Quasi-Gray Labelling for Grassmannian Constellations

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***Abstract*—This paper presents a technique for assigning binary labels to the points in an arbitrary Grassmannian constellation in a manner that approximates the Gray labelling. The idea behind this technique is to match the Grassmannian constellation of interest to the points in an auxiliary constellation that can be readily Gray labelled. In order to demonstrate the efficacy of the proposed technique, the labelled constellations are utilized in a BICM-encoded non-coherent MIMO communication system with iterative detection and decoding. Numerical simulations indicate that this labelling technique results in a non-coherent communication system that provides better bit error rate performance than systems that utilize the same constellation but employ labels that are generated either randomly or via a quasi-set-partitioning technique.**

I. Introduction

The compact Grassmann manifold, $\mathbb{G}_M(\mathbb{C}^T)$, is the set of equivalence classes of 'tall' $T \times M$ unitary matrices, where $T \geq 2M$ and two unitary matrices are said to be equivalent if they are related by the right multiplication of a square $M \times M$ unitary matrix [1]. A Grassmannian constellation, $\mathcal{C}$, is a set of discrete points on the Grassmann manifold, and if the distribution of these points is invariant under rotation, the constellation is said to be isotropically distributed.

Grassmannian constellations play an important role in multiple-input multiple-output (MIMO) communication systems. In particular, for non-coherent MIMO systems, in which no channel state information (CSI) is available at either the transmitter or the receiver, isotropically distributed Grassmannian constellations can achieve the high signal-to-noise ratio (SNR) ergodic capacity [2]. Designing such constellations can be quite challenging. However, using geometrical techniques, Grassmannian constellations with an approximately isotropic distribution can be efficiently generated [3]. In addition to non-coherent systems, Grassmannian constellations also arise in the quantization of channel state information and/or precoder designs in some coherent MIMO systems with limited feedback [4]–[6].

In this paper we will focus on non-coherent wireless MIMO communication systems operating at moderate-to-high SNRs. One way for such a system to operate close to the ergodic capacity is to envelop a Grassmannian constellation in a bit-interleaved coded-modulation (BICM) framework [7], [8]. In this scheme the transmitter maps the output of the BICM encoder to channel symbols that are drawn from a capacity-approaching Grassmannian constellation. However, the way in which the encoded bits are mapped to the Grassmannian constellation can have a significant impact on the performance of a BICM-encoded system. In particular, for coherent systems it has been shown that Gray mapping can provide a substantial SNR gain over other mapping techniques, including random and set-partitioning techniques [7].

For a given constellation to be Gray labelled, points that are close to each other in the signal space are assigned labels that are close in the Hamming distance sense. While it is possible to assign Gray labels to elementary two and three-dimensional constellations, assigning such labels to constellations of high dimensions is typically difficult [9]. This is especially true for non-uniform constellations or for constellations in which the number of neighbouring points is not known and possibly not identical for every point in the constellation. In such cases finding Gray labels becomes a formidable task that involves an exhaustive search over $|\mathcal{C}|!$ candidate labellings.

In this paper we devise a generic technique for generating quasi-Gray labels. Although these labels are not precisely Gray, their assignment follows the general Gray philosophy of mapping points that are close in the signal space to labels that are close in the Hamming distance sense. In particular, in this paper we show how quasi-Gray labels can be assigned to a Grassmannian constellation that has favourable geometric properties, but otherwise does not possess a particular structure. Our methodology is based on matching this constellation to a certain auxiliary constellation. In contrast to the original constellation, the auxiliary constellation does not possess the same favourable geometric properties, but it can be readily Gray labelled. The labelling of the original constellation is accomplished by assigning to each point in the original constellation the label of the point that matches it in the auxiliary constellation. Numerical simulations indicate that this labelling technique results in a non-coherent communication system that performs better than systems that utilize the same constellation but with labels that are generated either randomly or via a quasi-set-partitioning technique.

