BLIND SEPARATION OF BPSK SIGNALS USING NEWTON'S METHOD ON THE STIEFEL MANIFOLD

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ABSTRACT

We propose a new approach to solving the problem of blind separation of BPSK signals. Using the constant modulus property of the signal, we formulate this problem as a constrained minimization problem that can be solved efficiently using an extended Newton's method on the Stiefel manifold. Compared with the existing separation methods, the proposed method is quite robust to additive noise, achieves a low bit error rate, and enjoys a quadratic convergence rate and a low computational complexity. Simulation results show that our method is a competitive blind separation method.

1. INTRODUCTION

The blind signal separation (BSS) problem consists of recovering a set of statistically independent sources from a group of sensor observations. The challenge of this problem lies in the fact that the separation is attempted "blindly"; that is, without the knowledge of the sources nor the mixing environment. This paper considers a digital communication scenario in which d independent binary signals are transmitted to an antenna array of M antennas ($d \leq M$). We assume that the sources are synchronized and the delay spread is negligible. The received signals at the antenna array can then be modelled as

$$\mathbf{x}[n] = \mathbf{A}\mathbf{s}[n] + \mathbf{v}[n], \tag{1.1}$$

where $\mathbf{x}[n]$ is the sequence of vectors of the received signals at the antenna array, $\mathbf{A}$ is the channel (mixing) matrix, $\mathbf{s}[n]$ is the sequence of vectors of the transmitted signals, and $\mathbf{v}[n]$ is the sequence of vectors of additive noise at the antenna array. The goal is to blindly determine a separating matrix $\mathbf{B}$ such that $\mathbf{B}\mathbf{x}[n]$ resembles $\mathbf{s}[n]$. By 'resemble' we mean that as the influence of noise decreases, $\mathbf{B}\mathbf{x}[n]$ approaches $\mathbf{s}[n]$ up to a permutation and change of signs.

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In the general problem of blind signal separation from a linear mixture (1.1), the sources are arbitrary independent signals and higher-order statistics are usually required to determine the separating matrix $\mathbf{B}$. However, for digital communication signals, the sources are constrained to a finite alphabet and more efficient separation algorithms can be designed by exploiting this property [1, 4, 5, 8–10]. Unfortunately, the iterative least squares method in [8] and the 'hypercube' method in [4, 5] require the solution of non-convex optimization problems and hence these methods must include careful detection and management of locally optimal solutions. The clustering-based method in [1] is sensitive to mis-classification caused by additive noise, the analytic constant modulus method in [9] suffers from sensitivity to noise and occasional divergence and the 'polyhedral' method in [10] is quite sensitive to the direction of a randomly chosen initial vector. In this paper, we use the constant modulus property of the signal to formulate the BSS problem as a constrained optimization problem over the Stiefel manifold and solve it using Edelman's extended Newton's method [3]. Our method enjoys a quadratic convergence rate, and simulation results show that it is robust to additive noise and that it achieves a low bit error rate (BER).

2. PROBLEM FORMULATION

For clarity in the exposition, we will restrict our discussion to the case of binary phase shift keying (BPSK) for which all elements of $\mathbf{s}[n]$ are ± 1. However, our method can be easily generalized to the general M-ary PSK modulation case. We consider the model given in (1.1) and assume that the mixing matrix $\mathbf{A}$ has full column rank. Suppose N vectors of data samples have been collected at the antenna array. Then the model (1.1) can be re-written in matrix form as

$$\mathbf{X} = \mathbf{A}\mathbf{S} + \mathbf{V}, \tag{2.1}$$

where $\mathbf{X} = (\mathbf{x}[1], \ldots, \mathbf{x}[N])$ and $\mathbf{V} = (\mathbf{v}[1], \ldots, \mathbf{v}[N])$ are $M \times N$ matrices, and $\mathbf{S} = (\mathbf{s}[1], \ldots, \mathbf{s}[N])$ is a $d \times N$ matrix.

