check{L}_t = \check{L}_s = 0$. By observing the common form of $G(e^{j\theta})$ and $R(e^{j\theta})$ and that $\check{L}_t \leq \check{L}_s \leq \check{L}_p$ and $\check{U}_s \leq \check{U}_p$, the spectral mask constraint can be rewritten in a generic form as

$$\begin{cases} \check{\mathbf{I}} \check{\mathbf{x}} - \zeta^q \check{L}_p^q \mathbf{e}_0 \in \bar{\mathcal{K}}_{\text{real}}(2\pi f_p) \\ \check{\mathbf{I}} \check{\mathbf{x}} - \zeta^q \check{L}_t^q \mathbf{e}_0 \in \mathcal{K}_{\text{real}}(0) \\ \check{\mathbf{I}} \check{\mathbf{x}} - \zeta^q \check{L}_s^q \mathbf{e}_0 \in \mathcal{K}_{\text{real}}(2\pi f_s) \\ \zeta^q \check{U}_p^q \mathbf{e}_0 - \check{\mathbf{I}} \check{\mathbf{x}} \in \mathcal{K}_{\text{real}}(0) \\ \zeta^q \check{U}_s^q \mathbf{e}_0 - \check{\mathbf{I}} \check{\mathbf{x}} \in \mathcal{K}_{\text{real}}(2\pi f_s) \end{cases} \quad \text{and} \quad (46)$$

where $q = 1$ and $\check{\mathbf{x}} = \check{\mathbf{g}}$ when we design an odd-length symmetric filter and $q = 2$ and $\check{\mathbf{x}} = \check{\mathbf{r}}$ for autocorrelation designs.

For autocorrelation designs, $\check{L}_t = \check{L}_s = 0$, and hence, the constraint $\check{\mathbf{I}} \check{\mathbf{x}} - \zeta^q \check{L}_s^q \mathbf{e}_0 \in \mathcal{K}_{\text{real}}(2\pi f_s)$ is redundant.

The constraints in (46) define the set of feasible filters. The question that remains is which of these filters is the “best.” A large class of filter design objectives can be cast as the minimization of a convex quadratic function of the parameters. This class includes the weighted least-squares approximation of some desired magnitude response [3], [4], [9]. Filter design problems in this class take the following form: Given a positive semidefinite matrix \mathbf{Q} , a vector \mathbf{l} , and an integer $q \in \{1, 2\}$, find $\check{\mathbf{x}}$ achieving

$$\min_{\check{\mathbf{x}}} \check{\mathbf{x}}^T \mathbf{Q} \check{\mathbf{x}} - 2\mathbf{l}^T \check{\mathbf{x}} \quad (47)$$

subject to (46) and a linear normalization constraint on either $\check{\mathbf{x}}$ or ζ or show that none exist. This generic design problem can be solved by solving the following convex optimization problem.

Problem 1: Given q from (46) and given $\mathbf{Q} = \mathbf{L}\mathbf{L}^T$, \mathbf{l} , f_p , f_s , \check{L}_p , \check{L}_t , \check{L}_s , \check{U}_p , \check{U}_s , and M , find $\check{\mathbf{x}} \in \mathbb{R}^M$ achieving $\min \Gamma - 2\mathbf{l}^T \check{\mathbf{x}}$ over $\check{\mathbf{x}}$, $\zeta > 0$, $\mathbf{X}^{(pu)}$, $\mathbf{X}^{(pl)}$, $\mathbf{X}^{(t)}$, $\mathbf{X}^{(su)}$ $\in \mathcal{S}_+^{M \times M}$, $\mathbf{Z}^{(pl)}$, $\mathbf{Z}^{(su)}$ $\in \mathcal{S}_+^{(M-1) \times (M-1)}$, and if $q = 1$, $\mathbf{X}^{(sl)}$ $\in \mathcal{S}_+^{M \times M}$ and $\mathbf{Z}^{(sl)}$ $\in \mathcal{S}_+^{M-1 \times M-1}$, subject to $\|\mathbf{L}^T \check{\mathbf{x}}\|_2^2 \leq \Gamma$

$$\check{\mathbf{I}} \check{\mathbf{x}} - \zeta^q \check{L}_p^q \mathbf{e}_0 = \mathbf{L}^* \left(\mathbf{X}^{(pl)} \right) - \mathbf{\Lambda}^* \left(\mathbf{Z}^{(pl)}; 2\pi f_p, 2\pi(1-f_p) \right) \quad (48)$$

$$\check{\mathbf{I}} \check{\mathbf{x}} - \zeta^q \check{L}_t^q \mathbf{e}_0 = \mathbf{L}^* \left(\mathbf{X}^{(t)} \right) \quad (49)$$

