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Block Bialternate Sum and Associated Stability Formulae*

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The block bialternate sum is defined, analysed and used to solve some maximal stability problems.

Key Words—Stability limits; integral control; singular perturbations; matrix methods; Kronecker sum; structured matrices.

Abstract—A new block bialternate sum of a partitioned square matrix with itself is defined and some basic properties established. It has similar properties to the bialternate sum, but it preserves the block structure of the summand. It is used in place of the block Kronecker sum and the block Lyapunov sum to solve maximal stability problems of stable systems under low-gain integral control and of singularly perturbed systems. It is also used to find the 'critical gains' for a first-order controller with a scalar plant from which all stabilizing first-order controllers can be constructed. In each stability problem the internal dimensions of the resulting closed formulae are lower than those currently available.

1. INTRODUCTION

For stability analysis of systems in state space, the Kronecker sum of the system dynamics matrix with itself is a useful tool (see e.g. Fuller, 1968; Fu and Barmish, 1988; Genesio and Tesi, 1988; Saydy *et al.*, 1990). A key property is that the eigenvalues of the Kronecker sum of an $n \times n$ matrix with itself are the n^2 pair-sums of the eigenvalues of the matrix. This property facilitates the identification of $j\omega$-axis eigenvalues of dynamics matrices parameterized by real parameters. However, the Kronecker sum can be rather large and unwieldy. It has dimension $n^2 \times n^2$ and it destroys any block structure that the dynamics matrix may possess. As such, it is not particularly suited to the stability analysis of systems with a structured dynamics matrix. Examples of control problems in which the structure of the dynamics matrix can be used to advantage include calculation of the 'radius of integral controllability' of a system that is integral-controllable in the sense of Morari (1985), calculation of the maximal stability range of a singularly perturbed system and calculation of the 'critical gains' (Mustafa, 1994c) for the construction of all stabilizing first-order controllers.

Fuller (1968) showed that there are matrices of reduced dimension $\frac{1}{2}n(n+1) \times \frac{1}{2}n(n+1)$ (the Lyapunov sum) and $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ (the bialternate sum) that can be used in place of the Kronecker sum in many useful stability tests. The Lyapunov sum of a matrix with itself has only one copy of each pair-sum of eigenvalues of the matrix, and the bialternate sum of a matrix with itself has one copy of each pair-sum of eigenvalues, excluding those n sums of an eigenvalue with itself. However, these matrices also destroy any block structure that the matrix may possess.

Hyland and Collins (1989) and Mustafa (1995) showed how to modify the Kronecker and Lyapunov sums respectively so that they retain the block structure of their summands without losing their useful eigenvalue properties. Either of the resulting block-structured matrices can be used to find closed formulae that solve the above-mentioned structured stability problems (see Mustafa, 1994a–c, 1995). The purpose of the present paper is to define a block bialternate sum and to show that it can be used to reduce the internal dimensions of the closed formulae of Mustafa (1994a–c, 1995) for the stability problems described above.

The development closely follows that of the block Lyapunov sum in Mustafa (1995). In Section 2 we give a brief summary of the Kronecker and block Kronecker sums and some of the work in Chapters 3 and 6 of Magnus (1988). We also review an algebraic realization of the empty matrix concept that reduces the complexity of the exposition. In Section 3 we define the block bialternate sum using the work

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in Chapter 6 of Magnus (1988), paralleling the definition of the block Lyapunov sum in Mustafa (1995), and give some of its properties. In Section 4 we use the block bialternate sum to provide closed formulae of reduced complexity for the three structured stability problems discussed above.

2. PRELIMINARIES

Let $\mathbb{R}^{m\times n}$ denote the set of all real $m\times n$ matrices. For a matrix $X\in\mathbb{R}^{m\times n}$, let X^{T} denote the transpose of X and let $x_{ij}(1\le i\le m, 1\le j\le n)$ denote the (i,j)th scalar element of X. Let I_n denote the $n\times n$ identity matrix. Let $\bar{m}:=(m_1,m_2,\ldots,m_q)$ and $\bar{n}:=(n_1,n_2,\ldots,n_r)$ be vectors of non-negative integers and let $\mathbb{R}^{\bar{m}\times\bar{n}}$ denote the set of all real $m\times n$ matrices with *block structure* $(\bar{m};\bar{n})$, that is, all real matrices of the form

$$X=\begin{bmatrix} X_{11} & \cdots & X_{1r}\\ \vdots & & \vdots\\ X_{q1} & \cdots & X_{qr}\end{bmatrix},$$

where $X_{ij}\in\mathbb{R}^{m_i\times n_j}$ and $\sum_{i=1}^{q} m_i=m$ and $\sum_{i=1}^{r} n_i=n$. If $\bar{m}=\bar{n}$ then we say that the matrix has block structure $\bar{n}$. Let $\mathbb{S}_n$ denote the set of all skew-symmetric $X\in\mathbb{R}^{n\times n}$ and let $\mathbb{S}_{\bar{n}}$ denote the set of all skew-symmetric $X\in\mathbb{R}^{\bar{n}\times\bar{n}}$. That is, define

$$\mathbb{S}_n:=\{X: X=-X^{\mathrm{T}}\in\mathbb{R}^{n\times n}\},$$
$$\mathbb{S}_{\bar{n}}:=\{X=-X^{\mathrm{T}}\in\mathbb{R}^{\bar{n}\times\bar{n}}\}.$$

2.1. *Kronecker and block Kronecker sum*

For a matrix $X\in\mathbb{R}^{m\times n}$, let $\mathrm{vec}\,(X)$ denote the vector formed by stacking the columns of X, and for a matrix $X\in\mathbb{R}^{\bar{m}\times\bar{n}}$, let $\mathrm{vecb}\,(X)$ denote the vector formed by stacking the columns of X in the block-structured manner suggested by Hyland and Collins (1989). That is,

$$\mathrm{vec}\,(X):=\begin{bmatrix} x_{11}\\ x_{21}\\ \vdots\\ x_{m1}\\ \cdots\\ x_{12}\\ \vdots\\ x_{m2}\\ \cdots\\ \vdots\\ \cdots\\ x_{1n}\\ \vdots\\ x_{mn}\end{bmatrix},\quad \mathrm{vecb}\,(X):=\begin{bmatrix} \mathrm{vec}\,(X_{11})\\ \mathrm{vec}\,(X_{21})\\ \vdots\\ \mathrm{vec}\,(X_{q1})\\ \cdots\\ \mathrm{vec}\,(X_{12})\\ \vdots\\ \mathrm{vec}\,(X_{q2})\\ \cdots\\ \vdots\\ \cdots\\ \mathrm{vec}\,(X_{1r})\\ \vdots\\ \mathrm{vec}\,(X_{qr})\end{bmatrix}. \tag{1}$$

Let $\otimes$ and $\oplus$ denote the Kronecker product and the Kronecker sum respectively (for a survey, see Brewer, 1978) and let $\bar{\oplus}$ denote the block Kronecker sum as defined by Hyland and Collins (1989). The basic properties of the Kronecker product, the Kronecker sum and the block Kronecker sum (as given in Brewer, 1978; Hyland and Collins, 1989) will be used freely, but we draw attention to the following fact: if $A\in\mathbb{R}^{n\times n}$ then

$$\mathrm{vec}\,(AX+XA^{\mathrm{T}}) = (A\oplus A)\,\mathrm{vec}\,(X)\quad\text{for all } X\in\mathbb{R}^{n\times n}, \tag{2}$$

and if $A\in\mathbb{R}^{\bar{n}\times\bar{n}}$ then

$$\mathrm{vecb}\,(AX+XA^{\mathrm{T}}) = (A\bar{\oplus}A)\,\mathrm{vecb}\,(X)\quad\text{for all } X\in\mathbb{R}^{\bar{n}\times\bar{n}}.$$