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In common with many other BSS algorithms, our method starts with the pre-processing step of pre-whitening the received signals. By doing so, the cross-correlations among the received signals are removed, and the problem dimension is reduced to $d \times d$. Let the signal cross-correlation matrix be denoted by $\mathbf{R} = (1/N)\mathbf{X}\mathbf{X}^T$. The pre-whitening step can be carried out by first computing the eigen-decomposition of $\mathbf{R}$

$$\mathbf{R} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T = [\mathbf{U}_d \ \mathbf{U}_{M-d}] \begin{bmatrix} \mathbf{\Lambda}_d & 0 \\ 0 & \mathbf{\Lambda}_{M-d} \end{bmatrix} \begin{bmatrix} \mathbf{U}_d^T \\ \mathbf{U}_{M-d}^T \end{bmatrix}, \tag{2.2}$$

where $\mathbf{U}$ is orthonormal, the (nonnegative) diagonal elements of $\mathbf{\Lambda}$ are arranged in nonincreasing order, $\mathbf{\Lambda}_d$ and $\mathbf{\Lambda}_{M-d}$ consist of the first d and last $(M-d)$ diagonal elements of $\mathbf{\Lambda}$, respectively, and $\mathbf{U}$ is partitioned conformally with $\mathbf{\Lambda}$ to form $\mathbf{U}_d$ and $\mathbf{U}_{M-d}$. The pre-whitening matrix can be chosen as $\mathbf{W} = \mathbf{\Lambda}_d^{-1/2}\mathbf{U}_d^T$. The pre-whitened signal is then given by

$$\mathbf{Y} = \mathbf{W}\mathbf{X} = \mathbf{C}\mathbf{S} + \tilde{\mathbf{V}},$$

where $\mathbf{C} = \mathbf{\Lambda}_d^{-1/2}\mathbf{U}_d^T\mathbf{A}$ and $\tilde{\mathbf{V}} = \mathbf{\Lambda}_d^{-1/2}\mathbf{U}_d^T\mathbf{V}$. Since the BPSK sources are assumed to be zero-mean, of unit-variance and statistically independent, the matrix $\mathbf{C}$ is orthonormal. Our goal is then to determine an orthonormal separating matrix $\mathbf{Q}$ such that

$$\hat{\mathbf{S}} = \mathbf{Q}\mathbf{Y} = \mathbf{Q}\mathbf{C}\mathbf{S} + \mathbf{Q}\tilde{\mathbf{V}} \tag{2.3}$$

resembles the source signal $\mathbf{S}$. Mathematically, by 'resemble' we mean

$$\mathbf{Q}\mathbf{C} = \mathbf{D}\mathbf{P} \tag{2.4}$$

where $\mathbf{D}$ is a diagonal matrix with diagonal entries being ± 1, and $\mathbf{P}$ is a permutation matrix. Consequently, we have $\hat{\mathbf{S}} = \mathbf{D}\mathbf{P}\mathbf{S} + \mathbf{Q}\tilde{\mathbf{V}}$, implying that in the noise free case, $\hat{\mathbf{S}}$ and $\mathbf{S}$ are identical up to a row permutation and change of signs. Now the separated signals $\hat{\mathbf{S}}$ should be ± 1. Using $f_{\ell m}$ to denote the (ℓ, m)th element of a matrix $\mathbf{F}$, we can write $\hat{s}_{ij} = \sum_{k=1}^{d} q_{ik} y_{kj}$. We propose to solve the BSS problem using the following constant modulus formulation:

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{d}\sum_{j=1}^{N}\left(\left|\sum_{k=1}^{d} q_{ik} y_{kj}\right|^2 - 1\right)^2 \\ \text{subject to} \quad & \mathbf{Q}^T\mathbf{Q} = \mathbf{I}_d. \end{aligned} \tag{2.5}$$

The following proposition indicates that every optimal solution of problem (2.5) is a separating matrix when N is large enough. Therefore, we can solve this minimization problem (2.5) for the separating matrix $\mathbf{Q}$.