$$\zeta^q \check{U}_p^q \mathbf{e}_0 - \check{\mathbf{I}} \check{\mathbf{x}} = \mathbf{L}^* \left(\mathbf{X}^{(pu)} \right) \quad (50)$$

$$\zeta^q \check{U}_s^q \mathbf{e}_0 - \check{\mathbf{I}} \check{\mathbf{x}} = \mathbf{L}^* \left(\mathbf{X}^{(su)} \right) + \mathbf{\Lambda}^* \left(\mathbf{Z}^{(su)}; 2\pi f_s, 2\pi(1-f_s) \right) \quad (51)$$

and, if $q = 1$

$$\check{\mathbf{I}} \check{\mathbf{x}} - \zeta^q \check{L}_s^q \mathbf{e}_0 = \mathbf{L}^* \left(\mathbf{X}^{(sl)} \right) + \mathbf{\Lambda}^* \left(\mathbf{Z}^{(sl)}; 2\pi f_s, 2\pi(1-f_s) \right) \quad (52)$$

and one of the normalizations $\zeta = 1$ or $\mathbf{e}^T \check{\mathbf{x}} = 1$, for a given vector \mathbf{e} , or show that none exist.

In Problem 1, (48), (49), and (52) enforce the lower bound constraint of the spectral mask, and (50) and (51) enforce the upper bound constraint. Problem 1 consists of a linear objective, linear equality constraints [(48)–(51) and (52) where applicable], a linear inequality constraint on ζ , positive semi-definiteness constraints on the various \mathbf{X} and \mathbf{Z} matrices, and the constraint $\|\mathbf{L}^T \check{\mathbf{x}}\|_2^2 \leq \Gamma$. The set of vectors $[\Gamma, (\mathbf{L}^T \check{\mathbf{x}})^T]^T \in \mathbb{R}^{M+1}$ that satisfy this last constraint can be transformed to the intersection of a “rotated” second-order cone in \mathbb{R}^{M+2} and a hyperplane (e.g., [26]). Hence, Problem 1 is a convex symmetric cone program [27], [28], which can be efficiently solved using well-established interior point methods [29]. Furthermore, infeasibility can be reliably detected. If $\check{\mathbf{x}}$ represents the autocorrelation sequence of the filter, then an optimal filter can be obtained from the solution of Problem 1 by spectral factorization [9], [11].

We now demonstrate the flexibility of this design method by solving a number of filter design problems using small variations on Problem 1.

Example 1: Consider the design of a length 49 FIR filter which has the minimal “stopband energy” $E_s = (1/\pi) \int_{2\pi f_s}^{\pi} |G(e^{j\theta})|^2 d\theta$, subject to the spectral mask in (45), with $f_p = 590/4915.2$, $f_s = 740/4915.2$, $\check{L}_p^2 = 10^{-0.15}$, $\check{U}_p^2 = 10^{0.15}$, $\check{U}_s^2 = 10^{-4}$, i.e., $f_p \approx 0.12$,

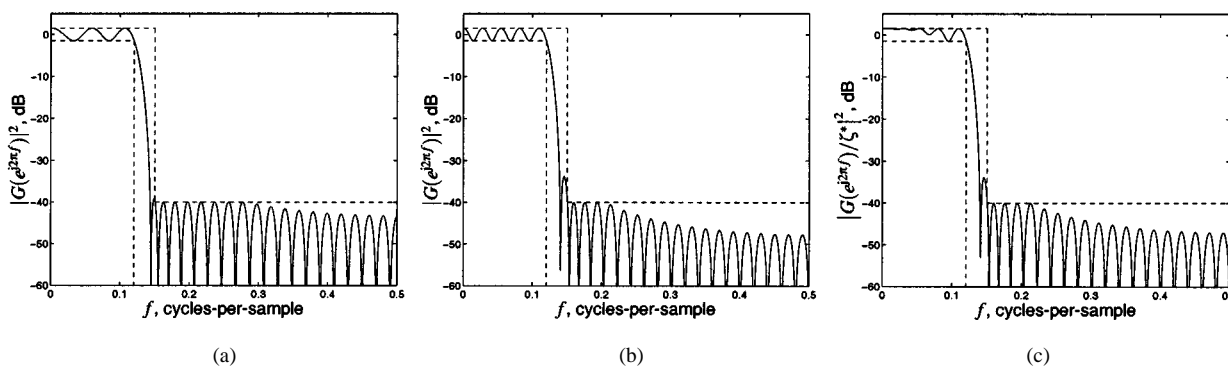


Fig. 1. Power spectra of length 49 filters from Example 1, along with the corresponding mask. In Fig. 1(c), ζ^* is the optimal value of ζ from Problem 1. (a) Linear-phase filter minimizing stopband energy. (b) Nonlinear-phase filter minimizing stopband energy. (c) Nonlinear-phase filter minimizing proportion of energy in the stopband.

