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Balanced approximation can be optimal

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Abstract: It is shown by example that there exist systems whose balanced approximations are optimal. Specifically, for various norms including the Hankel norm and the $\mathscr{L}_2$ norm, it is shown that the approximation error can achieve the minimum possible.

Keywords: Balanced approximation; model reduction; Hankel norm; finite differences; Hamiltonian systems.

1. Introduction

Since its introduction by Moore [7] in 1981, balanced approximation has become a popular and successful method for system approximation. Low-order models are simply obtained by discarding those states in a balanced realization that correspond to small Hankel singular values. These states are the most difficult to control and observe in an $\mathscr{L}_2$ sense. An advantage of the method is that the $\mathscr{L}_\infty$ approximation error is bounded above and below by simple functions of the Hankel singular values [3, 2]. The technique is therefore readily applicable to reduced-order $\mathscr{H}_\infty$ controller synthesis [1].

Hyland and Bernstein [6] have shown that, in general, balanced approximation is not optimal in the $\mathscr{L}_2$ norm. To complement that result, it is the purpose of the present paper to show that, in particular cases, balanced approximation *can* lead to approximations that minimize various error norms, including the $\mathscr{L}_2$ norm. Specifically, balanced approximations of a special class of systems with symmetric A matrices and unitary B and C matrices are shown to achieve the minimum possible error in the Hankel, Hilbert–Schmidt, nuclear and $\mathscr{L}_2$ norms. This property was first pointed out in [8] for the Hankel norm only.

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2. Preliminaries

Let $G(s) = C(sI - A)^{-1}B =: (A, B, C)$ be an n-state, stable, minimal system with controllability and observability gramians P and Q. The Hankel singular values of G are $\sigma_i = \lambda_i^{1/2}(PQ)$, $i = 1, 2, \ldots, n$ and are ordered $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$, as usual. We use the following standard norms of G to measure approximation errors:

- Hankel norm (see e.g. [3]), defined by $\|G\|_H := \sigma_1$
- Hilbert–Schmidt norm (see [4]), $\|G\|_{HS} := \sqrt{\sum_{i=1}^n \sigma_i^2}$
- nuclear norm (see [5]), $\|G\|_N := \sum_{i=1}^n \sigma_i$
- $\mathscr{L}_2$ norm, $\|G\|_2 = \sqrt{\operatorname{trace}(CPC^T)} = \sqrt{\operatorname{trace}(B^T Q B)}$.

The following lemma collects together some lower bounds on approximation errors with respect to the above norms. Our main result will be to show that these lower bounds can be attained by balanced approximation in some cases.

Lemma 2.1. *Let $G(s) = (A, B, C)$ be an n-state stable minimal $p \times m$ system with Hankel singular values, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Let $\tilde{G}(s)$ be any k-state stable $p \times m$ system, where $1 \leq k < n$. Then*

(a) $\|G - \tilde{G}\|_H \geq \sigma_{k+1}$,
(b) $\|G - \tilde{G}\|_{HS} \geq \sqrt{\sum_{i=k+1}^n \sigma_i^2}$,
(c) $\|G - \tilde{G}\|_N \geq \sum_{i=k+1}^n \sigma_i$,
(d) $\|G - \tilde{G}\|_2 \geq \sqrt{\sum_{i=k+1}^n \alpha_i^2}$,

where $\alpha_i := \lambda_i^{1/2}(P_s Q_s)$ and are ordered $\alpha_1 \geq \cdots \geq \alpha_n$. Here

$$P_s := \int_0^\infty e^{At} B B^T e^{A^T t} \frac{1}{\sqrt{\pi t}} dt,$$
$$Q_s := \int_0^\infty e^{A^T t} C^T C e^{At} \frac{1}{\sqrt{\pi t}} dt \tag{1}$$

are scaled controllability and observability gramians.

Proof. For (a) see [3] and for (b)–(d) see [4]. See also [14] for (b). □

We assume that the reader is familiar with balanced approximation and related concepts, as covered by Moore [7], Pernebo and Silverman [10] and Glover [3, Sections 2 and 4]. When we say that $\hat{G}$ is the *k-state balanced approximation* of an n-state stable system G, we mean, as usual, the k-state subsystem of a balanced realization of G that retains the k largest Hankel singular values.

