FLEXIBLE CODEBOOK DESIGN FOR LIMITED FEEDBACK DOWNLINK SYSTEMS VIA SMOOTH OPTIMIZATION ON THE GRASSMANNIAN MANIFOLD

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ABSTRACT

Grassmannian quantization codebooks play a central role in a number of limited feedback schemes for single and multi-user MIMO communication systems. In practice, it is often desirable that these codebooks possess additional properties that facilitate their implementation, beyond the provision of good quantization performance. Although some good codebooks exist, their design tends to be a rather intricate task. The goal of this paper is to suggest a flexible approach to the design of Grassmannian codebooks based on smooth optimization algorithms for the Grassmannian manifold and the use of smooth penalty functions to obtain additional desirable properties. As one example, rank-2 codebooks with a nested structure and elements from a discrete alphabet are designed. In some numerical comparisons, codebooks designed using the proposed approach have better Fubini-Study distance properties than some existing codebooks, and provide tangible performance gains when applied to a simple MIMO downlink scenario with zero-forcing beamforming, PU^{2}RC, and block diagonalization signalling. Furthermore, the proposed approach yields codebooks that attain desirable additional properties without incurring a substantial degradation in performance.

1. INTRODUCTION

One of the basic building blocks of conventional channel adaptation schemes in single and multi-user MIMO communication systems is a memoryless vector quantization scheme that is used to inform the transmitter of information that is available to the receiver(s) by feeding back indices of elements of a quantization codebook over a channel of limited rate [1]. In the multi-user case, that quantization scheme is typically partitioned, with one of the partitions being a quantization codebook that captures the channel direction information (CDI). The design of that codebook can be viewed as a lossy source compression problem on the Grassmannian manifold, which is the manifold on which subspaces are represented by a single matrix whose orthonormal columns span the subspace. Unfortunately, solving that source compression problem can be a rather intricate task, even when the scenario is such that the codebook should be uniformly distributed on the manifold. One reason for this is that with the exception of a few special cases (e.g., [2]), the problem has proven quite resistant to analysis of the structure of the optimal codebook. Some numerical design methods have been proposed, including a variant of the Lloyd algorithm [3] and an alternating projection method [4], and these have been quite successful in the case where the dimension of the subspaces is one. However, in the case of subspaces of dimension two or more, there are fewer results available, especially in the case in which the Fubini-Study distance is chosen as the metric on the manifold; see [5] for some examples.

In addition to pure quantization performance, practical considerations suggest that codebooks ought to possess additional properties that facilitate their implementation [6]. For example, (i) in the single-user case one might seek to constrain the elements of the codewords to have constant modulus, so as to avoid power imbalances at the transmitter; (ii) for reasons of storage and computational cost, it may be desirable to have the elements of the codewords come from a defined alphabet; and (iii) in applications in which there is the option of multiple signalling modes, it is desirable for the codebook to be nested, in the sense that structure is imposed on the codewords of a codebook of subspaces of higher dimension (higher rank) so that they generate codewords for codebooks of lower rank.

The desire to obtain codebooks that possess additional properties significantly complicates what is already quite a difficult design problem. As a result, approaches to designing such codebooks tend to be rather specialized. The goal of this paper is to suggest a flexible approach to codebook design based on smooth optimization on the Grassmannian manifold [7, 8] that is amenable to any of the commonly-used distance metrics. In order to tackle the codebook design constraints, we suggest the use of smooth penalty functions that enable the use of the algorithms for unconstrained optimization on the manifold. Numerical results for unconstrained subspace codebooks show that in the case of Fubini-Study distance, the obtained codebooks exhibit larger minimum distances than the known packings [5]. In the case of constrained codebooks, the generated codebooks attain the desired properties without incurring a substantial degradation in distance properties. Some simple simulations of a MIMO downlink with zero-forcing beamforming [9], PU^{2}RC [10], and block diagonalization [11] signalling show that the improved distance properties of the designed codebooks yield tangible performance gains.