$f_s \approx 0.15$, ± 1.5 dB passband ripple and 40 dB stopband suppression. (The choice of this particular mask is explained in Example 3.) For an odd-length symmetric filter, $E_s = \tilde{\mathbf{g}}^T \mathbf{Q} \tilde{\mathbf{g}}$, where $\mathbf{Q} = \tilde{\mathbf{I}} \tilde{\mathbf{Q}} \tilde{\mathbf{I}}$

$$[\tilde{\mathbf{Q}}]_{ij} = 2(\text{sinc}(i+j) + \text{sinc}(i-j))4f_c \left(\text{sinc}(2f_c(i+j)) + \text{sinc}(2f_c(i-j)) \right)$$

for $0 \leq i, j \leq M-1$ and $\text{sinc}(x) = \sin(\pi x)/(\pi x)$ for $x \neq 0$ and 1 for $x = 0$. For a general filter, $E_s = \tilde{\mathbf{r}}^T \tilde{\mathbf{r}}$, where $\tilde{l}_0 = 1/2 - f_c$, and $\tilde{l}_m = -2f_c \text{sinc}(2f_c m)$ for $1 \leq m \leq M-1$. Therefore, optimal filters can be designed using Problem 1 with the normalization constraint $\zeta = 1$. For $f_c = (f_p + f_s)/2$, the power spectrum of the optimal linear-phase filter is shown in Fig. 1(a), and that of an optimal nonlinear phase filter is shown in Fig. 1(b). Each design problem was solved using a Matlab-based general-purpose symmetric cone program solver called SeDuMi [26]. The linear-phase case was solved in 3.5 s on a 400 MHz Pentium II workstation, whereas the nonlinear phase case required 24 s. The sharper cutoff and improved high-frequency decay of the nonlinear-phase filter are clear from these figures. Although these filters minimize the stopband energy, they do not minimize the proportion of the total energy of the filter in the stopband. A nonlinear-phase filter that does so can be found by removing the constraint $\zeta = 1$ from Problem 1 (and, hence, allowing the mask to “float”) and replacing it with $r_0 = 1$. The resulting optimal autocorrelation was obtained in 25 s, and the power spectrum of an optimal filter is shown in Fig. 1(c). Observe that the flatter passband response in this case is achieved without greatly affecting the stopband decay. \square

In some applications, one may wish to enforce a spectral mask constraint that is not piecewise constant. For example, one may wish to have a “roll-off” zone that provides a more gradual transition between the passband and stopband in Fig. 1. (See Fig. 2 for an example.) We will now demonstrate how the large and diverse class of piecewise trigonometric polynomial masks can be precisely enforced using Theorem 3. For simplicity, we will restrict our attention to the case of enforcing a roll-off constraint on a lowpass filter, but the techniques can be easily generalized. Let f_r denote the frequency at the “left edge” of the roll-off portion of the mask, and let f_s denote the frequency at the left edge of the subsequent constant portion. Let the portion

of $\check{U}(e^{j\theta})^q$ in the roll-off region be described by the real part of a trigonometric polynomial, i.e., let $\check{U}(e^{j\theta})^q = \text{Re } B_q(e^{j\theta})$ for $\theta \in [2\pi f_r, 2\pi f_s]$, where $B_q(e^{j\theta}) = \sum_{k=0}^{M_{B_q}-1} b_{q,k} e^{-j\theta k}$. Then, this portion of the mask can be described in the notation of Section IV and Problem 1 by

$$\zeta^q \mathbf{b}_q - \tilde{\mathbf{I}} \tilde{\mathbf{x}} \in \mathcal{K}(2\pi f_r, 2\pi f_s) \quad (53)$$

where either \mathbf{b}_q or $\tilde{\mathbf{x}}$ is to be padded with zeros so that they are both of dimension $\tilde{M} = \max\{M_{B_q}, M\}$. For linear-phase filters, the rolloff must also be incorporated into the lower mask $\check{L}(e^{j\theta})$ because $G(e^{j\theta})$ is not constrained to be non-negative, i.e., $\check{L}(e^{j\theta}) = -\text{Re } B_1(e^{j\theta})$ for $\theta \in [2\pi f_r, 2\pi f_s]$. Hence, for the linear-phase case, we require (53) and

$$\tilde{\mathbf{I}} \tilde{\mathbf{x}} + \zeta \mathbf{b}_1 \in \mathcal{K}(2\pi f_r, 2\pi f_s). \quad (54)$$