3. Main result

Firstly, a definition of the relevant class of systems. The definition is essentially from [8, 9], but is strengthened here to have distinct Hankel singular values.

Definition 3.1. Define $\mathscr{S}^{p\times m}_{n;\,\text{stable}}$ to be the set of all n-state stable $p\times m$ systems ($n \le \min(m,p)$) with distinct Hankel singular values and a minimal realization (A,B,C) such that $A = A^{\mathrm{T}}$ and $BB^{\mathrm{T}} = C^{\mathrm{T}}C = I_n$.

Our main result is that balanced approximation of systems in the above class achieves equality in all the lower bounds in Lemma 2.1.

Theorem 3.2. *Let $G\in\mathscr{S}^{p\times m}_{n;\,\text{stable}}$ and let $\hat{G}$ be the k-state balanced approximation of G, where $1 \le k < n$. Then*

(a) $\|G-\hat{G}\|_{\mathrm{H}} = \sigma_{k+1}$,
(b) $\|G-\hat{G}\|_{\mathrm{HS}} = \sqrt{\sum_{i=k+1}^{n}\sigma_i^2}$,
(c) $\|G-\hat{G}\|_{\mathrm{N}} = \sum_{i=k+1}^{n}\sigma_i$,
(d) $\|G-\hat{G}\|_2 = \sqrt{\sum_{i=k+1}^{n}\alpha_i^2}$;

so $\hat{G}$ is simultaneously an optimal k-state approximation of G in the Hankel, Hilbert–Schmidt, nuclear and $\mathscr{L}_2$ norms.[1]

The proof may be found in the Appendix.

Clearly, $\mathscr{S}^{p\times m}_{n;\,\text{stable}}$ is a restricted class of systems – here our primary interest in $\mathscr{S}^{p\times m}_{n;\,\text{stable}}$ is to demonstrate that optimality may occur. However, practical situations leading to systems in $\mathscr{S}^{p\times m}_{n;\,\text{stable}}$ do exist, as the following two examples illustrate. (In both examples it turns out that $C = B^{\mathrm{T}}$, so the systems are also relaxation systems [13, Section 10].)

Example 3.3. (Taken from Mustafa [9]) Consider the following parabolic partial differential equation

$$\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2} + u$$

for $0 \le x \le 1$, $t > 0$, subject to boundary conditions $\theta(0,t) = \theta(1,t) = 0$ and given initial conditions. This equation can represent the temperature $\theta(x,t)$ in a one-dimensional normalized heat conduction problem, with heat input term $u(x,t)$. It can be approximated using a standard finite-difference technique as follows. Define a solution grid $x = lh$, where $l = 0,1,\ldots,N,N+1$ and $h(N+1) = 1$. At each interior grid point $x = ih$, $i = 1,2,\ldots,N$, apply the finite-difference approximation

$$\left.\frac{\partial^2\theta}{\partial x^2}\right|_{x=ih} = \frac{\theta((i-1)h,t) - 2\theta(ih,t) + \theta((i+1)h,t)}{h^2} + \mathrm{O}(h^2).$$

Then at each interior grid point, the exact solution $\theta(ih,t)$ is approximated by $v_i(t)$ that solves

$$\dot{v}_i(t) = \frac{v_{i-1}(t) - 2v_i(t) + v_{i+1}(t)}{h^2} + u_i(t), \qquad (2)$$

where $u_i(t) := u(ih,t)$, and $v_0(t) = v_{N+1}(t) = 0$. By defining a state vector $\tilde{x}(t) = (v_1(t),\ldots,v_N(t))^{\mathrm{T}}$, an input vector $\tilde{u}(t) = (u_1(t),\ldots,u_N(t))^{\mathrm{T}}$ and an output vector $\tilde{y}(t) = \tilde{x}(t)$, the N first-order differential equations in (2) can be written in a state space form

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t), \quad \tilde{y}(t) = C\tilde{x}(t),$$

where A is the stable $N\times N$ symmetric tridiagonal matrix

$$A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & & \\ 1 & -2 & 1 & \ddots & \\ 0 & 1 & -2 & \ddots & 0 \\ & \ddots & \ddots & \ddots & 1 \\ & & 0 & 1 & -2 \end{bmatrix}$$

and $B = C = I_N$. Furthermore, the Hankel singular values of (A,B,C) are shown in [9] to be distinct, so $(A,B,C)\in\mathscr{S}^{N,N}_{N;\,\text{stable}}$.