Furthermore, it is immediate from (1) that $\mathrm{vecb}\,(X)$ is just a rearrangement of elements of $\mathrm{vec}\,(X)$. Therefore there exists an $n^2\times n^2$ permutation matrix $P_{\bar{n}}$ such that

$$\mathrm{vec}\,(X)=P_{\bar{n}}\,\mathrm{vecb}\,(X)\quad\text{for all } X\in\mathbb{R}^{\bar{n}\times\bar{n}}.$$

Thus, for any $A\in\mathbb{R}^{\bar{n}\times\bar{n}}$,

$$\begin{aligned}\mathrm{vecb}\,(AX+XA^{\mathrm{T}}) &= P_{\bar{n}}^{\mathrm{T}}\,\mathrm{vec}\,(AX+XA^{\mathrm{T}})\\ &= P_{\bar{n}}^{\mathrm{T}}(A\oplus A)\,\mathrm{vec}\,(X)\\ &= P_{\bar{n}}^{\mathrm{T}}(A\oplus A)P_{\bar{n}}\,\mathrm{vecb}\,(X)\end{aligned} \tag{3}$$

for all $X\in\mathbb{R}^{\bar{n}\times\bar{n}}$. Since X in (3) is an arbitrary element of $\mathbb{R}^{\bar{n}\times\bar{n}}$, it follows that

$$A\bar{\oplus}A=P_{\bar{n}}^{\mathrm{T}}(A\oplus A)P_{\bar{n}}.$$

Therefore $A\bar{\oplus}A$ is similar to $A\oplus A$, and hence the eigenvalues of $A\bar{\oplus}A$ are the same as the eigenvalues of $A\oplus A$. They are the n^2 numbers $\lambda_i+\lambda_j$ $(1\le i,j\le n)$, where λ_i $(1\le i\le n)$ are the eigenvalues of A.

Following the nomenclature of Chapter 3 of Magnus (1988), we define the *commutation matrix* K_{mn} to be the $mn\times mn$ matrix such that

$$\mathrm{vec}\,(X^{\mathrm{T}})=K_{mn}\,\mathrm{vec}\,(X)\quad\text{for all } X\in\mathbb{R}^{m\times n}.$$

We note that $K_{mn} = K_{nm}^{\mathrm{T}} = K_{nm}^{-1}$ and that if L is $p \times m$ and R is $q \times n$ then

$$L \otimes R = K_{pq}(R \otimes L)K_{nm}.$$

Brewer (1978) called the commutation matrix the *permutation matrix* and denoted it by $U_{m \times n}$.

2.2. *Empty matrices*

The development of the block bialternate sum is simplified by the use of an algebraic realization of the concept of an empty matrix that is compatible with the usual matrix addition and multiplication operations. Such a realization has been developed independently by several researchers (see e.g. de Boor, 1990; Nett and Haddad, 1993; and references therein). We shall denote an empty matrix by $[\ ]_{p \times m}$, where it is implicit that at least one of the dimensions p or m is zero. For a scalar c and an $m \times p$ matrix $X_{m \times p}$, the following operations are defined (Nett and Haddad, 1993):

$$c \cdot [\ ]_{p\times m} = [\ ]_{p\times m} \cdot c = [\ ]_{p\times m},$$

$$[\ ]_{p\times m} + [\ ]_{p\times m} = [\ ]_{p\times m},$$

$$[\ ]_{0\times m} \cdot X_{m\times p} = [\ ]_{0\times p},$$

$$X_{m\times p} \cdot [\ ]_{p\times 0} = [\ ]_{m\times 0},$$

$$[\ ]_{p\times 0} \cdot [\ ]_{0\times m} = 0_{p\times m},$$

and hence $I_0 = [\ ]_{0\times 0}$ and $([\ ]_{0\times 0})^{-1} = [\ ]_{0\times 0}$. Furthermore, following de Boor (1990), we define $\det([\ ]_{0\times 0}) := 1$. For a matrix $X \in \mathbb{R}^{\bar{m}\times\bar{n}}$, if $m_i = 0$ for some i $(1 \le i \le q)$ then the ith block row of X is of zero height. Similarly, if $n_i = 0$ for some i $(1 \le i \le r)$ then the ith block column is of zero width.

2.3. *Bialternate sum*

Fuller (1968) introduced a matrix developed from Stephanos' bialternate product that he called the bialternate sum of a matrix with itself. The matrix was defined by *labelling* its scalar elements but without specifying the *arrangement* of the scalar elements to form the matrix. In this section we review Chapter 6 of Magnus (1988) and show that one of the matrices developed there is a specific arrangement of Fuller's elements. The development of the block bialternate sum in Section 3 will be based on the matrix of Magnus (1988), using some of the techniques of Mustafa (1995).

For any skew-symmetric matrix $X \in \mathbb{S}_n$, only $\frac{1}{2}n(n-1)$ of the n^2 elements are independent (say those below the diagonal). Define $\tilde{\mathrm{v}}(X)$ to be the $\frac{1}{2}n(n-1)$-element vector formed by stacking those parts of the columns of X that are below the main diagonal. That is,

$$\tilde{\mathrm{v}}(X) := \begin{cases} \begin{bmatrix} x_{21} \\ x_{31} \\ \vdots \\ x_{n1} \\ \cdots \\ x_{32} \\ \vdots \\ x_{n2} \\ \cdots \\ \vdots \\ \cdots \\ x_{n,n-1} \end{bmatrix} & \text{if } n > 1, \\ [\ ]_{0\times 1} & \text{if } n = 1. \end{cases} \tag{4}$$

A comparison with (1) shows that $\mathrm{vec}(X)$ consists of the elements of $\pm\tilde{\mathrm{v}}(X)$ and some additional zeros. Thus, following Chapter 6 of Magnus (1988), there exists a unique full-column-rank $n^2 \times \frac{1}{2}n(n-1)$ matrix of 0s and ± 1s, denoted by $\tilde{D}_n$, satisfying

$$\mathrm{vec}(X) = \tilde{D}_n \tilde{\mathrm{v}}(X) \quad \text{for all } X \in \mathbb{S}_n. \tag{5}$$

The matrix $\tilde{D}_n$ will be called the *duplication matrix for skew-symmetry* or the *skew duplication matrix* for short. Theorem 6.1 of Magnus (1988) gives a simple formula by which $\tilde{D}_n$ can be calculated (see (A.5) in the Appendix of the present paper). However, for the moment it is sufficient to note that since $\tilde{D}_n$ is of full column rank, it has a left inverse $\tilde{D}_n^+ = (\tilde{D}_n^{\mathrm{T}}\tilde{D}_n)^{-1}\tilde{D}_n^{\mathrm{T}}$, and

$$\tilde{\mathrm{v}}(X) = \tilde{D}_n^+ \,\mathrm{vec}(X) \quad \text{for all } X \in \mathbb{S}_n.$$

In fact, Theorem 6.2 of Magnus (1988) shows that $\tilde{D}_n^+ = \frac{1}{2}\tilde{D}_n^{\mathrm{T}}$.

Given a matrix $A \in \mathbb{R}^{n\times n}$, $AX + XA^{\mathrm{T}} \in \mathbb{S}_n$ for all $X \in \mathbb{S}_n$. Hence

$$\begin{aligned} \tilde{\mathrm{v}}(AX + XA^{\mathrm{T}}) &= \tilde{D}_n^+ \,\mathrm{vec}(AX + XA^{\mathrm{T}}) \\ &= \tilde{D}_n^+(A \oplus A)\,\mathrm{vec}(X) \\ &= \tilde{D}_n^+(A \oplus A)\tilde{D}_n \tilde{\mathrm{v}}(X) \end{aligned}$$

for all $X \in \mathbb{S}_n$. This motivates the definition of the $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ matrix

$$A \oplus A := \tilde{D}_n^+(A \oplus A)\tilde{D}_n,$$

so that, in a similar way to (2),

$$\tilde{v}(AX + XA^T) = (A \oplus A)\,\tilde{v}(X).$$

Note that in the case where $n = 1$, $A \oplus A = [\ ]_{0\times 0}$.