For this roll-off example, the complete spectral mask is described by (46), (53), and (54), with (54) being redundant in the case of autocorrelation design. Therefore, the design of nonlinear-phase filters that minimize the objective in (47) subject to the new mask can be achieved by adding two variables [$\mathbf{X}^{(ro)} \in \mathcal{H}_+^{\tilde{M} \times \tilde{M}}$ and $\mathbf{Z}^{(ro)} \in \mathcal{H}_+^{(\tilde{M}-1) \times (\tilde{M}-1)}$] to Problem 1, along with the additional constraint

$$\zeta^q \mathbf{b}_q - \tilde{\mathbf{I}} \tilde{\mathbf{x}} - \xi^{(ro)} \mathbf{j} \mathbf{e}_0 = \mathbf{L}^*(\mathbf{X}^{(ro)}) + \mathbf{\Lambda}^*(\mathbf{Z}^{(ro)}; 2\pi f_r, 2\pi f_s) \quad (55)$$

where $\xi^{(ro)} \in \mathbb{R}$ is unconstrained. (Note that in Problem 1, $\tilde{\mathbf{x}}$ is already constrained to be real.) For linear-phase filters, we require two more variables, namely, $\mathbf{X}^{(rol)} \in \mathcal{H}_+^{\tilde{M} \times \tilde{M}}$ and $\mathbf{Z}^{(rol)} \in \mathcal{H}_+^{(\tilde{M}-1) \times (\tilde{M}-1)}$, and the additional constraint

$$\tilde{\mathbf{I}} \tilde{\mathbf{x}} + \zeta \mathbf{b}_1 - \xi^{(rol)} \mathbf{j} \mathbf{e}_0 = \mathbf{L}^*(\mathbf{X}^{(rol)}) + \mathbf{\Lambda}^*(\mathbf{Z}^{(rol)}; 2\pi f_r, 2\pi f_s) \quad (56)$$

where $\xi^{(rol)} \in \mathbb{R}$ is unconstrained. In the following example, we revisit the designs in Example 1 with a new mask, which contains a roll-off section.

Example 2: Consider the mask from Example 1, and introduce a “tighter” stopband constraint consisting of a first-order trigonometric polynomial roll-off on the magnitude spectrum between $f_r = 740/4915.2$ and $f_s = 0.25$ and a constant bound on the magnitude for $0.25 \leq f \leq 0.5$. More specifically, in the roll-off region $\omega \in [2\pi f_r, 2\pi f_s]$, we enforce $|G(e^{j\omega})| \leq \zeta B_1(e^{j\omega})$, where the first-order trigonometric polynomial $B_1(e^{j\omega})$ has real coefficients $b_{1,0}$ and $b_{1,1}$ chosen such that $B_1(e^{j2\pi f_r}) = 10^{-2}$ and $B_1(e^{j2\pi f_s}) = 10^{-2.5}$, and

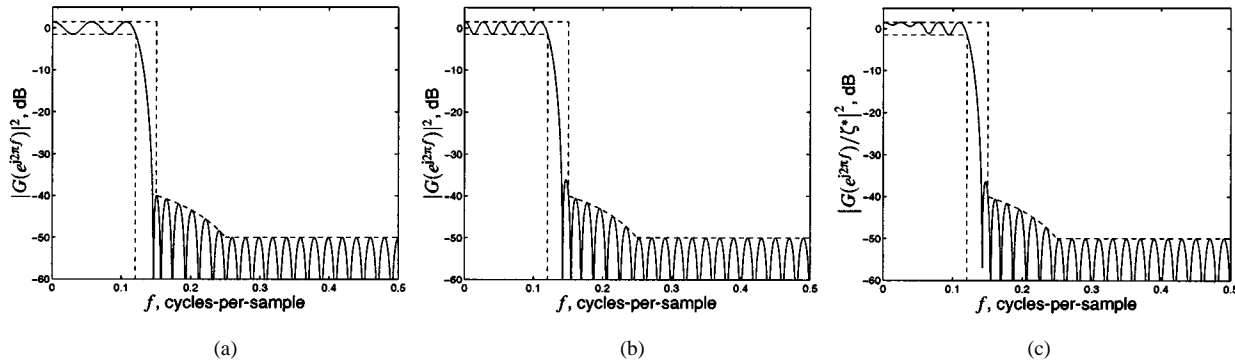


Fig. 2. Power spectra of length 49 filters from Example 2, along with the corresponding mask. In (c), ζ^* is the optimal value of ζ . (a) Linear-phase filter minimizing stopband energy. (b) Nonlinear-phase filter minimizing stopband energy. (c) Nonlinear-phase filter minimizing proportion of energy in the stopband.