2. SYSTEM MODEL

Consider a MU-MIMO downlink system with a base station with M_t antennas communicating to K users, the kth of which has M_{r_k} antennas. At each channel use, the transmitter sends $P_k \leq \min\{M_t, M_{r_k}\}$ symbols to the kth user, whose received signal is

$$\mathbf{y}_k = \sqrt{\frac{E_s}{M}}\,\mathbf{H}_k\mathbf{V}\mathbf{s} + \mathbf{n}_k, \tag{1}$$

where $\mathbf{H}_k \in \mathbb{C}^{M_{r_k}\times M_t}$ is the channel matrix from the transmitter to the kth user, $\mathbf{s}$ contains the $P = \sum_k P_k$ transmitted symbols and is normalized so that $E[\mathbf{s}\mathbf{s}^H] = \mathbf{I}_P$, $\mathbf{V} \in \mathbb{C}^{M_t\times P}$ is the transmitter's preprocessing matrix, and $\mathbf{n}_k$ is the vector of additive noise samples at the kth receiver.

This work was supported in part by NSERC and the Canada Research Chairs program.

We consider a scenario in which the channel changes in a block fading manner and the variations are on a time scale that makes it viable for the receivers to feed back channel direction information (CDI), and possibly an indication of the quality of the channel, to the transmitter, and for the transmitter to adapt the preprocessing matrix $\mathbf{V}$, and possibly the the coding and modulation schemes that produce $\mathbf{s}$, to the information it receives. We will focus on the design of a Grassmannian quantization codebook for the CDI, $\mathcal{F} = \{\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_N\}$, in which the matrices $\mathbf{F}_i \in \mathbb{C}^{M_t \times M}$ represent the subspaces of dimension $M \leq M_t$ in $\mathbb{C}^{M_t}$ that their orthonormal columns span. As such, $\mathcal{F}$ is referred to as a rank-M codebook. We will denote the corresponding Grassmannian manifold by $\mathbb{G}_{M_t,M}$. In order to focus on the key principles we will consider the case in which we seek to design a codebook that provides near-uniform quantization of the subspaces spanned by $\mathbf{F}_i$ (e.g., [1]), while also possessing additional properties; e.g., [6]. The suggested approach can be extended to the case of non-uniform quantization, but in the interests of space we will leave that implicit.

3. GRASSMANNIAN SUBSPACE PACKINGS

The starting point for our approach to the design of Grassmannian codebooks is the problem of finding a codebook such that the minimum distance between codeword pairs is maximized. That is find a codebook $\mathcal{F} = \{\mathbf{F}_i\}_{i=1}^N$ that solves

$$\max_{\{\mathbf{F}_i\},\, \mathbf{F}_i \in \mathbb{G}_{M_t,M}} \; \min_{i \neq j} \; d(\mathbf{F}_i, \mathbf{F}_j), \tag{2}$$

where $d(\mathbf{F}_i, \mathbf{F}_j)$ is a measure of the distance between the subspaces spanned by its arguments. Three aspects of this problem make it difficult to solve. First, the constraint that the codewords $\mathbf{F}_i$ lie on the manifold is non-convex. Second, the $\min$ operator is not differentiable, and third, many distance metrics for the manifold are both non-convex and non-differentiable. These aspects suggest that the basic problem in (2) may be difficult to solve, even before we seek to modify it to obtain codebooks with additional features.

Our strategy for constructing an effective technique for finding good solutions to the codebook design problem in (2) is to use algorithms developed for optimization of smooth functions on the Grassmannian manifold [7, 8]. These algorithms are adaptations of conventional algorithms for unconstrained optimization of smooth functions to the manifold, and so in order to use them we need to construct a smooth approximation of the objective.