It is shown in Theorem 6.14 of Magnus (1988) that the eigenvalues of $A \oplus A$ are the $\frac{1}{2}n(n-1)$ numbers $\lambda_i + \lambda_j$ $(1 \le j < i \le n)$, where λ_i $(1 \le i \le n)$ are the eigenvalues of A. Hence $A \oplus A$ has the same eigenvalues as Fuller's (1968) bialternate sum. In fact, the connection with Fuller's bialternate sum is much stronger, as shown in the following fact.

Fact 1. The matrix $A \oplus A$ is a particular arrangement of the labelled elements in Fuller's definition of the bialternate sum.

Proof. See the Appendix. □

In the light of Fact 1 we shall call the matrix $A \oplus A$ the *bialternate sum of A* (with itself).

It is useful to define the $n^2 \times n^2$ matrix

$$\tilde{N}_n := \tfrac{1}{2}(I_{n^2} - K_{nn}).$$

For future reference, the following lemma collects some useful properties of K_{nn}, $\tilde{D}_n$, $\tilde{N}_n$ and their relationships. They are taken or easily follow from Theorems 3.13, 6.2, 6.3, 6.11, 6.14 and 6.16 of Magnus (1988). (See Lemma 2.1 of Mustafa (1995) for a similar collection of results for the duplication matrix.)

Lemma 1. Given the definitions of $\tilde{D}_n$, K_{nn} and $\tilde{N}_n$ above,

(i) $\tilde{D}_n^+\tilde{D}_n = I_{n(n-1)/2}$;
(ii) $\tilde{D}_n\tilde{D}_n^+ = \tilde{N}_n = -K_{nn}\tilde{N}_n = -\tilde{N}_nK_{nn}$;
(iii) $\tilde{D}_n = -K_{nn}\tilde{D}_n = \tilde{N}_n\tilde{D}_n$;
(iv) $\tilde{D}_n^+ = \tilde{D}_n^+\tilde{N}_n = -\tilde{D}_n^+K_{nn} = \frac{1}{2}\tilde{D}_n^T$.

If A and B are $n \times n$ matrices then

(v) $(A \oplus A)^{-1} = \tilde{D}_n^+(A \oplus A)^{-1}\tilde{D}_n$ if $A \oplus A$ is nonsingular;
(vi) $2\tilde{D}_n^+(A \otimes B)\tilde{D}_n = 2\tilde{D}_n^+(B \otimes A)\tilde{D}_n = \tilde{D}_n^+(A \otimes B + B \otimes A)\tilde{D}_n$;
(vii) $(A - B) \oplus (A - B) = (A \oplus A) - (B \oplus B) = A \oplus A - \tilde{D}_n^+(I_n \otimes B)\tilde{D}_n$;
(viii) $(kA) \oplus (kA) = k(A \oplus A)$ for $k \in \mathbb{R}$.

If M is an $m \times n$ matrix then

(ix) $\tilde{D}_m^+(M \otimes M) = \tilde{D}_m^+(M \otimes M)\tilde{N}_n$.

3. BLOCK BIALTERNATE SUM

3.1. *Definition*

Consider $X \in \mathbb{S}_{\bar{n}}$, a skew-symmetric matrix with block structure $\bar{n}$. Define $\tilde{v}b(X)$ to be the following $\frac{1}{2}n(n-1)$ element vector with $\frac{1}{2}r(r+1)$ block-rows:

$$\tilde{v}b(X) := \begin{bmatrix} \tilde{v}(X_{11}) \\ \text{vec}(X_{21}) \\ \vdots \\ \text{vec}(X_{r1}) \\ \cdots \\ \tilde{v}(X_{22}) \\ \text{vec}(X_{32}) \\ \vdots \\ \text{vec}(X_{r2}) \\ \cdots \\ \vdots \\ \cdots \\ \tilde{v}(X_{r-1,r-1}) \\ \text{vec}(X_{r,r-1}) \\ \cdots \\ \tilde{v}(X_{rr}) \end{bmatrix}, \quad (6)$$

which is a block structured version of $\tilde{v}(X)$. Recall from Section 2.3 that if $n_i = 1$ then $\tilde{v}(X_{ii}) = [\ ]_{0\times 1}$, so some of the block-rows in $\tilde{v}b(X)$ may be of zero height.

By comparison with (1) and recalling (5), it can be seen that $\text{vecb}(X)$ consists of the elements of $\pm\tilde{v}b(X)$ and some additional zeros. Hence there is a full-column-rank $n^2 \times \frac{1}{2}n(n-1)$ matrix of 0s and ±1s, denoted by $\tilde{D}_{\bar{n}}$, with r^2 block rows and $\frac{1}{2}r(r+1)$ block-columns (some of which may be of zero width) such that

$$\text{vecb}(X) = \tilde{D}_{\bar{n}}\tilde{v}b(X) \quad \text{for all } X \in \mathbb{S}_{\bar{n}}. \quad (7)$$

We shall call $\tilde{D}_{\bar{n}}$ the *block skew duplication matrix*. Details of how to calculate $\tilde{D}_{\bar{n}}$ will be given in Section 3.2, but for the moment it suffices to note that since $\tilde{D}_{\bar{n}}$ is of full column rank, it has a left inverse $\tilde{D}_{\bar{n}}^+ = (\tilde{D}_{\bar{n}}^T\tilde{D}_{\bar{n}})^{-1}\tilde{D}_{\bar{n}}^T$, and

$$\tilde{v}b(X) = \tilde{D}_{\bar{n}}^+\,\text{vecb}(X) \quad \text{for all } X \in \mathbb{S}_{\bar{n}}.$$

Given a matrix $A \in \mathbb{R}^{\bar{n}\times\bar{n}}$,

$$\begin{aligned}\tilde{v}b(AX + XA^T) &= \tilde{D}_{\bar{n}}^+\,\text{vecb}(AX + XA^T) \\ &= \tilde{D}_{\bar{n}}^+(A\bar{\oplus}A)\,\text{vecb}(X) \\ &= \tilde{D}_{\bar{n}}^+(A\bar{\oplus}A)\tilde{D}_{\bar{n}}\tilde{v}b(X)\end{aligned}$$

for all $X \in \mathbb{S}_{\bar{n}}$. This motivates the definition of the $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ matrix with $\frac{1}{2}r(r+1)$ block-rows (some of which may be of zero height) and $\frac{1}{2}r(r+1)$ block-columns (some of which may be of zero width)

$$A \bar{\bar{\oplus}} A := \tilde{D}_{\bar{n}}^{+}(A \bar{\oplus} A)\tilde{D}_{\bar{n}},$$

which we call the *block bialternate sum of A for block structure* $\bar{n} = (n_1, n_2, \ldots, n_r)$.