Proposition 1 *Suppose the noise power is zero. Let* $\mathbf{S}$ *be a* $d \times 2^d$ *matrix containing all* 2^d *possible combinations of* ± 1 *in its columns. Let* $\mathbf{Q}$ *be an optimal solution of* (2.5)*. Then* $\mathbf{Q}$ *is a separating matrix;* i.e. (2.4) *holds.*

Proof: Let $\mathbf{E} = \mathbf{Q}\mathbf{C}$. Then $\hat{\mathbf{S}} = \mathbf{E}\mathbf{S}$. Since $\mathbf{Q}$ is the optimal solution of (2.5), $\mathbf{Q}$ must be orthonormal and the entries of $\hat{\mathbf{S}}$ are ± 1. Since $\mathbf{C}$ is orthonormal, it follows that $\mathbf{E}$ is also orthonormal. We will show that there is only one non-zero element on each row and column of $\mathbf{E}$, and this element can only be ± 1. Then $\mathbf{E} = \mathbf{D}\mathbf{P}$. To show this, we only need to show that there is only one non-zero element on each row of $\mathbf{E}$. Once this is established, we can use the property $\mathbf{E}\mathbf{E}^T = \mathbf{I}$ to conclude that all other elements on the same column as this non-zero element must be zero, and this non-zero element can only be ± 1.

Let $\mathbf{e}^T$ be a row of matrix $\mathbf{E}$. In the absence of noise, the corresponding row in $\hat{\mathbf{S}}$ is

$$\hat{\mathbf{s}}^T = \mathbf{e}^T\mathbf{S}, \quad \text{or} \quad \hat{\mathbf{s}} = \mathbf{S}^T\mathbf{e}, \tag{2.6}$$

We will use induction on d to show that there exists only one non-zero entry in $\mathbf{e}$. In the case of $d = 1$, (2.6) reduces to $\hat{\mathbf{s}} = \mathbf{s}e$, where $\hat{\mathbf{s}}$ and $\mathbf{s}$ are 2×1 vectors, and e is a scalar. Since the entries of $\hat{\mathbf{s}}$ and $\mathbf{s}$ are ± 1, e can only be ± 1. The proposition holds true in this case.

Suppose that the proposition holds true for $d = k$; *i.e.*, if $\hat{\mathbf{s}}_k = \mathbf{S}_k^T\mathbf{e}_k$, then $\mathbf{e}_k$ has only one non-zero element. Now we will prove that the statement is true for $d = k + 1$ case; *i.e.*, for

$$\hat{\mathbf{s}}_{k+1} = \mathbf{S}_{k+1}^T\mathbf{e}_{k+1}, \tag{2.7}$$

there is only one non-zero entry in vector $\mathbf{e}_{k+1}$. Rewrite (2.7) as

$$\begin{bmatrix} \hat{\mathbf{s}}^1 \\ \hat{\mathbf{s}}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{S}_k^T \\ -\mathbf{1} & \mathbf{S}_k^T \end{bmatrix} \begin{bmatrix} e_1 \\ \mathbf{e}_2 \end{bmatrix}, \tag{2.8}$$

where $\hat{\mathbf{s}}^1$ and $\hat{\mathbf{s}}^2$ are $2^k \times 1$ vectors, $\mathbf{1}$ is a $2^k \times 1$ vector with all its entries being 1's, e_1 is the first element of $\mathbf{e}_{k+1}$, and $\mathbf{e}_2$ contains the remaining entries of $\mathbf{e}_{k+1}$. If $e_1 = 0$, the first row of (2.8) becomes $\hat{\mathbf{s}}^1 = \mathbf{S}_k^T\mathbf{e}_2$. From the inductive hypothesis we know that there is only one non-zero entry in $\mathbf{e}_2$. Therefore, only one element in $\mathbf{e}_{k+1}$ can be non-zero, proving the proposition in this case. If $e_1 \neq 0$, subtracting the second row of (2.8) from the first, we have