In this paper, we will focus on the Fubini-Study and chordal distances between subspaces as they are the most commonly used in limited feedback systems. However, the suggested procedures can be applied to any valid subspace distance or smoothed version thereof. The Fubini-Study and chordal distances are, respectively,

$$d_{\text{FS}}(\mathbf{F}_i, \mathbf{F}_j) = \arccos\big|\det(\mathbf{F}_j^H \mathbf{F}_i)\big|,$$

$$d_{\text{ch}}(\mathbf{F}_i, \mathbf{F}_j) = \frac{1}{\sqrt{2}} \|\mathbf{F}_i \mathbf{F}_i^H - \mathbf{F}_j \mathbf{F}_j^H\|_F = \big(M - \|\mathbf{F}_j^H \mathbf{F}_i\|_F^2\big)^{1/2}.$$

If we define: $\tilde{d}_{\text{FS}}(\mathbf{F}_i, \mathbf{F}_j) = \big|\det(\mathbf{F}_j^H \mathbf{F}_i)\big|$ and $\tilde{d}_{\text{ch}}(\mathbf{F}_i, \mathbf{F}_j) = \|\mathbf{F}_j^H \mathbf{F}_i\|_F^2$ then the problem in (2) can be rewritten as:

$$\min_{\{\mathbf{F}_i\}, \mathbf{F}_i \in \mathbb{G}_{M_t,M}} \; \max_{i \neq j} \; \tilde{d}(\mathbf{F}_i, \mathbf{F}_j). \tag{3}$$

In the Fubini-Study case, $\tilde{d}_{FS}(\mathbf{F}_i, \mathbf{F}_j)$ is not smooth, and consistent with our general approach we seek a smooth approximation. One approximation that is well suited to the problem in (3) is:

$$\hat{\tilde{d}}_{\text{FS}}(\mathbf{F}_i, \mathbf{F}_j) = (1 + \det(\mathbf{X}^H \mathbf{X})) \log(1 + \det(\mathbf{X}^H \mathbf{X})), \tag{4}$$

where $\mathbf{X} = \mathbf{F}_j^H \mathbf{F}_i$. For ease of notation we set $\hat{\tilde{d}}_{\text{ch}}(\cdot,\cdot) = \tilde{d}_{\text{ch}}(\cdot,\cdot)$.

There are number of ways in which the expression $\max_{i \neq j} \hat{\tilde{d}}(\mathbf{F}_i, \mathbf{F}_j)$ in (3) can be smoothly approximated. One is to use the approximation $\max\{a, b\} \approx \log(e^a + e^b)$. (The reverse approximation is similar to the "max-log" approximation that is often used in soft decoding algorithms.) This approximation was successfully used in [12] for the construction of Grassmannian constellations for non-coherent MIMO communication and is also applicable here. In this paper, we will take a different approach by approximating $\max_{i \neq j} \hat{\tilde{d}}(\mathbf{F}_i, \mathbf{F}_j)$ by:

$$J_1\big(\{\mathbf{F}_i\}\big) = \Big(\sum_{i \neq j} \hat{\tilde{d}}(\mathbf{F}_i, \mathbf{F}_j)^\beta\Big)^{1/\beta}, \tag{5}$$

for some $\beta \geq 1$. Thus, we approximate the ∞–norm of the vector of distances by its β–norm. A related idea was mentioned in [13].

Even though the problem

$$\min_{\{\mathbf{F}_i\},\, \mathbf{F}_i \in \mathbb{G}_{M_t,M}} J_1\big(\{\mathbf{F}_i\}\big) \tag{6}$$

is smooth, it remains non-convex, due, in part, to the nature of the manifold. As such, a straightforward descent method on the manifold will yield, at best, a locally optimal solution. However, we have found that the following approach typically yields good solutions.

The suggested approach to obtaining good solutions to (6), and hence to (3) and the original problem in (2), is based on a sequential approximation procedure on the Grassmannian manifold, in which we first solve (6) for a small value of β, and then we use the output of that optimization step as a starting point for an optimization process with a larger value for β. This sequential procedure takes advantage of the better conditioning of $J_1(\cdot)$ for smaller values of β. In particular, our basic procedure will be:

Basic Procedure

1. Set $\beta = 2$ and randomly select an initial codebook.
2. Starting from the codebook obtained in the previous iteration, obtain a good solution to (6) using an algorithm for smooth unconstrained optimization on the manifold [7, 8].
3. Evaluate the quality of the codebook against a known bound (if any), and evaluate the progress of the algorithm in terms of the rate of increase of the minimum distance.
4. Terminate if desired, else $\beta \leftarrow \beta + 2$ and return to step 2.