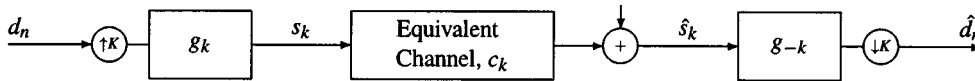


Fig. 3. Equivalent discrete-time model of baseband PAM.

in the constant portion of the stopband $\omega \in [2\pi f_s, \pi]$, we enforce $|G(e^{j\omega})| \leq \zeta 10^{-2.5}$. That is, the rolloff “starts” at $f_r \approx 0.15$ with a suppression level of 40 dB and rolls off to a suppression level of 50 dB at $f_s = 0.25$ from which point, the required suppression level remains 50 dB (see Fig. 2). Note that since $R(e^{j\omega}) = |G(e^{j\omega})|^2$, $|G(e^{j\omega})| \leq \zeta B_1(e^{j\omega})$ for $\omega \in [2\pi f_r, 2\pi f_s]$ can be imposed directly on the power spectrum by enforcing $R(e^{j\omega}) \leq \zeta^2 B_2(e^{j\omega})$ for $\omega \in [2\pi f_r, 2\pi f_s]$, where $B_2(e^{j\omega})$ is a second-order trigonometric polynomial with coefficients $b_{2,0} = b_{1,0}^2 + b_{1,1}^2/2$, $b_{2,1} = 2b_{1,0}b_{1,1}$ and $b_{2,2} = b_{1,1}^2/2$. The power spectra of the length 49 linear and nonlinear phase filters that minimize the stopband energy with $f_c = (f_p + f_r)/2$, subject to the spectral mask, are shown in Fig. 2(a) and (b), and that of the nonlinear-phase filter that minimizes the proportion of the total energy of the filter in the stopband is shown in Fig. 2(c). (The optimal designs were obtained in 14, 95, and 99 s, respectively, using the setup described in Example 1.) Note that these filters have similar passband characteristics to the corresponding filters in Example 1 but that the stopband characteristics are substantially altered by the new mask. \square

In the transmission of digital data by pulse amplitude modulation (PAM), we often encounter design specifications in terms of a spectral mask of the form in Fig. 1. In fact, the mask in Fig. 1 is that specified for the “chip” waveform in the IS95 digital cellular communication standard [30]. A simplified block diagram of a PAM scheme is shown in Fig. 3. The transmitted power is normalized to unity, i.e., $r_0 = 1$. In order to control the intersymbol interference (ISI) in a distortionless channel, we can enforce the constraint

$$2 \sum_{i>0} r_{Ki}^2 \leq \epsilon \tag{57}$$

for some (small) $\epsilon \geq 0$, [31]. This term is the mean square error (MSE) in \hat{d}_n in Fig. 3 in the absence of noise and channel distortion. When $\epsilon = 0$, this constraint is equivalent to self-orthogonality (that is, to g_k being a “root-Nyquist” filter), but when $\epsilon > 0$, it allows us to trade ISI for other system properties. One of these properties might be the sensitivity of the MSE in

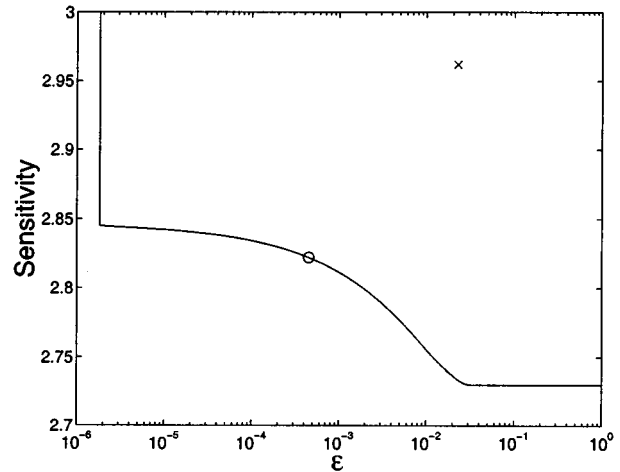


Fig. 4. Tradeoff between the sensitivity and ϵ for the IS95 standard for Example 3. The “x” and “o” denote the positions achieved by the IS95 filter and the robust filter in Fig. 5(b), respectively.

\hat{d}_n to unknown channel distortion. If the unknown channel is modeled as $c_k = \delta_k + c_k^{(e)}$, where δ_k is the Kronecker delta (which is the impulse response of a distortionless channel), then an appropriate measure of the sensitivity of the PAM scheme is the worst-case MSE over a bounded set of $c_k^{(e)}$'s [31]. For a given bound ϵ on the ISI, this sensitivity can be minimized by solving Problem 1 with $\tilde{\mathbf{x}} = \tilde{\mathbf{r}}$, $r_0 = 1$, $\mathbf{Q} = \tilde{\mathbf{I}}$, $\mathbf{l} = \mathbf{0}$, and the additional constraint in (57) [31].