$$\hat{\mathbf{s}}^1 - \hat{\mathbf{s}}^2 = 2e_1\mathbf{1}. \tag{2.9}$$

Since the entries of $\hat{\mathbf{s}}^1$ and $\hat{\mathbf{s}}^2$ are ± 1, e_1 can only be ± 1. Suppose $e_1 = 1$, then from (2.9) we know there can only be $\hat{\mathbf{s}}^1 = \mathbf{1}$ and $\hat{\mathbf{s}}^2 = -\mathbf{1}$. Substituting $e_1 = 1$ and $\hat{\mathbf{s}}^1 = \mathbf{1}$ into the first row of (2.8), we get $\mathbf{1} = \mathbf{1} + \mathbf{S}_k^T\mathbf{e}_2$. This implies $\mathbf{e}_2 = \mathbf{0}$ because $\mathbf{S}_k^T$ has full column rank. Therefore, vector $\mathbf{e}_{k+1}$ has only one non-zero entry, namely e_1, proving that the proposition is true in this case. The case $e_1 = -1$ can be treated similarly. This completes the induction on d and the proof of proposition. **Q.E.D.**

Notice that the objective function in (2.5) is smooth in $\mathbf{Q}$. This makes it possible to apply existing optimization techniques to solve (2.5). However, the orthonormal constraint $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ is cumbersome to handle computationally. Geometrically, the feasible set defined by $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ corresponds to the so called *Stiefel manifold.* Our approach is to apply the extended Newton's method over the Stiefel manifold [3] to this problem by following the geodesic directions (*i.e.,* the shortest curve between two points on the manifold). Since the iterates are confined to the Stiefel manifold, the problem is effectively reduced to an unconstrained one [3]. (See [6] for an alternative algorithm which does not take geodesic steps and [7] for the application of that algorithm to a different BSS problem.) If we let $\phi(\mathbf{Q})$ denote the objective in (2.5), the application of Edelman's algorithm to our blind signal separation problem is as follows:

1: Choose an initial separating matrix $\mathbf{Q}$ such that $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$;

2: Compute the gradient of $\phi(\mathbf{Q})$ at point $\mathbf{Q}$, which is given by $\mathbf{G} = \mathbf{\Phi_Q} - \mathbf{Q}\mathbf{\Phi_Q^T}\mathbf{Q}$, where $\mathbf{\Phi_Q}$ is the $d \times d$ matrix of partial derivatives of $\phi(\mathbf{Q})$ with respect to the elements of $\mathbf{Q}$; *i.e.* $[\mathbf{\Phi_Q}]_{ij} = \frac{\partial \phi}{\partial q_{ij}}$;

3: Compute the Newton direction $\mathbf{\Delta}$ such that $\mathbf{Q}^T\mathbf{\Delta} = -\mathbf{\Delta}^T\mathbf{Q}$ and

$$\mathbf{\Phi_{QQ}}(\mathbf{\Delta}) - \mathbf{Q}\,\mathrm{skew}(\mathbf{\Phi_Q^T}\mathbf{\Delta}) - \mathrm{skew}(\mathbf{\Delta}\mathbf{\Phi_Q^T})\mathbf{Q} - \frac{1}{2}\mathbf{\Pi}\mathbf{\Delta}\mathbf{Q}^T\mathbf{\Phi_Q} = -\mathbf{G},$$

where $\mathrm{skew}(\mathbf{X}) = (\mathbf{X} - \mathbf{X}^T)/2$, $\mathbf{\Pi} = \mathbf{I} - \mathbf{Q}\mathbf{Q}^T$ and $\mathbf{\Phi_{QQ}}(\mathbf{\Delta})$ is defined by

$$[\mathbf{\Phi_{QQ}}(\mathbf{\Delta})]_{kl} = 2\sum_{i,j} \frac{\partial^2 \phi}{\partial q_{ij}\partial q_{kl}}[\mathbf{\Delta}]_{ij};$$

4: Move along the Newton direction $\mathbf{\Delta}$ from $\mathbf{Q}$ to $\mathbf{Q}(t)$ using the geodesic formula, $\mathbf{Q}(t) = \mathbf{Q}\exp(t\,\mathbf{Q}^T\mathbf{\Delta})$, where the step size t is determined via an Armijo-type line search [2];

5: Repeat from step 2 until the norm of the Newton direction $\mathbf{\Delta}$ is smaller than a pre-set threshold.