For certain distance metrics, including the chordal distance, the Rankin bound (e.g., [4]) can be used as the bound in step 3. Indeed, in cases where a bound is available, the basic procedure can be refined by setting α to the value of the known bound and replacing $J_1(\cdot)$ in (5) by

$$J_2\big(\{\mathbf{F}_i\}\big) = \Big(\sum_{i \neq j} \big(\hat{\tilde{d}}(\mathbf{F}_i, \mathbf{F}_j) - \alpha\big)^\beta\Big)^{1/\beta}. \tag{7}$$

Our numerical experience suggests that doing so can reduce the number of required iterations of the basic procedure. Actually, even when non-trivial bounds are not known, such as in the Fubini-Study case, the outcome of the basic procedure can sometimes be improved by using $J_2(\cdot)$ while adapting α based on the outcome of the previous iteration and keeping track of the best codebook found so far.

In order to assess the basic design procedure, in Table 1 we compare the minimum Fubini-Study distances of codebooks designed with the basic procedure against those of the corresponding codebooks in [5]. In each case, there is a tangible increase in the minimum distance of the codebook. In the case of designs based on the chordal distance, our results are essentially the same as those in [4].

Table 1. Minimum Fubini-Study distances of codebooks.

N	$M_t \times M$	Our codebook	Codebook in [5]
4	4×2	1.5708	1.2451
8	4×2	1.3418	1.0414
16	4×2	1.2123	0.8654
64	4×2	0.9613	0.6059
16	6×2	1.4812	N/A
32	6×2	1.3636	N/A
64	6×2	1.2986	N/A
16	6×3	1.5261	1.1936
32	6×3	1.4187	1.0724
64	6×3	1.3514	0.9722
32	8×2	1.4738	1.3153

4. LINE PACKINGS WITH CONSTANT MODULUS

In scenarios in which at most one data stream is sent to each user, the generic codebook design problem in (2) reduces to the line packing problem of finding a codebook of vectors $\mathbf{f}_i \in \mathbb{G}_{M_t,1}$, that are maximally separated with respect to the distance metric $d_{\text{line}}(\mathbf{f}_i, \mathbf{f}_j) = (1 - |\mathbf{f}_j^H \mathbf{f}_i|^2)^{1/2}$. Following the derivations in the previous section, that packing problem can be written in the form in (3) with $\tilde{d}_{\text{line}}(\mathbf{f}_i, \mathbf{f}_j) = |\mathbf{f}_j^H \mathbf{f}_i|$. By defining the tailored smooth approximation $\hat{\tilde{d}}_{\text{line}}(\mathbf{f}_i, \mathbf{f}_j) = \big(1 + |\mathbf{f}_j^H \mathbf{f}_i|^2\big) \log \big(1 + |\mathbf{f}_j^H \mathbf{f}_i|^2\big)$, we can obtain good solutions to the problem in (3) and hence good solutions to the original problem in (2) by applying the basic design procedure to a problem of the form in (7), where α can be initialized to the value of the corresponding Rankin bound. That procedure produces codebooks that exhibit essentially the same distance properties as the best of the existing codebooks; e.g., [4]. The goal of this section is to leverage the basic procedure to obtain codebooks with similar distance properties and the additional property that the elements of the vectors of the codebook have constant modulus.