Example 3: The filter specified for the synthesis of the chip waveform in IS95 has length 48 and $K = 4$ and satisfies the spectral mask specified in the standard (and illustrated in Fig. 1), but it generates a large MSE in a distortionless channel. To efficiently determine whether this MSE can be reduced while simultaneously reducing the sensitivity to unknown channel distortion, the modified version of Problem 1 was solved for various values of ϵ . (Each solution was obtained in about 23 s.) The tradeoff is shown in Fig. 4, from which it is clear that the IS95 filter can be greatly improved upon. The spectra of the IS95 filter and a representative optimal filter are plotted in Fig. 5. The robust filter provides a substantially lower “chip error rate” than

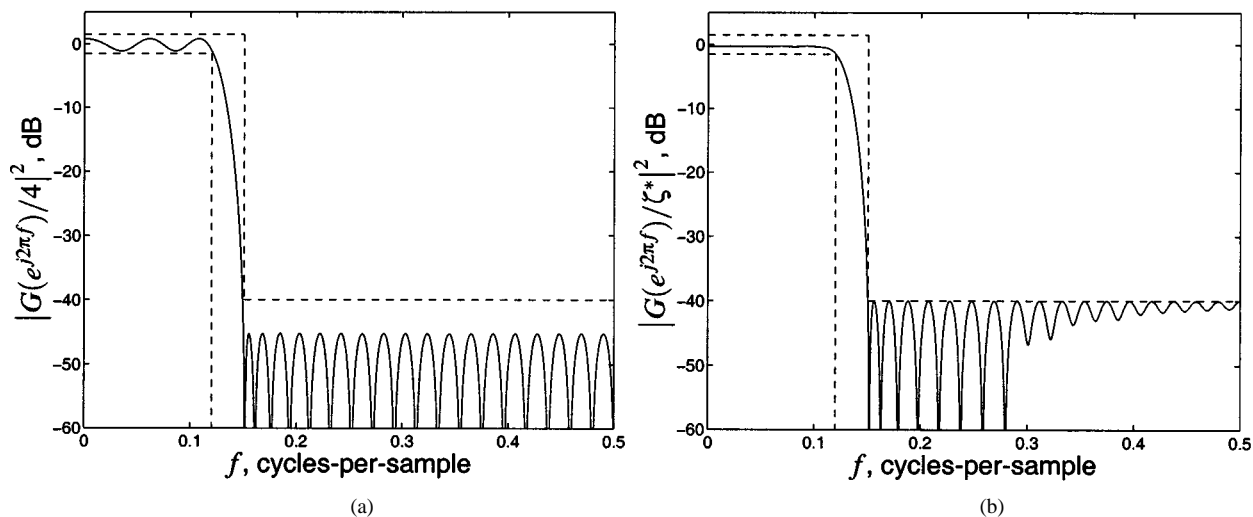


Fig. 5. Power spectra (in decibels) of the filters in Example 3 with the IS95 mask. Here, ζ^* is the optimal value of ζ from the modified version of Problem 1. (a) IS95 filter. (b) Robust filter.

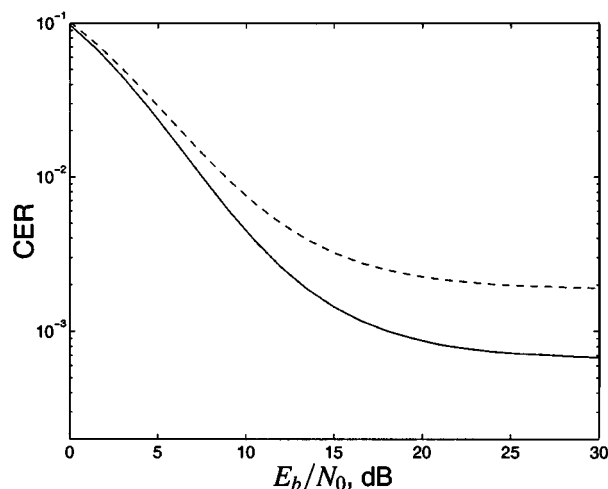


Fig. 6. Simulated chip error rates (CER) in a slowly varying frequency-selective Rician channel against signal-to-noise ratio for Example 3. Legend—Dashed: IS95 filter. Solid: robust filter.

the IS95 filter in a slowly varying frequency-selective Rician fading channel, as shown in Fig. 6. (See [31] for the details). \square

Example 4: In addition to tradeoffs between ISI and sensitivity, tradeoffs between ISI and bandwidth are also of interest in the design of PAM schemes. For a given level of ISI, a filter achieving the minimum bandwidth can be efficiently found using a bisection-based search on the stopband edge of the mask for the feasibility boundary of a convex cone feasibility problem [31]. That feasibility problem is based on the modified version of Problem 1 used in Example 3. (This is a variation of the method used to find minimum bandwidth self-orthogonal filters in [10].) The resulting tradeoff for the IS95 spectral mask is plotted in Fig. 7, from which it is clear that the IS95 filter is some distance from the optimal filters. \square

B. Beamformer Design

In standard narrowband beamforming applications, the outputs of each antenna element at a given instant are linearly combined to form the array output at that instant [32]. If $x_k(n)$ denotes the complex envelope of the output of the k th antenna ele-