3. SIMULATION RESULTS

We now compare the performance of our method with two methods which exploit the geometry of the pre-whitened BPSK signal separation problem, namely Hansen's hypercube method [4, 5] and Xavier's polyhedral method [10]. The scenario we consider is that in [4]; *i.e.,* $d = 3$ BPSK signals, $M = 5$ antennas and $N = 100$ data samples, with the elements of the mixing matrix $\mathbf{A}$ being independent zero-mean Gaussian random variables of the same variance. As in [4] the columns of $\mathbf{A}$ are normalized to have unit norm, and matrices with a correlation between a pair of columns greater than 0.95 are excluded. A total of 5000 Monte Carlo runs were performed for each SNR point, and the ambiguity in the BSS problem (2.4) was resolved via differential encoding and best fit selection. Since these three blind methods are 'zero-forcing' in nature, we provide results for a 'zero forcing' separator with full channel knowledge as a benchmark. In that case, $\hat{\mathbf{S}} = \mathbf{S} + \mathbf{A}^\dagger\mathbf{V}$, where $\mathbf{A}^\dagger$ is the pseudo inverse of the matrix $\mathbf{A}$.

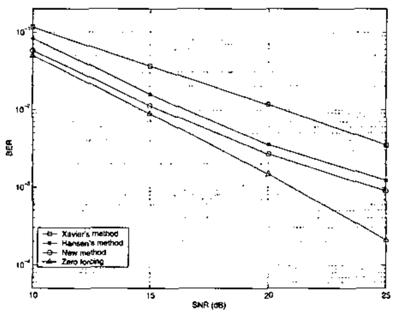

Fig. 1. Raw bit error rates

Fig. 1 shows the raw bit error rates (BERs) of the three blind methods of interest and the reference zero-forcing method, from which it is clear that our method is much closer to the benchmark than the other two methods. However, the raw BER can be distorted by 'failures' of the blind methods; *i.e.,* when $\mathbf{QC}$ is not close to $\mathbf{DP}$ [see (2.4)]. By computing the Frobenius norm of the difference between $\mathbf{QC}$ and the nearest matrix of the form $\mathbf{DP}$ we can identify such failures and remove the corresponding records. The resulting BER curves are shown in Fig. 2 and the corresponding failure rates are shown in Fig. 3. Again our method appears to have a clear advantage.

Finally, we compare the number of floating point operations (FLOPs) required to calculate $\mathbf{Q}$ in our implementations of the blind methods (see Fig. 4). Our method appears to have a significant advantage here too. It is interesting to note that Hansen's hypercube methods requires more FLOPs at higher SNR. This is because the hypercube method employs an SNR dependent statistical test to determine whether a globally optional solution to the embedded non-convex optimization problem has been achieved. If the test is negative, the optimization routine is repeated, at additional computational cost.

4. REFERENCES

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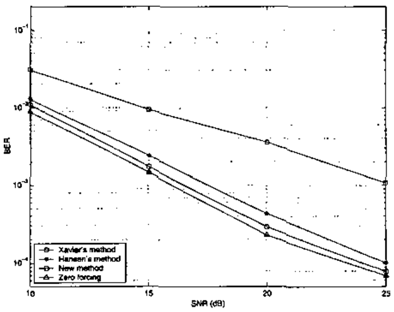

Fig. 2. Bit error rates excluding records for which any blind method failed.

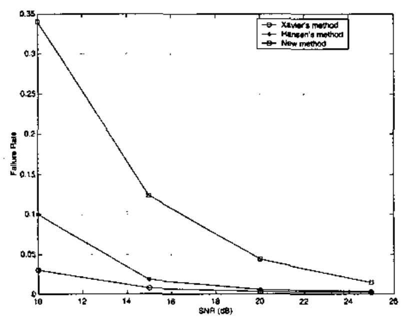

Fig. 3. Failure rates.

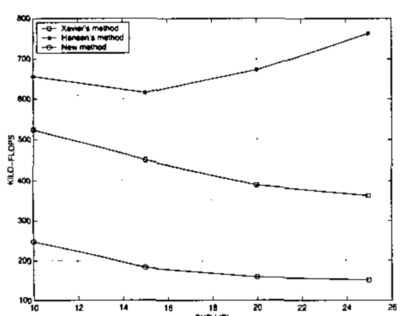

Fig. 4. Floating point operation counts.

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