By comparing (6) and (4), it can be seen that for $X \in \mathbb{S}_{\bar{n}}$, $\tilde{\mathrm{v}}\mathrm{b}\,(X)$ is just a rearrangement of the elements of $\tilde{\mathrm{v}}\,(X)$, and vice versa. Hence there exists an $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ permutation matrix $\tilde{Q}_{\bar{n}}$ such that

$$\tilde{\mathrm{v}}\,(X) = \tilde{Q}_{\bar{n}}\tilde{\mathrm{v}}\mathrm{b}\,(X) \quad \text{for all } X \in \mathbb{S}_{\bar{n}}.$$

Hence, for any $A \in \mathbb{R}^{\bar{n} \times \bar{n}}$,

$$\begin{aligned} \tilde{\mathrm{v}}\mathrm{b}\,(AX + XA^{\mathrm{T}}) &= \tilde{Q}_{\bar{n}}^{\mathrm{T}}\tilde{\mathrm{v}}\,(AX + XA^{\mathrm{T}}) \\ &= \tilde{Q}_{\bar{n}}^{\mathrm{T}}(A \oplus A)\,\tilde{\mathrm{v}}\,(X) \\ &= \tilde{Q}_{\bar{n}}^{\mathrm{T}}(A \oplus A)\tilde{Q}_{\bar{n}}\tilde{\mathrm{v}}\mathrm{b}\,(X) \quad (8) \end{aligned}$$

for all $X \in \mathbb{S}_{\bar{n}}$. Since X in (8) is an arbitrary element of $\mathbb{S}_{\bar{n}}$, it follows that

$$A \bar{\bar{\oplus}} A = \tilde{Q}_{\bar{n}}^{\mathrm{T}}(A \oplus A)\tilde{Q}_{\bar{n}}.$$

Therefore $A \bar{\bar{\oplus}} A$ is similar to $A \oplus A$, and in particular the eigenvalues of $A \bar{\bar{\oplus}} A$ are the $\frac{1}{2}n(n-1)$ numbers $\lambda_i + \lambda_j$ $(1 \le j < i \le n)$, where λ_i $(1 \le i \le n)$ are the eigenvalues of A.

3.2. *Calculation of the block skew duplication matrix*

We now study the block skew duplication matrix in more detail, and provide a conceptual algorithm for its construction.

If $X \in \mathbb{S}_{\bar{n}}$ then, using the definition of $\tilde{\mathrm{v}}\mathrm{b}\,(X)$ in (6), $\mathrm{vec}\,(X_{ij})$ $(1 \le i, j \le r)$ is related to the block elements of $\tilde{\mathrm{v}}\mathrm{b}\,(X)$ in the following way:

$$\mathrm{vec}\,(X_{ij}) = \begin{cases} \mathrm{vec}\,(X_{ij}) & \text{if } i > j, \\ \tilde{D}_{n_i}\tilde{\mathrm{v}}\,(X_{ii}) & \text{if } i = j, \\ -K_{n_jn_i}\mathrm{vec}\,(X_{ij}) & \text{if } i < j. \end{cases} \quad (9)$$

Using the definitions of $\tilde{D}_{\bar{n}}$ in (7) and $\mathrm{vecb}\,(\cdot)$ in (1), and the relationships in (9), the following procedure for the construction of $\tilde{D}_{\bar{n}}$ can be derived. It is a straightforward modification of the procedure given by Mustafa (1995) for the construction of the block duplication matrix.

Construction of $\tilde{D}_{\bar{n}}$ *for* $\bar{n} = (n_1, n_2, \ldots, n_r)$. The block skew duplication matrix $\tilde{D}_{\bar{n}}$ has r^2 block-rows and $\frac{1}{2}r(r+1)$ block-columns (some of which may be of zero width). To find the pth block-row, where $1 \le p \le r^2$, use the following procedure.

- Let (i, j) be the pth pair of integers in the sequence

$$(1,1), (2,1), \ldots, (r,1); (1,2), (2,2), \ldots, (r,2); \ldots; (1,r), (2,r), \ldots, (r-1,r), (r,r).$$

- If $i \ge j$, define $m(r, i, j)$ to be the position that (i, j) takes in the sequence

$$(1,1), (2,1), \ldots, (r,1); (2,2), (3,2), \ldots, (r,2); \ldots; (r-1,r-1), (r,r-1); (r,r).$$

- If $i < j$, define $m(r, i, j)$ to be the position that (i, j) takes in the sequence

$$(1,1), (1,2), \ldots, (1,r); (2,2), (2,3), \ldots, (2,r); \ldots; (r-1,r-1), (r-1,r); (r,r).$$

- The pth block row of $\tilde{D}_{\bar{n}}$ has in the $m(r, i, j)$th block-column the matrix $\tilde{M}_{pm}$, where

$$\tilde{M}_{pm} := \begin{cases} I_{n_in_j} & \text{if } i > j, \\ \tilde{D}_{n_i} & \text{if } i = j, \\ -K_{n_jn_i} & \text{if } i < j. \end{cases}$$

and zeros elsewhere.

Note that if $i = j$ and $n_i = 1$ then $\tilde{M}_{pm} = [\ ]_{1\times 0}$, and hence the $m(r, i, j)$th block-column of $\tilde{D}_{\bar{n}}$ has zero width.

An alternative method for finding $m(r, i, j)$ is to use the following formula (Vetter, 1975):

$$m(r, i, j) = \begin{cases} r(j-1) + i - j - \frac{1}{2}j(j-3) & \text{if } i \ge j, \\ r(i-1) + j - i - \frac{1}{2}i(i-3) & \text{if } i < j. \end{cases}$$

In order to form $A \bar{\bar{\oplus}} A$, we also need to construct $\tilde{D}_{\bar{n}}^{+} = (\tilde{D}_{\bar{n}}^{\mathrm{T}}\tilde{D}_{\bar{n}})^{-1}\tilde{D}_{\bar{n}}^{\mathrm{T}}$. Since $\tilde{D}_{\bar{n}}$ has only one non-zero block in each block-row, $\tilde{D}_{\bar{n}}^{\mathrm{T}}\tilde{D}_{\bar{n}}$ is block-diagonal. Inspection of the procedure for the calculation of $\tilde{D}_{\bar{n}}$ above reveals that the $m(r, i, j)$th element on the block-diagonal is

$$\begin{matrix} \tilde{D}_{n_i}^{\mathrm{T}}\tilde{D}_{n_i} & \text{if } i = j, \\ I_{n_in_j} + K_{n_jn_i}^{\mathrm{T}}K_{n_jn_i} & \text{if } i \ne j. \end{matrix}$$

Using Lemma 1 and the properties of the commutation matrix given in Section 2.1, it follows that

$$\tilde{D}_{\bar{n}}^{+} = (\tilde{D}_{\bar{n}}^{\mathrm{T}}\tilde{D}_{\bar{n}})^{-1}\tilde{D}_{\bar{n}}^{\mathrm{T}} = \tfrac{1}{2}\tilde{D}_{\bar{n}}^{\mathrm{T}}.$$

If $\bar{n} = (1, 1, \ldots, 1)$, so that the block structure is that of the individual elements, then $\tilde{D}_{\bar{n}}$ reduces to the unstructured form $\tilde{D}_n$. To show this, observe that $\tilde{D}_1 = [\ ]_{1\times 0}$ and $K_{11} = 1$. Substitution of these quantities into the procedure for the construction of $\tilde{D}_{\bar{n}}$ above gives a matrix with $\frac{1}{2}n(n+1)$ block-columns, exactly n of which are of zero width. It is then a simple, but tedious, matter of evaluating the pth rows

$(1 \le p \le n^2)$ of $\tilde{D}_n$ and $\tilde{D}_{\bar{n}}$, using the formula for $\tilde{D}_n$ in Theorem 6.1 of Magnus (1988) and the procedure for the construction of $\tilde{D}_{\bar{n}}$ above, and showing that they are equal.

As an aside, we point out that the permutation and skew duplication matrices $P_{\bar{n}}$, $\tilde{D}_{\bar{n}}$, $\tilde{D}_n$ and $\tilde{Q}_{\bar{n}}$ are related by

$$P_{\bar{n}}\tilde{D}_{\bar{n}} = \tilde{D}_n\tilde{Q}_{\bar{n}}.$$

This result follows by noting that for all $X \in \mathbb{S}_{\bar{n}}$, $\mathrm{vec}(X) = P_{\bar{n}}\,\mathrm{vecb}(X) = P_{\bar{n}}\tilde{D}_{\bar{n}}\,\bar{\mathrm{v}}\mathrm{b}(X)$ and $\mathrm{vec}(X) = \tilde{D}_n\,\bar{\mathrm{v}}(X) = \tilde{D}_n\tilde{Q}_{\bar{n}}\,\bar{\mathrm{v}}\mathrm{b}(X)$.