[1] Note added in proof: $\hat{G}$ can also be made into an H_∞ optimal approximation by adding a suitable constant term using [15].

Example 3.4. In [12] van der Schaft and Oeloff developed a model reduction procedure, known as pseudo-balanced truncation, for the subclass of Hamiltonian systems which are linear, time-reversible [11], and have all their poles on the $j\omega$-axis, excluding the origin. These systems have the property that $G(s) = G^{\mathrm{T}}(s) = G(-s)$ and occur in linear conservative systems with collocated sensors and actuators. Such systems always have a minimal $2n$-state realization

$$G(s) = \left(\begin{bmatrix} 0 & M \\ -R & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ L \end{bmatrix}, [L^{\mathrm{T}} \quad 0] \right), \tag{3}$$

where $M, R > 0$. Using the relation

$$G_*(s^2) = G(s) \tag{4}$$

one can associate with $G(s)$ an n-state *gradient system*, $G_{\mathrm{g}}(s)$. The gradient system is stable and has a minimal realization $G_{\mathrm{g}}(s) = (-MR, ML, L^{\mathrm{T}})$.

Let S be a balancing state transformation for the gradient system. Then it turns out that $SMS^{\mathrm{T}} = I_n$, and a balanced realization of the gradient system is $G_{\mathrm{g}} = (\bar{A}_{\mathrm{g}}, \bar{B}_{\mathrm{g}}, \bar{C}_{\mathrm{g}})$, where $\bar{A}_{\mathrm{g}} = -S^{-\mathrm{T}}RS^{-1}$, $\bar{B}_{\mathrm{g}} = S^{-\mathrm{T}}L$, $\bar{C}_{\mathrm{g}} = L^{\mathrm{T}}S^{-1}$. Under the relation in (4) the balancing state transformation S for the gradient system induces a symplectic state transformation, blockdiag $(S, S^{-\mathrm{T}})$, on G in (3). This results in a $2n$-state *pseudo-balanced* realization of G,

$$G(s) = \left(\begin{bmatrix} 0 & I_n \\ \bar{A}_g & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \bar{B}_g \end{bmatrix}, [\bar{B}_g^{\mathrm{T}} \quad 0] \right).$$

The model reduction method proposed in [12] involves finding a k-state balanced approximation of G_{g}, then mapping back to the corresponding $2k$-state pseudo-balanced Hamiltonian system. The resulting $2k$-state Hamiltonian system is said to be a *$2k$-state pseudo-balanced approximation* of the original Hamiltonian system. By inspection, $\bar{A}_{\mathrm{g}} = \bar{A}_{\mathrm{g}}^{\mathrm{T}} < 0$ and $\bar{C}_{\mathrm{g}} = \bar{B}_{\mathrm{g}}^{\mathrm{T}}$. If $\bar{B}_{\mathrm{g}}\bar{B}_{\mathrm{g}}^{\mathrm{T}} = I_n$ and the Hankel singular values of G_g are distinct, then $G_{\mathrm{g}} \in \mathcal{S}^{p\times m}_{n;\,\mathrm{stable}}$.

4. Conclusion

It has been shown in this paper that for a certain class of systems balanced approximation is optimal with respect to several measures of approximation error. It remains an open question whether a wider class of systems behave similarly.

Appendix

Proof of Theorem 3.2. To show (a)–(c) it is enough to show that the system $G - \hat{G}$ has Hankel singular values $\sigma_{k+1}, \ldots, \sigma_n$. (Many of the properties of systems in $\mathcal{S}^{p\times m}_{n;\,\mathrm{stable}}$ required to show this appeared more formally in [8].)