For the line packing case, the constant modulus constraint is that the ℓth element of each vector $\mathbf{f}_i$ has modulus $1/\sqrt{M_t}$; i.e., $\big|[\mathbf{f}_k]_\ell\big| = 1/\sqrt{M_t}$. However, our basic design procedure is based on unconstrained optimization on the manifold. In order to use that procedure we define the smooth penalty term

$$P_{\text{cm}}\big(\{\mathbf{f}_i\}\big) = \Big(\sum_{k,\ell}\big(\big|[\mathbf{f}_k]_\ell\big|^2 - \tfrac{1}{M_t}\big)^\beta\Big)^{1/\beta}, \tag{8}$$

and apply the basic procedure with the cost function

$$J_3\big(\{\mathbf{f}_i\}\big) = w_1 J_2\big(\{\mathbf{f}_i\}\big) + w_2 P_{\text{cm}}\big(\{\mathbf{f}_i\}\big) \tag{9}$$

for appropriately chosen weights w_1 and w_2. The basic procedure can also be augmented by a sequential scheme in which w_2 is increased at each step and the solution from the previous step is used to initialize the smooth unconstrained optimization on the manifold. Our numerical experience with this technique, not formally reported here, has shown that for codebooks of the order of those in Table 1, the suggested procedure yields (essentially) constant modulus codebooks with approximately the same minimum distance as the unconstrained codebooks in [4], and does so with only minimal tuning of the weights.

5. LINE PACKINGS WITH DEFINED ALPHABET

In this section, we will build on the previous design by adding the condition that the elements of the codewords come from a defined (constant modulus) alphabet. In the case of simple alphabets, such as 4-PSK, this greatly reduces both the storage requirements of the codebook and the computational costs imposed on the receiver.

The restriction to a defined alphabet $\mathcal{A}$ offers the possibility to design codebooks based on exhaustively evaluating each admissible codebook. However, there are $|\mathcal{A}|^{M_t N}$ admissible codebooks and even for modestly sized codebooks the computational cost of this approach exceeds the computational resources that one could reasonably apply to the problem. The goal in this section is to use the approach of the previous section to generate partial codebooks with good properties. These partial codebooks are then completed by exhaustive search, but that exhaustive search is typically over a much smaller dimension.

To describe the principles of the approach, let $\mathcal{A}$ be scaled so that each element has modulus $1/\sqrt{M_t}$. The procedure is based on a notion of a satisfactory codebook. One way in which this can be assessed is to compare the achieved minimum distance of the finite alphabet codebook to the largest quantized distance that is smaller than the Rankin bound. A satisfactory codebook would achieve a large fraction of this bound. The procedure is as follows: First, we fix the first element of the codebook to be a randomly generated vector with elements from $\mathcal{A}$. Then we relax the finite alphabet constraint on the remaining $N-1$ codewords and use the procedure in the previous section to design a good codebook of size N with constant modulus elements. The elements of the codewords in that codebook are then quantized to the nearest point in the alphabet.[1] The quantized codebook is then analyzed to determine whether there are any codewords that induce distances that are deemed to be unsatisfactory. Those $\bar{N}$ codewords are removed and an exhaustive search over $\bar{N}$ codewords of the defined alphabet is performed to replace them. Typically, for codebooks of practical sizes, $\bar{N} \ll N$ and hence this exhaustive search of reduced dimension is often viable. In cases in which $\bar{N}$ is deemed to be too large, the basic relaxation–quantization approach can be applied to the design of the $\bar{N}$ replacement codewords. This procedure is dependent on the number of codewords $\bar{N}$ that are removed after the initial design. Typically, this would be a reasonably small number, but if the replacement procedure does not yield a satisfactory codebook, one can repeat the partitioning procedure, eliminating more codewords, and then perform a replacement procedure.

In our numerical experiments, the above procedure produced quantized codebooks of practical sizes with excellent distance properties. In Table 2 we have presented the minimum distance of codebooks designed for the QPSK alphabet and we have compared this to the Rankin bound. In all cases the designed codebooks achieve a known upper bound on the minimum distance for a defined alphabet codebook (e.g., the largest quantized distance below the Rankin bound), and hence can be considered to be optimal.