3.3. *The* 2×2-*block case*

As will be illustrated in Section 4, there are a number of applications of the block bialternate sum in which there is a 2×2-block structure. That is, $\bar{n} = (n_1, n_2)$. We therefore provide the details of that case here. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

with $A_{11} \in \mathbb{R}^{n_1\times n_1}$ and $A_{22} \in \mathbb{R}^{n_2\times n_2}$. Using the procedure in Section 3.2, with $\bar{n} = (n_1, n_2)$, gives

$$\tilde{D}_{\bar{n}} = \begin{bmatrix} \tilde{D}_{n_1} & 0 & 0 \\ 0 & I_{n_1n_2} & 0 \\ 0 & -K_{n_2n_1} & 0 \\ 0 & 0 & \tilde{D}_{n_2} \end{bmatrix},$$

and hence

$$\tilde{D}_{\bar{n}}^{+} = \begin{bmatrix} \tfrac{1}{2}\tilde{D}_{n_1}^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & \tfrac{1}{2}I_{n_1n_2} & -\tfrac{1}{2}K_{n_1n_2} & 0 \\ 0 & 0 & 0 & \tfrac{1}{2}\tilde{D}_{n_2}^{\mathrm{T}} \end{bmatrix}.$$

The 2×2-block Kronecker sum is (Hyland and Collins, 1989)

$$A\bar{\oplus}A = \begin{bmatrix} A_{11}\oplus A_{11} & I_{n_1}\otimes A_{12} & A_{12}\otimes I_{n_1} & 0 \\ I_{n_1}\otimes A_{21} & A_{11}\oplus A_{22} & 0 & A_{12}\otimes I_{n_2} \\ A_{21}\otimes I_{n_1} & 0 & A_{22}\oplus A_{11} & I_{n_2}\otimes A_{12} \\ 0 & A_{21}\otimes I_{n_2} & I_{n_2}\otimes A_{21} & A_{22}\oplus A_{22} \end{bmatrix}.$$

Therefore, using the definition $A\,\bar{\diamond}\,A = \tilde{D}_{\bar{n}}^{+}(A\bar{\oplus}A)\tilde{D}_{\bar{n}}$ and Lemma 1, it follows that

$$A\,\bar{\diamond}\,A = \begin{bmatrix} A_{11}\diamond A_{11} & \tilde{D}_{n_1}^{\mathrm{T}}(I_{n_1}\otimes A_{12}) & 0 \\ (I_{n_1}\otimes A_{21})\tilde{D}_{n_1} & A_{11}\oplus A_{22} & (A_{12}\otimes I_{n_2})\tilde{D}_{n_2} \\ 0 & \tilde{D}_{n_2}^{T}(A_{21}\otimes I_{n_2}) & A_{22}\diamond A_{22} \end{bmatrix}.$$

4. STABILITY FORMULAE

In this section we use the block bialternate sum to derive closed-formula solutions of the three stability problems discussed in Section 1. In each case the the internal dimensions of the derived formulae are lower than those currently available. To state the results, we make the following definitions. Given a square matrix M, we define $\lambda_{\max}^{+}(M)$ to be the largest positive real eigenvalue of M, or 0^{+} if there are no positive real eigenvalues, and we define $\lambda_{\mathrm{real}}(M)$ to be any real eigenvalue of M. Given another square matrix N of the same dimension, we define $\lambda_{\mathrm{real}}(M, N)$ to be any finite real generalized eigenvalue of M (i.e. any finite real solution of $\det(M - \lambda N) = 0$). A matrix will be said to be *(asymptotically) stable* if all its eigenvalues have strictly negative real parts, and a system $G(s) = D + C(sI - A)^{-1}B$ will be said to be *stable* if A is stable.

4.1. *Radius of integral controllability*

Consider the negative-feedback connection of the integral controller kI_m/s to an n-state stable $m\times m$ system $G(s)$. Morari (1985) showed that there exists a k^* such that kI_m/s stabilizes $G(s)$ for all $k \in (0, k^*)$ if $G(0)$ has all its eigenvalues in the open right half-plane, and only if $G(0)$ has all its eigenvalues in the closed right half-plane minus the origin. If there exists such a k^* then $G(s)$ is said to be *integral controllable* (for related work, see also Lunze, 1985; Campo and Morari, 1994). The largest possible value of k^* is known as the *radius of integral controllability,* and will be denoted by $k_{\max}^*$. A $2mn$-dimensional eigenvalue formula for $k_{\max}^*$ was derived under mild conditions by Mustafa (1994a). Its construction involved the use of the block Kronecker sum within the guardian map framework (Saydy *et al.,* 1990). Mustafa (1995, 1994b) showed that the dimension of the eigenvalue problem can be halved by using the block Lyapunov sum and that the mild conditions can be removed by using a Schur-type determinantal formula from the Appendix of Mustafa and Davidson (1994).

In the following proposition we derive an improved mn-dimensional eigenvalue formula for the radius of integral controllability, using the block bialternate sum. The formula improves on that of Mustafa (1995, 1994b) in that it involves the inverses of square matrices of dimensions $\tfrac{1}{2}n(n-1)$, $\tfrac{1}{2}m(m-1)$ and n, whereas the corresponding matrices in the formula of Mustafa (1995, 1994b) are of dimensions $\tfrac{1}{2}n(n+1)$, $\tfrac{1}{2}m(m+1)$ and n.

Proposition 1. Let $G(s) = D + C(sI - A)^{-1}B$ be an n-state $m \times m$ system. Furthermore, let $G(s)$ be stable and let $G(0)$ have all its eigenvalues in the open right half-plane. The radius of integral controllability of $G(s)$ is

$$k^*_{\max} = \frac{1}{\lambda^+_{\max}(Z)},$$

where Z is the $mn \times mn$ matrix

$$\begin{aligned} Z := {} & (A^{-1} \otimes D) \\ & + [(A^{-1}B \otimes I_m)\tilde{D}_m \quad (A^{-1} \otimes C)\tilde{D}_n] \\ & \times \begin{bmatrix} [G(0) \oplus G(0)]^{-1} & 0 \\ 0 & -(A \oplus A)^{-1} \end{bmatrix} \\ & \times \begin{bmatrix} \tilde{D}^T_m[CA^{-1} \otimes G(0)] \\ \tilde{D}^T_n(I_n \otimes B) \end{bmatrix}. \end{aligned}$$

Proof. See the Appendix. □

We now provide an example of the application of Proposition 1 to determine the radius of integral controllability of an interval plant family (for background, see e.g., Barmish *et al.*, 1992).

Example 1. Consider the following interval plant family taken from Hollot and Tempo (1994):

$$G(s) = \frac{N(s)}{D(s)} = \frac{39\,060(q_3s^3 + q_2s^2 + q_1s + q_0)}{s^7 + r_6s^6 + r_5s^5 + r_4s^4 + r_3s^3 + r_2s^2 + r_1s + r_0}, \tag{10}$$

where

$$q_3 \in [0.8, 8], \quad q_2 \in [3.28, 32.8],$$
$$q_1 \in [3.52, 35.2], \quad q_0 \in [0.32, 3.2],$$
$$r_6 \in [81.31, 81.92], \quad r_5 \in [3910.38, 3960.94],$$
$$r_4 \in [61\,412.23, 63\,897.28],$$
$$r_3 \in [343\,379.7, 386\,727.7],$$
$$r_2 \in [620\,977.75, 915\,287.75],$$
$$r_1 \in [1\,324\,350, 1\,978\,350],$$
$$r_0 \in [126\,562.5, 189\,062.5].$$

As pointed out by Hollot and Yang (1990), Corollary 1.5 of Ghosh (1985) can be easily modified to show that an interval plant family that contains no members with a zero at the origin is stabilized by a positive-grain integral controller k/s if and only if the integral controller k/s stabilizes the following plants:

$$\frac{N_3(s)}{D_1(s)}, \quad \frac{N_4(s)}{D_2(s)}, \quad \frac{N_2(s)}{D_3(s)}, \quad \frac{N_1(s)}{D_4(s)}, \tag{11}$$

where $N_i(s)$ and $D_i(s)$ $(i = 1, 2, 3, 4)$ are the Kharitonov polynomials (Kharitonov, 1978) with ordering as in Barmish *et al.* (1992) for the numerator and denominator of the interval plant respectively.