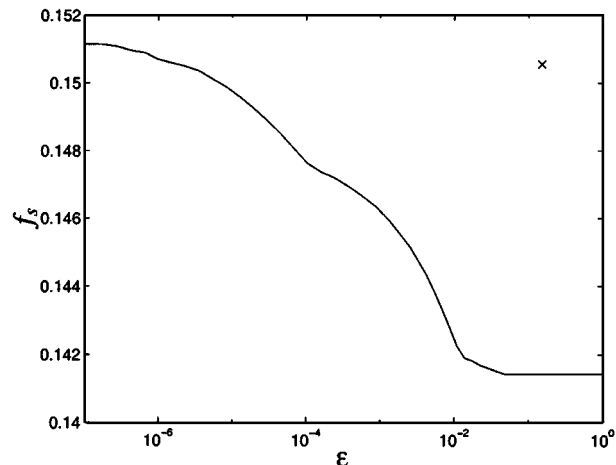


Fig. 7. Minimal stopband edge f_s against ISI bound ϵ for the IS95 mask. “ \times ” denotes the position of the IS95 filter.

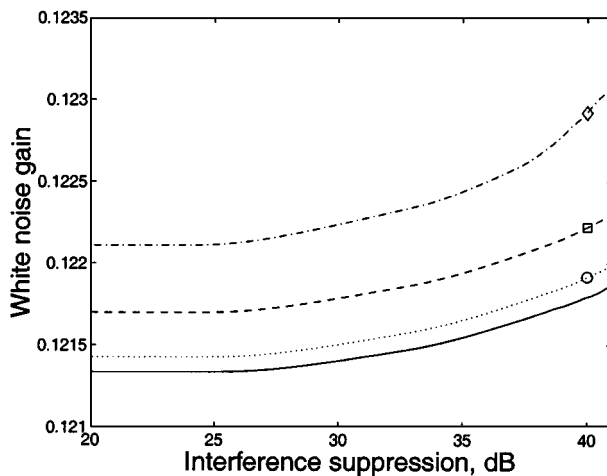


Fig. 8. Trade-off between the minimum white noise gain $\mathbf{w}^H \mathbf{w}$ and the interference suppression $|\Delta_s|$ for maximum sidelobe levels Δ_s of 0.1 dB (solid), -18 dB (dotted), -20 dB (dashed), and -22 dB (dot-dashed) for Example 5. The o , \square , and \diamond denote the tradeoffs achieved by the beamformers in Fig. 9.

ment at the n th instant and if $y(n)$ denotes the complex envelope of the array output at that instant, then

$$y(n) = \mathbf{x}(n)^T \mathbf{w} = \mathbf{w}_c^H \mathbf{x}(n)$$

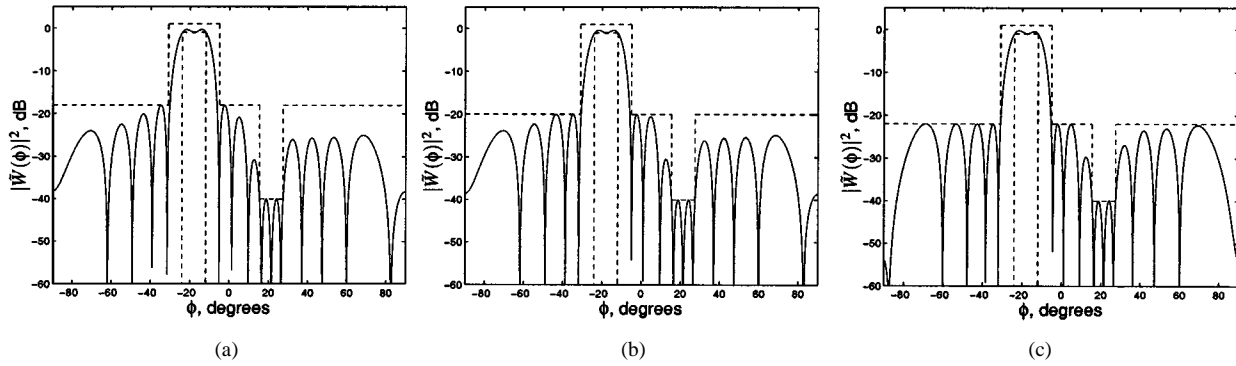


Fig. 9. Beam patterns with minimal white noise gain, subject to 40 dB interference suppression and different maximum sidelobe levels, Δ_s , for Example 5, along with the corresponding masks. (a) $\Delta_s = -18$ dB. (b) $\Delta_s = -20$ dB. (c) $\Delta_s = -22$ dB.

where w_k is the “weight” applied to the output of the k th antenna element, and $[\mathbf{w}_c]_k = \bar{w}_k$. It is well known [32] that if the array geometry is linear, with equi-spaced elements with separation d and if the array operates on signals with wavelength λ , then the (complex) “gain” of the array for a signal arriving at an angle ϕ to broadside (perpendicular to the array) is

$$\tilde{W}(\phi) = e^{j\lambda} W \left(e^{j2\pi(d/\lambda) \sin \phi} \right)$$

where $W(e^{j\theta})$ is the Fourier transform of w_k , and $e^{j\lambda}$ determines the “phase center” of the array. For simplicity, we will focus on the standard case where $d = \lambda/2$.