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II. Grassmannian Constellations

A. Geometrically designed constellations

Achieving the ergodic capacity of a non-coherent MIMO communication system operating at high SNRs is equivalent to sphere packing on the Grassmann manifold [2]. Using this observation, the problem of designing rate-efficient Grassmannian constellations can be cast as an optimization problem in which the minimum distance between any pair of constellation points is maximized. Such a problem can be cast as

$$\max_{\{\mathbf{Q}_{X_i}\}_{i=1}^{|\mathcal{C}|}} \quad \min_{i\neq j} \quad d(\mathbf{Q}_{X_i}, \mathbf{Q}_{X_j}),$$
$$\text{subject to} \quad \mathbf{Q}_{X_i} \in \mathbb{G}_M(\mathbb{C}^T), \quad \forall i \in \{1, \dots, |\mathcal{C}|\},$$

where $d(\mathbf{Q}_{X_i}, \mathbf{Q}_{X_j})$ is the distance between the points $\mathbf{Q}_{X_i}$ and $\mathbf{Q}_{X_j}$. In [3] it was shown that an appropriate metric for designing these constellations is the so-called chordal Frobenius norm, which is given by [10]

$$d(\mathbf{Q}_{X_i}, \mathbf{Q}_{X_j}) = \sqrt{M - \mathrm{Tr}(\mathbf{\Sigma}_{ij})}, \tag{1}$$

where $\mathbf{\Sigma}_{ij}$ is the diagonal matrix of singular values of $\mathbf{Q}_{X_i}^\dagger \mathbf{Q}_{X_j}$, and $(\cdot)^\dagger$ and $\mathrm{Tr}(\cdot)$ denote the Hermitian transpose and the matrix trace operators, respectively. Using this metric, several approaches for designing Grassmannian constellation were developed in [3]. Those approaches rely on using derivative-based optimization techniques that exploit the smooth geometry of the Grassmann manifold. In addition to maximizing ergodic rate objectives, geometric techniques have been used to design Grassmannian constellations that meet alternate objectives; for example, see [11], [12].

In spite of their favourable geometric properties, the constellations generated by the aforementioned techniques do not have a known structure that enables the system designer to Gray label them in a systematic manner. Addressing this drawback will be the focus of the remainder of the paper.

B. Auxiliary constellations

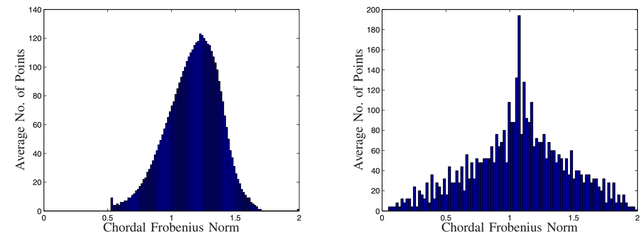
An algebraic technique for designing Grassmannian constellations for non-coherent signalling over MIMO channels has been provided in [13]. Regarded as a generalization of pilot-assisted designs [14], this technique constructs the ‘tall’ unitary matrix, $\mathbf{Q}_{Z_i}$, that represents each constellation point i in the following way:

$$\mathbf{Q}_{Z_i} = \begin{bmatrix} \mathbf{G}^T & \mathbf{D}_i^T \end{bmatrix}^T, \tag{2}$$

where $\mathbf{G}$ is a constant matrix common for all constellation points and $\mathbf{D}_i$ is a matrix that is distinct for each constellation point. The size of $\mathbf{G}$ and $\mathbf{D}_i$ are selected to satisfy the orthogonality condition. For example, for $M = 2$ and $T = 4$,

$$\mathbf{G} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \text{ and } \mathbf{D}_i = \frac{1}{2}\begin{bmatrix} e^{\jmath\frac{2\pi}{Q}k(i)} & e^{\jmath\frac{2\pi}{Q}\ell(i)} \\ e^{-\jmath\frac{2\pi}{Q}\ell(i)} & -e^{-\jmath\frac{2\pi}{Q}k(i)} \end{bmatrix},$$

where $(k(i), \ell(i))$ is a pair integers that is distinct for each constellation point $i \in \{1, \dots, |\mathcal{C}_A|\}$, where $\mathcal{C}_A$ is the auxiliary constellation generated by (2), $k(i), \ell(i) \in \{0, \dots, Q-1\}$ and $Q = \sqrt{|\mathcal{C}_A|}$. Noting that the matrices $\{\mathbf{D}_i\}_i^{|\mathcal{C}_A|}$ are analogous to the standard MIMO orthogonal designs [15], it was shown in [13] that the construction in (2) can be extended to other values of M and T. However, for ease of exposition, in this paper we will restrict our attention to the case of $M = 2$ and $T = 4$.