Since the interval plant family in (10) is stable and $G(0) > 0$ for all members of the family, it is integral controllable. Therefore its radius of integral controllability is the smallest of the radii of integral controllability of the plants in (11). By applying the formula of Proposition 1 to each of the four plants in (11), we obtain

$$k^*_{\max} = \min\{0.4971, 4.7196, 4.2202, 0.9995\} = 0.4971.$$

To illustrate that $k^*_{\max} = 0.4971$ is indeed the maximal gain, the cross-section of the robust root locus of $G(s)/s$ at a gain of 0.4971 was calculated using the method of Barmish and Tempo (1990). A detail of the cross-section is shown in Fig. 1. It is clear from this figure that when $k = k^*_{\max}$, there is one pair of $j\omega$-axis poles in the cross-section of the robust root locus. Hence the closed-loop system of $k^*_{\max}/s$ with the interval plant in (10) is indeed on the stability/instability threshold.

4.2. *Maximal stability range of singularly perturbed systems*

Consider the singularly perturbed system

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2,$$
$$\epsilon\dot{x}_2 = A_{21}x_1 + A_{22}x_2,$$

where

$$x_1 \in \mathbb{R}^{n_1}, \quad x_2 \in \mathbb{R}^{n_2}, \quad \epsilon > 0. \tag{12}$$

It is well known (Klimushchev and Krasovskii, 1962) that if both A_{22} and $A_{11} - A_{12}A_{22}^{-1}A_{21}$ are asymptotically stable matrices then there exists an $\epsilon^* > 0$ such that the system in (12) is

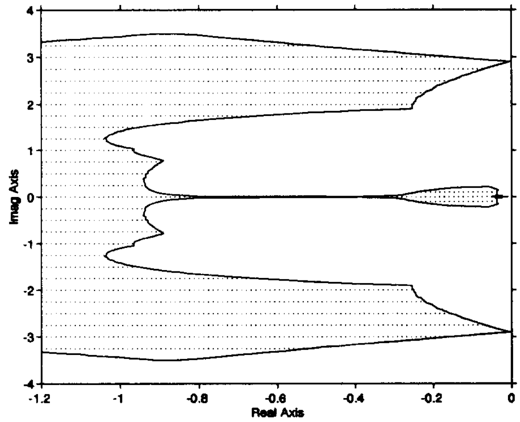

Fig. 1. The cross-section of the robust root locus of $G(s)/s$ at $k^*_{\max}$ for Example 1.

asymptotically stable for all $\epsilon \in (0, \epsilon^*)$. The maximal value of ϵ^* is known as the maximal stability range of the singularly perturbed system, and will be denoted by ϵ^*_{max}. For some time, only estimates of ϵ^*_{max} were available (see e.g. Chen and Lin, 1990, and references therein). More recently, however, Feng (1988) and Abed *et al.* (1990) characterized ϵ^*_{max} using Nyquist and guardian map techniques respectively. Mustafa (1994a) derived an exact $2n_1n_2$-dimensional eigenvalue formula for ϵ^*_{max} under mild conditions and, in independent work, Sen and Datta (1993) used the (unstructured) bialternate sum to derive an $\frac{1}{2}(n_1+n_2)(n_1+n_2-1)$ dimensional eigenvalue formula for ϵ^*_{max}. Furthermore, Mustafa (1994b) used the block Lyapunov sum to derive an n_1n_2-dimensional eigenvalue formula without the mild conditions required in Mustafa (1994a).

In the following proposition we use the block bialternate sum to give an improved n_1n_2-dimensional eigenvalue formula for ϵ^*_{max}. The proof is by applying the transformation of Section 4.2 of Mustafa (1995) to obtain an equivalent k^*_{max} problem, and then applying Proposition 1.

Proposition 2. Consider the singularly perturbed system in (12) where A_{22} and $A_0 := A_{11} - A_{12}A_{22}^{-1}A_{21}$ are asymptotically stable matrices. The maximal ϵ^* such that the system is stable for all $\epsilon \in (0, \epsilon^*)$ is given by

$$\epsilon^*_{max} = \frac{1}{\lambda^+_{max}(W)},$$

where W is the $n_1n_2 \times n_1n_2$ matrix

$$W := -(A_{22}^{-1} \otimes A_{11}) + [(A_{22}^{-1}A_{21} \otimes I_{n_1})\tilde{D}_{n_1} \quad (A_{22}^{-1} \otimes A_{12})\tilde{D}_{n_2}] \times \begin{bmatrix} (A_0 \oplus A_0)^{-1} & 0 \\ 0 & -(A_{22} \oplus A_{22})^{-1} \end{bmatrix} \times \begin{bmatrix} \tilde{D}^T_{n_1}(A_{12}A_{22}^{-1} \otimes A_0) \\ \tilde{D}^T_{n_2}(I_{n_2} \otimes A_{21}) \end{bmatrix}.$$

We demonstrate the advantages of Proposition 2 over previous work in the following example.

Example 2. Consider the singularly perturbed system for which

$$A_{11} = \begin{bmatrix} -3 & 4 \\ 0 & 2 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -3 & 4 \\ -1 & -2 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -2 & 3 \\ 0 & -3 \end{bmatrix}.$$

This system has also been used by Chen and Lin (1990), Sen and Datta (1993) and Mustafa (1994b, 1995). Applying Proposition 2, the eigenvalues of W are

$$-0.7976, \quad -0.4865, \quad 0.2021, \quad 1.0201,$$

and hence

$$\epsilon^*_{max} = 1/1.0201 = 0.9803,$$

as was obtained by Sen and Datta (1993) and Mustafa (1994b, 1995). Note that in this example W is a 4×4 matrix, as was the case in the application of the formula of Mustafa (1994b) to the present problem. The corresponding matrix in the method of Sen and Datta (1993) is of dimension 6×6. Furthermore, the expression for W involves the inverse of a 2×2 matrix and the inverses of two scalars, whereas the formula of Mustafa (1994b) involves the inverses of a 2×2 and two 3×3 matrices. By contrast, the formula of Sen and Datta (1993) involves the inverse of a 6×6 matrix. The computational savings of the formula in Proposition 2 over the formulae of Mustafa (1994b) and Sen and Datta (1993) can be even more significant in other cases.

4.3. *All stabilizing first-order controllers via the critical gains*

Consider the negative-feedback connection of a general first-order controller $k(s) = (\alpha s + \beta)/(s + \gamma)$ to an n-state scalar plant $g(s) = c(sI - A)^{-1}b$. Let λ_i $(1 \le i \le n+1)$ denote the closed-loop eigenvalues (the eigenvalues of the closed-loop dynamics matrix), and, with a slight abuse of terminology, let the controller parameters α, β and γ be known as *gains*. Mustafa (1994c) showed that all stabilizing first-order controllers for $g(s)$ can be found by dividing the (α, β, γ) gain-space into distinct regions whose boundaries are 'critical gains'. The *critical gains* are defined to be those (finite) real values of α, β and γ (if any) such that $\lambda_i + \lambda_j = 0$ for some i, j $(1 \le i, j \le n+1)$. Using some elements of guardian map theory (Saydy *et al.*, 1990), Mustafa (1994c) showed that all the controller gains within a given region result in the same number of unstable closed-loop eigenvalues. Therefore, once the gain-space has been divided as above, it is easy to identify the regions (if any) that result in no unstable closed-loop eigenvalues. The union of those regions that result in no unstable closed-loop eigenvalues contains all the controller gains α, β and γ that stabilize $g(s)$.