In many applications, we would like to control the “beam pattern” of the array $|\tilde{W}(\phi)|^2$, but that results in nonconvex constraints on w_k . Using the autocorrelation of the weights $r_m = \sum_k w_k \bar{w}_{k-m}$, we have that $\tilde{R}(\phi) = \sum_m r_m e^{-jm\pi \sin \phi} = |\tilde{W}(\phi)|^2$, and therefore, bound constraints on $|\tilde{W}(\phi)|^2$ result in linear constraints on r_m . For an M -element array

$$\tilde{R}(\phi) = \text{Re} \mathbf{v}(\pi \sin \phi; M-1)^H \tilde{\mathbf{I}} \tilde{\mathbf{r}}$$

where $[\tilde{\mathbf{r}}]_m = r_m$, $0 \leq m \leq M-1$ and $\tilde{\mathbf{I}}$ was defined in Section V-A. Therefore, piecewise constant and piecewise trigonometric polynomial constraints on $|\tilde{W}(\phi)|^2$ can be compactly enforced in an analogous way to that for the spectral masks in Section V-A, as we now demonstrate in a simple example derived from [32, Fig. 2.5].

Example 5: Suppose that a desired signal impinges on a 16-element linear equi-spaced array with element separation $d = \lambda/2$ from an angle of $\phi_d = -18^\circ \pm 6^\circ$ and that interfering signals arrive from angles in the range $\phi_i = 21.5^\circ \pm 6^\circ$. An interesting data independent [32] beamforming problem is to minimize the response to (spatially) white noise (i.e., $\mathbf{w}^H \mathbf{w} = r_0$) subject to the gain in the direction of the desired signal being within $\pm \Delta_d$ dB and to the gain in the direction of the interferers being less than Δ_i dB. Furthermore, to guard against unexpected interferers from other directions, we would like to keep the sidelobes below Δ_s dB and to constrain the main lobe (as determined by Δ_s) to be within $-18 \pm 13^\circ$. In short, our objective is to minimize the white noise gain, subject to a mask of the shape in Fig. 9. This problem can be cast in a similar way to Problem 1 with $\tilde{\mathbf{x}} = \tilde{\mathbf{r}}$, $\zeta = 1$, $\mathbf{Q} = \mathbf{0}$ and $\mathbf{l} = -e_0/2$, except that the vector $\tilde{\mathbf{r}}$ and the various \mathbf{X} and \mathbf{Z} matrices may be complex. Therefore, the tradeoffs between the white noise gain and the level of interference suppression, for

different values of the maximum sidelobe level and the “look direction ripple” Δ_d can be efficiently found. Examples of such tradeoffs (for $\Delta_d = 0.1$ dB) are presented in Fig. 8, and examples of the resulting beam patterns are shown in Fig. 9. (Each optimal $\tilde{\mathbf{r}}$ was computed in about 7 s.) These examples clearly demonstrate the role that the sidelobe level constraint plays in determining the shape of the beam pattern. \square

VI. CONCLUDING REMARKS

In this paper, we have provided a compact representation of piecewise constant and piecewise trigonometric polynomial spectral mask constraints via linear matrix inequalities. This representation is precise and avoids the heuristic approximation of the mask incurred when discretization techniques are used. The representation is also convex, and it generates practically competitive design algorithms (based on well-established interior point methods) for a diverse class of FIR filtering and narrowband beamforming problems. Using such algorithms, (in)feasibility of the spectral mask can be detected reliably, which is especially important when the design problem is solved iteratively in a binary search scheme (such as in minimal length filter design). In addition to these applications, generalizations of our results to rational filters (i.e., infinite impulse response filters) and to multidimensional filters are of interest in control theory, as well as signal and image processing, and are currently being pursued. In closing, we point out that we have efficiently solved the design problems that result from our compact representation (e.g., Problem 1) using a sophisticated, but general purpose, convex cone program solver [26]. Although this is convenient from a practitioner’s perspective, we believe that more efficient implementations of our design approach can be obtained by developing an application-specific solver that exploits the extensive algebraic structure that our design problems possess. Recent work on application-specific solvers for problems from the same class [33], [34] suggests that the resulting reductions in computational and memory requirements can be substantial.

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