6. SUBSPACE PACKINGS WITH NESTED STRUCTURE AND DEFINED ALPHABET

In this section, we extend the ideas used in the previous sections to generate nested codebooks with defined alphabet; i.e., defined alphabet codebooks that are required to contain codebooks of lower rank. For simplicity we will focus on the case of codebooks of

[1] This simple rounding procedure can be replaced by a more sophisticated randomized rounding procedure, but we will not do that here.

Table 2. Line packings with QPSK alphabet

N	M_t	Our Packings	Rankin Bound
8	4	0.86603	0.92582
16	4	0.86603	0.89443
32	4	0.70711	0.87988
8	6	0.94281	0.9759
16	6	0.94281	0.94281
32	6	0.88192	0.92748
16	8	0.93541	0.96609
32	8	0.93541	0.95038

rank $M = 2$. In that case we seek a codebook $\mathcal{F} = \{\mathbf{F}_i\}_{i=1}^N$ in which each $\mathbf{F}_i = [\mathbf{f}_i, \check{\mathbf{f}}_i]$, where $\{\mathbf{f}_i\}_{i=1}^N$ forms a codebook of rank $M = 1$.

In the spirit of the previous approaches, the design procedure will involve a relaxation of the defined alphabet constraint, smooth optimization over the manifold, followed by rounding to the defined alphabet, and subsequent evaluation and iteration. The design procedure will be done using two main steps. The first step is to find a rank-1 codebook $\{\mathbf{f}_i\}_{i=1}^N$ using the procedures presented in Section 5. The elements of the rank-2 codebook must have orthonormal columns and should exhibit large chordal distances, and we capture these properties in the smooth objective:

$$J_4\big(\{\mathbf{F}_i\}\big) = w_1 J_2\big(\{\mathbf{F}_i\}\big) + w_2 P_{\text{cm}}\big(\{\check{\mathbf{f}}_i\}\big) + w_3\Big(\sum_i |f_i^H \check{f}_i|^\beta\Big)^{1/\beta}. \quad (10)$$

The first term in (10) captures the quantization performance of the rank-2 codebook, the second term penalizes codeword elements that deviate from the modulus $1/\sqrt{M_t}$ and the final term encourages orthogonality between the columns of $\mathbf{F}_i$. As in the basic procedure, a sequence of relaxed problems of the form $\max_{\{\mathbf{F}_i\}} J_4\big(\{\mathbf{F}_i\}\big)$ will be solved for increasing values of β. As in the previous section, when the solution to the final relaxed problem is rounded to the defined alphabet, there may be codewords that induce distances that are deemed unsatisfactory, or may result in rank-2 codewords that are not orthogonal. These can be removed from the codebook and replaced by codewords obtained via exhaustive search, or through another iteration of the relaxation approach starting from the first step again. However, in both cases the size of the new problem is the number of codewords to be replaced, which is typically much smaller than N.

We have summarized some of the distance results for precoders designed using this technique in Table 3. In this case, the distance metric in the optimization of the rank-2 codebook is the chordal distance. This table shows that in spite of the nesting and finite alphabet constraints, our approach can find codebooks that come reasonably close to the Rankin bound at both ranks.

7. SIMULATION RESULTS

We now provide a preliminary evaluation of the performance of codebooks obtained using the suggested approach in some simulations of the MIMO downlink with zero-forcing beamforming (ZFBF) [9], PU^{2}RC [10], and block diagonalization [11] signalling under a simple i.i.d. Rayleigh fading channel model. First, we consider the ZFBF case for a system with $M_t = 4$ transmitter antennas,

Table 3. Nested codebooks with QPSK alphabet

N	$M_t \times M$	Rankin bnd for Rank-1	Rank-1 ach'd dist.	Rankin bnd for Rank-2	Rank-2 ach'd chord. dist.
8	4×2	0.92582	0.86603	1.069	1
16	4×2	0.89443	0.86603	1.0328	1
8	6×2	0.9759	0.94281	1.2344	1.0541
16	6×2	0.94281	0.94281	1.1926	1
8	8×2	1	1	1.3093	1.1456
16	8×2	0.96609	0.93541	1.2649	1.0897