A particular attraction of the approach of Mustafa (1994c) is that closed generalized eigenvalue formulae for the critical gains were given. Furthermore, these formulae were shown to simplify in certain special cases. In the following proposition and its corollaries the block bialternate sum is used to give more compact formulae for the critical gains. In order to state the results, we define the $\frac{1}{2}n(n+1)\times\frac{1}{2}n(n+1)$ matrices with block structure $(\frac{1}{2}n(n-1), n)$

$$P_1 := \begin{bmatrix} A\oplus A & 0 \\ -(I_n\otimes c)\tilde{D}_n & A \end{bmatrix},\quad Q_1 := \begin{bmatrix} bc\oplus bc & 0 \\ 0 & bc \end{bmatrix},$$
$$R_1 := \begin{bmatrix} 0 & -\tilde{D}_n^{\mathrm{T}}(I_n\otimes b) \\ 0 & 0 \end{bmatrix},\quad S_1 := \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix}, \tag{13}$$

and the $(n+1)\times(n+1)$ matrices with block structure $(n, 1)$

$$P_2 := \begin{bmatrix} A & 0 \\ -c & 0 \end{bmatrix},\quad Q_2 := \begin{bmatrix} bc & 0 \\ 0 & 0 \end{bmatrix},$$
$$R_2 := \begin{bmatrix} 0 & -b \\ 0 & 0 \end{bmatrix},\quad S_2 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{14}$$

Let $\tilde{\alpha}(\beta,\gamma)$ denote the critical values for α for fixed values of β and γ. Similarly, let $\tilde{\beta}(\alpha,\gamma)$ and $\tilde{\gamma}(\alpha,\beta)$ denote the critical values for β and γ respectively.

Proposition 3. Let $g(s) = c(sI - A)b$ be an n-state scalar system. For the controller $k(s) = (\alpha s + \beta)/(s+\gamma)$ connected to $g(s)$, the critical gains are

- $\tilde{\alpha}(\beta,\gamma) = \lambda_{\mathrm{real}}(P_i - \beta R_i - \gamma S_i, Q_i - \gamma R_i)$ for $i = 1, 2$ and arbitrary β, γ;
- $\tilde{\beta}(\alpha,\gamma) = \lambda_{\mathrm{real}}(P_i - \alpha Q_i - \gamma S_i + \alpha\gamma R_i, R_i)$ for $i = 1, 2$ and arbitrary α, γ;
- $\tilde{\gamma}(\alpha,\beta) = \lambda_{\mathrm{real}}(P_i - \alpha Q_i - \beta R_i,\ S - \alpha R_i)$ for $i = 1, 2$ and arbitrary α, β;

where P_i, Q_i, R_i and S_i, for $i = 1, 2$ are defined in (13) and (14).

Proof. See the Appendix. □

As pointed out by Mustafa (1994c), there are several special cases in which further matrix analysis can simplify the formulae for the critical gains. For example, consider the case where $\gamma = 0$, so that $k(s) = \alpha + \beta/s$ is a PI controller. To simplify the terminology, the critical gains $\tilde{\beta}(\alpha, 0)$ will be called the *critical integral gains*.

Corollary 1. Let $g(s) = c(sI - A)^{-1}b$ be an n-state system and define $\hat{A} := A - \alpha bc$. If α is any fixed real number such that $\hat{A}$ and $\hat{A}\oplus\hat{A}$ are non-singular and $c\hat{A}^{-1}b \neq 0$ then the critical integral gains for the PI controller $k(s) = \alpha + \beta/s$ connected to $g(s)$ are the finite values of

$$\frac{1}{\lambda_{\mathrm{real}}(Y(\alpha))}$$

together with zero, where $Y(\alpha)$ is the $n\times n$ matrix $Y(\alpha) := -(\hat{A}^{-1}\otimes c)\tilde{D}_n(\hat{A}\oplus\hat{A})^{-1}\tilde{D}_n(I_n\otimes b)$.

The proof of Corollary 1 is straightforward (and so is omitted)—one simply applies the same determinantal manipulations that are used in the proof of Proposition 1 in the Appendix (which are all valid under the assumptions of Corollary 1). The conditions of Corollary 1 are easy to check. Indeed, $\hat{A}$ is non-singular provided $\alpha \neq \lambda_{\mathrm{real}}(A, bc)$, and $\hat{A}\oplus\hat{A}$ is non-singular provided $\alpha \neq \lambda_{\mathrm{real}}(A\oplus A, bc\oplus bc)$. If A is non-singular then $\hat{A}$ is invertible provided $\alpha \neq -1/g(0)$. Also, if A is non-singular and $g(0)\neq 0$ then $c\hat{A}^{-1}b \neq 0$ for all α such that $\hat{A}$ is invertible.

Another case where simplifications occur is when α is fixed and β and γ are varied. This case includes controllers of the form $k(s) = \beta/(s+\gamma)$ (where $\alpha = 0$) and $k(s) = (s+\beta)/(s+\gamma)$ (where $\alpha = 1$). Formulae for the critical values of β and γ in this case are given in the following corollary.

Corollary 2. Let $g(s) = c(sI - A)^{-1}b$ be an n-state system and define $\hat{A} := A - \alpha bc$. Let α be any fixed real number such that $\hat{A}$ and $\hat{A}\oplus\hat{A}$ are non-singular and define $\hat{g}_0 := -c\hat{A}^{-1}b$. For the controller $k(s) = (\alpha s + \beta)/(s+\gamma)$ connected to $g(s)$, the critical gains are

- $\tilde{\beta}(\alpha,\gamma) = \lambda_{\mathrm{real}}(\hat{A} - \gamma I_n - \alpha\gamma V(\alpha), -V(\alpha))$, and if $\hat{g}_0 \neq 0$ then also $\tilde{\beta}(\alpha,\gamma) = -\gamma(1 - \alpha\hat{g}_0)/\hat{g}_0$;
- $\tilde{\gamma}(\alpha,\beta) = \lambda_{\mathrm{real}}(\hat{A} + \beta V(\alpha), I_n + \alpha V(\alpha))$, and if $\hat{g}_0 \neq 1/\alpha$ then also $\tilde{\gamma}(\alpha,\beta) = -\beta\hat{g}_0/(1 - \alpha\hat{g}_0)$;

where $V(\alpha)$ is the $n\times n$ matrix $V(\alpha) := (I_n\otimes c)\tilde{D}_n(\hat{A}\oplus\hat{A})^{-1}\tilde{D}_n^{T}(I_n\otimes b)$.

Proof. See the Appendix. □

This section is completed with the following illustrative example.

Example 3. Consider the plant

$$g(s) = \frac{-1.645s^3 - 1.03446s^2 - 0.04075s}{s^4 + 1.0603s^3 - 1.1154s^2 - 0.0565s - 0.0512},$$

taken from Bhattacharyya *et al.* (1988). This plant is not stabilizable by a proportional gain (Bhattacharyya *et al.*, 1988), and furthermore, by using the formulae of Proposition 3 and Corollary 1 to plot the critical gains $\tilde{\alpha}(\beta, 0)$ and $\tilde{\beta}(\alpha, 0)$, it can be shown that it is not stabilizable by a PI controller. Bhattacharyya *et al.* (1988) solved a linear programming problem to show that

$$k_0(s) = \frac{\alpha_0 s + \beta_0}{s + \gamma_0},$$

where $\alpha_0 \approx -0.7982$, $\beta_0 \approx -1.4951$ and $\gamma_0 \approx -0.6366$, is a stabilizing controller. Since stabilization is normally not the only goal of controller design, it is of interest to find the maximal set of stabilizing controllers around $k_0(s)$. To illustrate how this can be done using the methods of this section, we consider controllers of the form $(\alpha_0 s + \beta)/(s + \gamma)$.