(a) Target constellation (b) Auxiliary constellation

Fig. 1. Distance spectra of target and auxiliary 4096-point Grassmannian constellations

Let (k, ℓ) and (k', ℓ') be two pairs of integers that correspond to the distinct points i and i' in $\mathcal{C}_A$, respectively. For this example, the two singular values of $Q_{Z_i}^\dagger Q_{Z_i'}$ are equal and are given by

$$\frac{1}{2}\sqrt{2 + \cos\left(\tfrac{2\pi}{Q}(k - k')\right) + \cos\left(\tfrac{2\pi}{Q}(\ell - \ell')\right)}. \tag{3}$$

From (3) and (1) it was observed in [16] that for each constellation point with indices (k, ℓ) there exist exactly four closest neighbours with indices $(k, (\ell \pm 1) \mod Q)$ and $((k \pm 1) \mod Q, \ell)$. Now, each of the integers k, ℓ, k', ℓ' corresponds to a point in a phase-shift keying (PSK) constellation. Using these observations, it was concluded in [16, Theorem 1] that if the PSK constellations indexed by the integers k, ℓ, k', ℓ' are identically Gray labelled, the resulting Grassmannian constellation will also be Gray labelled.

III. The Match-and-Label Algorithm

We now use the Gray labelled auxiliary constellation, $\mathcal{C}_A$, to provide quasi-Gray labels for our capacity-approaching constellation, $\mathcal{C}$, which we will refer to as the target constellation. First, we generate $\mathcal{C}_A$ such that $|\mathcal{C}_A| = |\mathcal{C}|$. We note that if $\mathcal{C}_A$ and $\mathcal{C}$ had the same geometric structure, it would have been possible to Gray label $\mathcal{C}$, by assigning to each point the label of the corresponding point in the auxiliary constellation. In Figure 1 we have plotted the distance spectrum of a 4096-point target constellation that will be used in our numerical example in Section IV-C and the distance spectrum of the corresponding auxiliary constellation. The target constellation was generated using the rotation-based version [3] of the geometric design technique of Section II-A and the corresponding auxiliary constellation was generated using the technique given in Section II-B. From this figure it can be seen that these constellations have significantly different geometric structures. Hence, to assign quasi-Gray labels to the target constellation it is required to find a map for associating points in this with points in the auxiliary one.

Let $\mathcal{G}_A(i)$ be the operator that assigns a Gray label to the i-th point of $\mathcal{C}_A$ and $\mathcal{G}(j)$ be the operator that assigns a label to the j-th point of $\mathcal{C}$. A greedy technique for matching the auxiliary and target constellations can be constructed by taking an arbitrary point from the target constellation and associate it with the point in the auxiliary constellation that lies at the smallest chordal Frobenius norm. In particular, we begin by forming two lists: $\mathcal{L}$ and $\mathcal{L}_A$. Initially these lists include all the points in $\mathcal{C}$ and $\mathcal{C}_A$, respectively. After matching the first point from $\mathcal{L}$ with the corresponding one from $\mathcal{L}_A$, these points are labelled and eliminated from their respective lists. This process is repeated until the two lists are exhausted. Having matched the two constellations, we now assign the label of each point in $\mathcal{C}_A$ to the corresponding point in $\mathcal{C}$. This procedure can be formally stated as follows:

Algorithm 1:

- Initialize: $\mathcal{L} = \{i|\mathbf{Q}_{X_i} \in \mathcal{C}\}$ and $\mathcal{L}_A = \{r|\mathbf{Q}_{Z_r} \in \mathcal{C}_A\}$.
- For every $i = 1, \ldots, |\mathcal{C}|$,
  - Find $r \in \mathcal{L}_A$ such that
  $$\mathbf{Q}_{Z_r} = \arg\min_{s\in\mathcal{L}_A} d(\mathbf{Q}_{X_i}, \mathbf{Q}_{Z_s}).$$
  - Set $\mathcal{G}(i) = \mathcal{G}_A(r)$.
  - $\mathcal{L} \leftarrow \mathcal{L} \setminus \{i\}$ and $\mathcal{L}_A \leftarrow \mathcal{L}_A \setminus \{r\}$.

In the next section we will test this labelling technique in a BICM-encoded non-coherent MIMO system.

IV. Application to BICM-encoded non-coherent MIMO systems

A. A non-coherent MIMO communication system

In a non-coherent MIMO communication system neither the transmitter nor the receiver has access to channel state information (CSI). We consider such a system operating over a frequency-flat richly-scattered block-fading[1] channel of coherence time $T \geq \min(M, N) + N$, where M and N are the numbers of transmit and receive antennas, respectively, with $M = \min\{\lfloor T/2 \rfloor, N\}$; cf. [2]. We will denote the signal vector transmitted at each channel use by the rows of a $T \times M$ matrix $\mathbf{Q}_X$, and hence the $T \times N$ received signal matrix $\mathbf{Y}$ is

$$\mathbf{Y} = \mathbf{Q}_X\mathbf{H} + \mathbf{V}, \tag{4}$$

where $\mathbf{H}$ is the $M \times N$ channel matrix whose entries are drawn independently from the standard complex Gaussian distribution $\mathcal{CN}(0, 1)$, and $\mathbf{V}$ is the $T \times N$ additive noise matrix whose entries are drawn independently from $\mathcal{CN}(0, M/\rho T)$, where ρ is the signal-to-noise ratio.