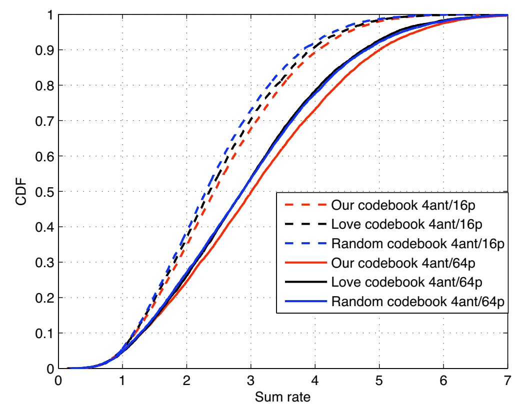

Fig. 1. CDF of the sum rate for a ZFBF system.

4 receivers and rank-1 codebooks of size $N = 16$, and 64. In Figure 1 we have plotted the cumulative distribution (cdf) of the sum of the rates that can be achieved using ZFBF and Gaussian signalling at an average SNR of 15 dB. We consider codebooks designed using our basic procedure with the Fubini-Study distance, Love's codebooks [5] and the average performance of a set of codebooks generated randomly using the uniform distribution on the manifold. Figure 1 demonstrates that the improved Fubini-Study distance properties of our codebooks (see Table 1) generate improved sum rate statistics. In Figure 2 we plot the corresponding results for PU^{2}RC signalling. In that case, the tangible performance advantages of our codebooks extend over a broader range of rates. Figure 3 examines the performance of defined alphabet codebooks of size $N = 16$ and 32 with $\mathcal{A}$ being the QPSK constellation in the PU^{2}RC scheme. Our defined alphabet codebooks provide performance that is close to that of the unconstrained codebooks and better than the average performance of randomly generated codebooks (with unconstrained coefficients).

Finally, we consider a system with $M_t = 4$ transmit antennas, 2 users, and rank-2 codebooks of size $N = 16$. The CDF of the sum rate at an SNR of 15 dB is plotted in Figure 4 for Love's codebook [5], an unconstrained codebook and a nested codebook with QPSK alphabet designed using the approach suggested in Section 6, and the average of a set of randomly generated unconstrained codebooks. Although our unconstrained codebook does provide marginally better performance than the codebook in [5], the interesting feature of Figure 4 is the excellent performance of the nested defined alphabet codebook.

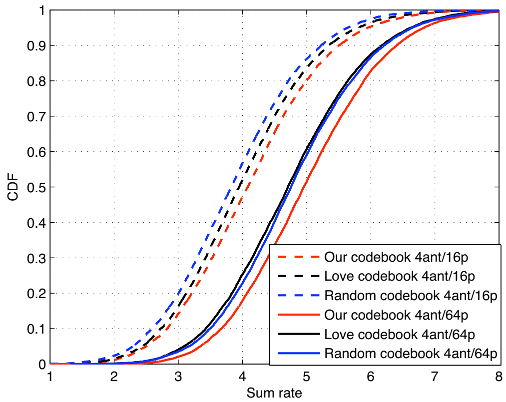

Fig. 2. CDF of the sum rate for a PU^{2}RC system.

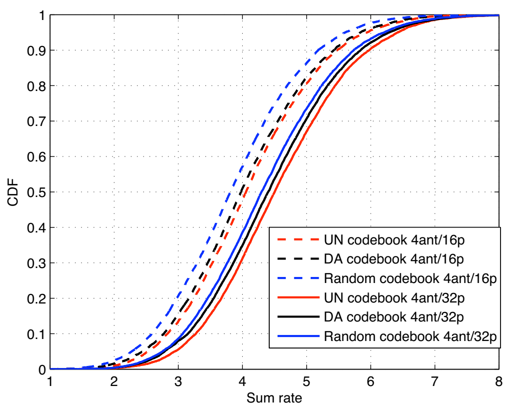

Fig. 3. CDF of the sum rate for a PU^{2}RC system with unconstrained (UN), QPSK alphabet (DA) and random codebooks.