By simple algebraic manipulations, it can be shown that for $G \in \mathcal{S}^{p\times m}_{n;\,\mathrm{stable}}$ a balanced realization is $G = (\Lambda, W^{\mathrm{T}}B, CW)$, where $A = W\Lambda W^{\mathrm{T}}$ is a spectral decomposition of A. That is, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, where λ_i are the eigenvalues of A ordered $\lambda_1 > \cdots > \lambda_n$ and W is the unitary matrix of eigenvectors. Note that the Hankel singular values are given by $\sigma_i = -(2\lambda_i)^{-1}$, [8, Corollary 2.4], and are distinct by assumption. Using this realization, and partitioning $\Lambda = \mathrm{diag}(\Lambda_1, \Lambda_2)$ and $W = [W_1 \quad W_2]$ conformally with the balanced approximation, a realization of $G - \hat{G}$ is

$$G - \hat{G} = \left(\begin{bmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_1 \end{bmatrix}, \begin{bmatrix} W_1^{\mathrm{T}}B \\ W_2^{\mathrm{T}}B \\ -W_1^{\mathrm{T}}B \end{bmatrix}, \right.$$

$$\left. [CW_1 \quad CW_2 \quad CW_1] \right).$$

Applying the nonsingular state transformation

$$\begin{bmatrix} I_k & 0 & 0 \\ 0 & I_{n-k} & 0 \\ -I_k & 0 & I_k \end{bmatrix}$$

gives the realization

$$G - \hat{G} = \left(\begin{bmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_1 \end{bmatrix}, \begin{bmatrix} W_1^{\mathrm{T}}B \\ W_2^{\mathrm{T}}B \\ 0 \end{bmatrix}, \right.$$

$$\left. [0 \quad CW_2 \quad CW_1] \right),$$

from which the uncontrollable and unobservable states (which are stable) can be removed to give a minimal realization

$$G - \hat{G} = (\Lambda_2, W_2^{\mathrm{T}}B, CW_2). \tag{5}$$

This is a stable balanced system with gramians equal to

$$\Sigma_2 = \operatorname{diag}(-(2\lambda_{k+1})^{-1}, \ldots, -(2\lambda_n)^{-1})$$
$$= \operatorname{diag}(\sigma_{k+1}, \ldots, \sigma_n).$$

Hence, for systems $G \in \mathscr{S}^{p\times m}_{n;\text{stable}}$ the Hankel singular values of $G - \hat{G}$ are $\sigma_{k+1}, \ldots, \sigma_n$. Now, (a)–(c) follow immediately from the definitions of the norms in Section 2.

To show (d), we first evaluate $\|G - \hat{G}\|_2$ for $G \in \mathscr{S}^{p\times m}_{n;\text{stable}}$. Using (5),

$$\|G - \hat{G}\|_2 = \sqrt{\operatorname{trace}(CW_2\Sigma_2 W_2^{\mathrm{T}} C^{\mathrm{T}})}$$
$$= \sqrt{\operatorname{trace}(\Sigma_2 W_2^{\mathrm{T}} C^{\mathrm{T}} C W_2)}.$$

Noting that $C^{\mathrm{T}}C = I_n$ and $W_2^{\mathrm{T}} W_2 = I_{n-k}$, this simplifies to

$$\|G - \hat{G}\|_2 = \sqrt{\textstyle\sum_{i=k+1}^{n} \sigma_i}.$$

We now show that for $G \in \mathscr{S}^{p\times m}_{n;\text{stable}}$, we have $\sigma_i = \alpha_i^2$. From [4, Section 3.2], for an n-state system $(A_{\mathrm{d}}, B_{\mathrm{d}}, C_{\mathrm{d}})$ with $A_{\mathrm{d}} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, the expressions in (1) become

$$[P_{\mathrm{s}}]_{ij} = \frac{[B_{\mathrm{d}} B_{\mathrm{d}}^{\mathrm{T}}]_{ij}}{\sqrt{-(\bar{\lambda}_i + \lambda_j)}},$$

$$[Q_{\mathrm{s}}]_{ij} = \frac{[C_{\mathrm{d}}^{\mathrm{T}} C_{\mathrm{d}}]_{ij}}{\sqrt{-(\lambda_i + \bar{\lambda}_j)}}, \qquad i, j = 1, 2, \ldots, n.$$