In this scenario the capacity-achieving input signals for high-SNR operation are $T \times M$ unitary matrices that are isotropically distributed on $\mathbb{G}_M(\mathbb{C}^T)$, [2]. The likelihood of the received signal given a transmitted unitary matrix satisfies [2], [17]

$$p(\mathbf{Y}|\mathbf{Q}_X) \propto \exp\Big(-\tfrac{\rho T}{M}\mathrm{Tr}\Big(\mathbf{Y}^\dagger\Big(\mathbf{I}_T - \tfrac{\rho T}{\rho T+M}\mathbf{Q}_X\mathbf{Q}_X^\dagger\Big)\mathbf{Y}\Big)\Big). \tag{5}$$

[1]In this model, the channel remains constant for a block of T channel uses, and in each block the channel coefficients are statistically independent of those in other blocks; e.g., [2].

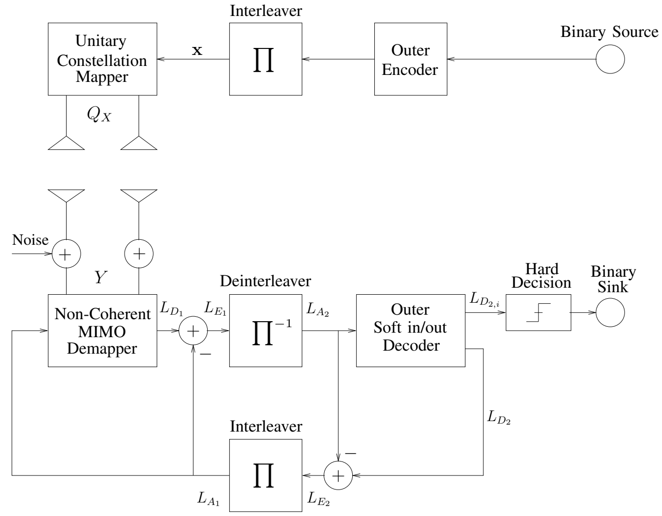


Fig. 2. A BICM-IDD scheme for non-coherent MIMO communication [8].

B. A BICM-IDD scheme

BICM is known to be sensitive to the labelling of the channel symbols [7]. When employed with iterative (soft) detection and decoding (IDD), BICM-encoded systems were shown in [18] to be capable of providing effective communication at rates close to the ergodic capacity of coherent MIMO communication systems. In [8] the BICM-IDD scheme was extended to non-coherent MIMO systems that use Grassmannian constellations. A generic BICM-IDD system adapted to non-coherent communication is shown in Figure 2.

In this figure, $\mathbf{x}$ denotes a length-n vector of encoded interleaved bits. Let the k-th element of this vector, x_k, correspond to a certain channel use for which $\mathbf{Y}$ is the received signal matrix and $\mathbf{Q}_{X_i}$ is the transmitted signal matrix. The conditioned log likelihood ratio of x_k is given by $L_{D_1}(x_k|\mathbf{Y})$, where [18]

$$L_{D_1}(x_k|\mathbf{Y}) = \log\frac{P(x_k = +1|\mathbf{Y})}{P(x_k = -1|\mathbf{Y})} \tag{6}$$

$$= \log\frac{\sum_{\mathbf{Q}_{X_i}\in\mathcal{X}_{k,+1}} p(\mathbf{Y}|\mathbf{Q}_{X_i})P(\mathbf{Q}_{X_i})}{\sum_{\mathbf{Q}_{X_i}\in\mathcal{X}_{k,-1}} p(\mathbf{Y}|\mathbf{Q}_{X_i})P(\mathbf{Q}_{X_i})}, \tag{7}$$

where $p(\mathbf{Y}|\mathbf{Q}_{X_i})$ is given in (5). An approximation to $P(\mathbf{Q}_{X_i})$ can be obtained from the decoder outputs at the previous iteration using the standard assumption of independence of the interleaved encoded bits; e.g. [18], $P(\mathbf{Q}_{X_i}) \approx \prod_{k=1}^n P(x_k = [\mathcal{G}(i)]_k)$, where $[\mathcal{G}(i)]_k$ denotes the k-th element of the label $\mathcal{G}(i)$. The set $\mathcal{X}_{k,+1}$ contains all the matrices $\{\mathbf{Q}_{X_i}\}$ in the constellation whose indices have $x_k = +1$; i.e., $\mathcal{X}_{k,\pm1} = \{\mathbf{Q}_{X_i} \in \mathcal{C}|x_k = [\mathcal{G}(i)]_k = \pm1\}$.