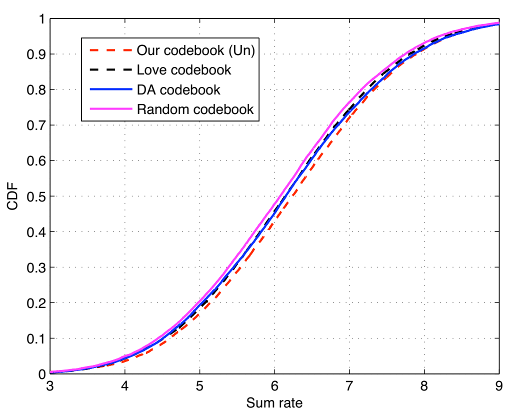

Fig. 4. CDF of the sum rate for rank-2 codebooks, including unconstrained (UN), and nested QPSK alphabet (DA) codebooks, for a block diagonalization scheme.

8. REFERENCES

[1] D. J. Love, R. W. Heath, Jr., V. K. N. Lau, D. Gesbert, B. D. Rao, and M. Andrews, "An overview of limited feedback in wireless communication systems," *IEEE J. Sel. Areas Commun.*, vol. 26, no. 8, pp. 1341–1365, 2008.

[2] R. A. Pitaval, H. L. Määttänen, K. Schober, O. Tirkkonen, and R.Wichman, "Beamforming codebooks for two transmit antenna systems based on optimum Grassmannian packings," *IEEE Trans. Inf. Theory*, vol. 57, no. 10, pp. 6591–6602, 2011.

[3] P. Xia and G. Giannakis, "Design and analysis of transmit-beamforming based on limited-rate feedback," *IEEE Trans. Signal Process.*, vol. 54, no. 5, pp. 1853–1863, 2006.

[4] I. S. Dhillon, R. W. Heath, T. Strohmer, and J. A. Tropp, "Constructing packings in Grassmannian manifolds via alternating projection," *Exper. Math*, vol. 17, no. 1, pp. 9–35, 2008.

[5] D. J. Love, "Grassmannian subspace packing webpage," [Online]: `https://engineering.purdue.edu/~djlove/grass.html`.

[6] B. Clerckx, Y. Zhou, and S. Kim, "Practical codebook design for limited feedback spatial multiplexing," in *Proc. IEEE Int. Conf. Commun.*, Beijing, 2008.

[7] A. Edelman, T. Arias, and S. T. Smith, "The geometry of algorithms with orthogonality constraints," *SIAM J. Matrix Anal. Appl.*, vol. 20, no. 2, pp. 303–353, 1998.

[8] J. H. Manton, "Optimization algorithms exploiting unitary constraints," *IEEE Trans. Signal Process.*, vol. 50, no. 3, pp. 635–650, 2002.

[9] N. Jindal, "MIMO broadcast channels with finite-rate feedback," *IEEE Trans. Inf. Theory*, vol. 52, no. 11, pp. 5045–5060, 2006.

[10] H. Lee, I. Sohn, and K. B. Lee, "Low-complexity unitary preprocessing scheme for limited feedback multiuser MIMO systems," in *Proc. 15th Asia-Pac. Conf. Commun.*, Shanghai, 2009.

[11] N. Ravindran and N. Jindal, "Limited feedback-based block diagonalization for the MIMO broadcast channel," *IEEE J. Sel. Areas Commun.*, vol. 26, no. 8, pp. 1473–1482, 2008.

[12] R. H. Gohary and T. N. Davidson, "Noncoherent MIMO communication: Grassmannian constellations and efficient detection," *IEEE Trans. Inf. Theory*, vol. 55, no. 3, pp. 1176–1205, 2009.

[13] J. H. Conway, R. H. Hardin, and N. J. A. Sloane, "Packing lines, planes, etc.: Packings in Grassmannian spaces," *Exper. Math*, vol. 5, pp. 139–159, 1996.