Hence, with $G = (\Lambda, W^{\mathrm{T}}B, CW)$, the scaled gramians become

$$P_{\mathrm{s}} = Q_{\mathrm{s}} = \operatorname{diag}((-2\lambda_1)^{-1/2}, \ldots, (-2\lambda_n)^{-1/2})$$

and by definition $\alpha_i^2 = -(2\lambda_i)^{-1}$. But since $\sigma_i = -(2\lambda_i)^{-1}$, we have $\alpha_i^2 = \sigma_i$ and therefore

$$\|G - \hat{G}\|_2 = \sqrt{\textstyle\sum_{i=k+1}^{n} \sigma_i} = \sqrt{\textstyle\sum_{i=k+1}^{n} \alpha_i^2}$$

and $\hat{G}$ is an optimal $\mathscr{L}_2$ approximation of G.

References

[1] B.D.O. Anderson and Y. Liu, Controller reduction: Concepts and approaches, *IEEE Trans. Automat. Control* **34** (1989) 802–812.

[2] D.F. Enns, Model reduction with balanced realizations: An error bound and a frequency weighted generalization, in: *Proc. 23rd IEEE Conf. on Decision and Control*, Las Vegas, NV (1984) 127–132.

[3] K. Glover, All optimal Hankel-norm approximations of linear multivariable systems and their $\mathscr{L}_\infty$-error bounds, *Int. J. Control* **39** (1984) 1115–1193.

[4] K. Glover, Model reduction: A tutorial on Hankel-norm methods and lower bounds on $\mathscr{L}_2$ errors, in: *Proc. 10th IFAC Cong.*, Munich (1987).

[5] K. Glover, R.F. Curtain and J.R. Partington, Realisation and approximation of linear infinite-dimensional systems with error bounds, *SIAM J. Control Optim.* **26** (1988) 863–898.

[6] D.C. Hyland and D.S. Bernstein, The optimal projection equations for model reduction and the relationships among the methods of Wilson, Skelton and Moore, *IEEE Trans. Automat. Control.* **30** (1985) 1201–1211.

[7] B.C. Moore, Principal component analysis in linear systems: controllability, observability and model reduction, *IEEE Trans. Automat. Control* **26** (1981) 17–32.

[8] D. Mustafa, A class of systems for which balanced truncation is Hankel-norm optimal, in: *Proc. 30th IEEE Conf. on Decision and Control*, Brighton, England (1991) 1943–1948.

[9] D. Mustafa, Balanced truncation is Hankel-norm optimal when used to approximate a finite-difference model of a parabolic PDE, in: *Proc. 31st IEEE Conf. on Decision and Control*, Tucson, AZ (1992).

[10] L. Pernebo and L.M. Silverman, Model reduction via balanced state-space representations, *IEEE Trans. Automat. Control* **27** (1982) 382–387.

[11] A.J. van der Schaft, Time-reversible Hamiltonian systems, *Systems and Control Lett.* **1** (1982) 295–300.

[12] A.J. van der Schaft and J.E. Oeloff, Model reduction of linear conservative mechanical systems. *IEEE Trans. Automat. Control* **35** (1990) 729–733.

[13] J.C. Willems, Dissipative dynamic systems. Part II: Linear systems with quadratic supply rates, *Archive Rat. Mech. Anal.* **45** (1972) 352–393.

[14] D.A. Wilson, System order reduction using the Hilbert–Schmidt norm of the Hankel operator, in: *Proc. 12th IMACS World Congr.*, Paris (1988).

[15] D. Kavranoğlu, H, norm approximation of systems by constant matrices, in: *Proc. 31st IEEE Conf. on Decision and Control*, Tucson, AZ (1992).