C. A numerical example

We consider a system with $M = N = 2$ and a coherence time $T = 4$. The outer encoder in Figure 2 was chosen to be a systematic parallel concatenated turbo code with identical recursive convolutional constituent codes, and the BICM and "turbo" interleavers were selected from a set of pseudo-randomly generated candidates. At the receiver, four

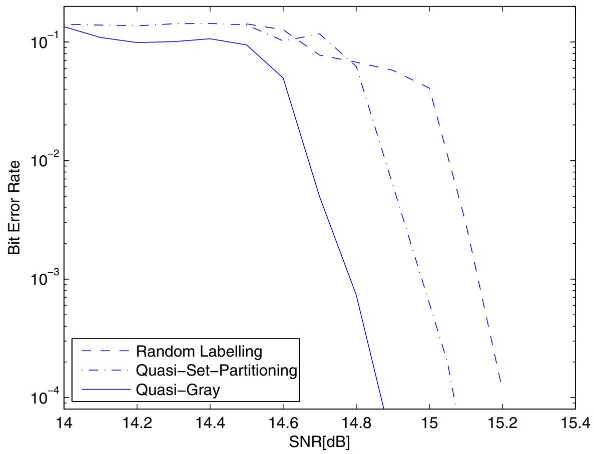


Fig. 3. Bit error rate performance of the proposed Grassmannian BICM-IDD scheme with random, quasi-set-partitioning, and quasi-Gray labelling. The approximate SNR threshold for a rate of 2.4 bpcu is about 11.7 dB.

detection-decoding iterations were performed for each block, with eight BCJR-based [19] turbo iterations being performed within the outer decoder for each detection-decoding iteration.

In Figure 3 we provide a comparison between the performance of the quasi-Gray labelling scheme proposed herein with random labelling and the quasi-set-partitioning labelling scheme proposed in [3]. For the simulations presented in this figure, the outer code was a rate-4/5 punctured turbo code with input block length $n = 32016$ bits. Using the notation in [20], the partition of each constituent convolutional code was $(2, 1, 1, 1)$, and its octal generators were $z^{\{i1\}} = (6, 0, 2, 3)$, $h^{\{1i\}} = (0, 6, 0, 5)$, $h^{\{2i\}} = (0, 3, 0, 1)$, $h^{\{3i\}} = (1, 2, 0, 3)$, and $h^{\{4i\}} = (2, 3, 3, 2)$. A rotation-based version (see [3]) of the geometric design technique outlined in Section II-A was used to generate the 4096-point Grassmannian constellation used in this figure, yielding an overall data rate of 2.4 bits per channel use (bpcu). The auxiliary constellation used in generating the quasi-Gray labels was designed as described in Section II-B using two 64-PSK constellations. The distance spectra of these constellations are given in Figure 1.

From Figure 3, it is seen that although quasi-set-partitioning labelling can provide an SNR gain of about 0.15 dB over a randomly chosen labelling at a bit error rate of 10^{-3}, a more significant gain of about 0.35 dB at the same bit error rate can be obtained by quasi-Gray labelling. Further gain may be achievable by employing the same labelling idea, but with a more sophisticated match-and-label algorithm than the one presented in Section III. Since no convenient expression is available for the ergodic capacity of a non-coherent MIMO block-fading channel, the high-SNR capacity approximation derived in [2] can be used to provide an approximate SNR threshold for the rate of interest. For the current example this threshold is about 11.7 dB.

V. Conclusion

This paper presented a novel technique for assigning quasi-Gray labels to arbitrary Grassmannian constellations. The main idea that underlies this technique is to match the target constellation with an auxiliary constellation that has a different geometric construction but can be readily Gray labelled. The proposed labelling technique was tested in a BICM-encoded non-coherent MIMO communication system and was shown to yield an appreciable SNR gain over both random and quasi-set-partitioning labelling.

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