Using Corollary 2, the critical gains $\tilde{\gamma}(\alpha_0, \beta)$ were computed for a grid of values of β between -5 and 0, and the critical gains $\tilde{\beta}(\alpha_0, \gamma)$ were computed for a grid of values of γ between -1.5 and 0.5. These critical gains divide the given window of (β, γ) gain-space into regions, as shown in Fig. 2. Since $k_0(s)$ is a stabilizing controller, the shaded region in Fig. 2 is the maximal stabilizing region in the given window of (β, γ) gain-space that contains (β_0, γ_0). Calculation of the closed-loop eigenvalues for a single point in each of the other regions shows that the other regions in Fig. 2 do not contain any stabilizing gain pairs. As can be seen from Fig. 2, there is considerable freedom in the selection of β and γ such that the controller $(\alpha_0 s + \beta)/(s + \gamma)$ stabilizes $g(s)$, which is not evident from the linear programming approach of Bhattacharyya *et al.* (1988). The freedom in β and γ exposed by the techniques of this section is then available to satisfy other performance constraints.

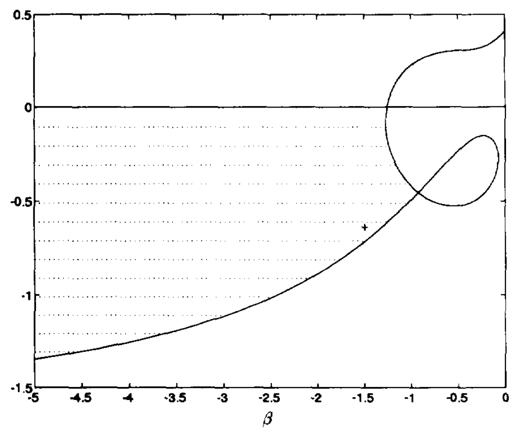

Fig. 2. The maximal stabilizing region (shaded) in the given window of (β, γ) gain-space that contains (β_0, γ_0) (marked with the '+') for Example 3.

5. CONCLUSIONS

The block bialternate sum defined in this paper retains the useful eigenvalue properties of the bialternate sum whilst preserving the block structure of the summand. As such, it is a logical progression from the development of the block Kronecker sum (Hyland and Collins, 1989) and the block Lyapunov sum (Mustafa, 1995). To illustrate some of its applications, the block bialternate sum has been used to find closed eigenvalue formulae for the radius of integral controllability and the maximal stability range of a singularly perturbed system. It has also been used to find formulae for the 'critical gains', from which all stabilizing first-order controllers for a scalar plant can be easily obtained. In each problem the internal dimensions of the derived formulae are lower than those previously available.

In the design of control systems it is often useful to consider a generalized notion of stability. This allows the inclusion of performance constraints such as a minimum damping factor or a maximum oscillation frequency. However, to solve the stability problems considered in this paper with respect to such generalized stability domains, the block Kronecker sum is needed rather than the block bialternate sum, as shown by Mustafa and Davidson (1995).

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APPENDIX—PROOFS

Proof of Fact 1

Fuller (1968) defined the bialternate sum of a matrix A with itself to be the $\frac{1}{2}n(n-1)\times\frac{1}{2}n(n-1)$ matrix G, with rows labelled pq $(p=2,3,\ldots,n; q=1,2,\ldots,p-1)$ and columns labelled rs $(r=2,3,\ldots,n; s=1,2,\ldots,r-1)$, such that the element at the intersection of the row labelled pq and the column labelled rs is

$$g_{pq,rs}:=\det\begin{bmatrix}a_{pr} & a_{ps}\\ \delta_{qr} & \delta_{qs}\end{bmatrix}+\det\begin{bmatrix}\delta_{pr} & \delta_{ps}\\ a_{qr} & a_{qs}\end{bmatrix}, \qquad (A.1)$$

where δ_{ij} is the Kronecker delta.

For a matrix M, let $[M]_{i,j}$ denote the (i,j)th scalar element of M and let $[M]_{\cdot j}$ denote the jth column of M. Careful examination of $A\oplus A$, where $A\in\mathbb{R}^{n\times n}$, reveals that

$$[A\oplus A]_{(i-1)n+j,(k-1)n+l}=a_{ik}\delta_{jl}+a_{jl}\delta_{ik} \qquad (A.2)$$

for $1\le i,j,k,l\le n$.

Given integers x, y $(1\le x,y\le\frac{1}{2}n(n-1))$, the (x,y)th element of $A\,\hat{\oplus}\,A$ is

$$[A\,\hat{\oplus}\,A]_{x,y}=\tfrac{1}{2}([\tilde{D}_n]_{\cdot x})^{\mathrm{T}}(A\oplus A)[\tilde{D}_n]_{\cdot y}. \qquad (A.3)$$

To proceed further, we use the expression for $\tilde{D}_n$ given in Theorem 6.1 of Magnus (1988). Let e_i^n denote the ith column of I_n and let $E_{ij}:=e_i^n(e_j^n)^{\mathrm{T}}$. Define $\tilde{T}_{ij}:=E_{ij}-E_{ji}$. Define $\tilde{m}(n,i,j)$ $(1\le j<i\le n)$ to be the position which the pair (i,j) takes in the sequence

$$(2,1),(3,1),\ldots,(n,1);(3,2),(4,2),\ldots,$$
$$(n,2);\ldots;(n-1,n-2),(n,n-2);(n,n-1), \qquad (A.4)$$

and note that an alternative way to find $\tilde{m}(n,i,j)$ is to use the formula (see Magnus, 1988, p. 93)

$$\tilde{m}(n,i,j)=(j-1)n+i-\tfrac{1}{2}j(j+1).$$

Define $\tilde{u}_{ij}$ to be the $\tilde{m}(n,i,j)$th column of $I_{n(n-1)/2}$. Then

$$\tilde{D}_n=\begin{cases}\displaystyle\sum_{j=1}^{n-1}\sum_{i=j+1}^{n}\mathrm{vec}(\tilde{T}_{ij})\,\tilde{u}_{ij}^{\mathrm{T}} & \text{if } n>1,\\ [\ ]_{1\times 0} & \text{if } n=1.\end{cases} \qquad (A.5)$$

Furthermore, define $\tilde{E}_{i,j}:=e_i^{n^2}(e_j^{n(n-1)/2})^{\mathrm{T}}$. Then

$$\mathrm{vec}(\tilde{T}_{ij})\,\tilde{u}_{ij}^{\mathrm{T}}=\tilde{E}_{(j-1)n+i,\tilde{m}(n,i,j)}-\tilde{E}_{(i-1)n+j,\tilde{m}(n,i,j)}, \qquad (A.6)$$

and hence

$$[\tilde{D}_n]_{\cdot,\tilde{m}(n,i,j)}=e^{n^2}_{(j-i)n+i}-e^{n^2}_{(i-1)n+j}. \qquad (A.7)$$

Now define integer pairs (p,q) and (r,s) such that (p,q) and (r,s) are respectively the xth and yth pairs of integers in the sequence in (A.4). Hence

$$\tilde{m}(n,p,q)=x,\quad \tilde{m}(n,r,s)=y. \qquad (A.8)$$

Substituting (A.7) and (A.8) into (A3) gives

$$\begin{aligned}[A\,\hat{\oplus}\,A]_{x,y}=\tfrac{1}{2}([A\oplus A]_{(q-1)n+p,(s-1)n+r}\\ -[A\oplus A]_{(p-1)n+q,(s-1)n+r}\\ -[A\oplus A]_{(q-1)n+p,(r-1)n+s}\\ +[A\oplus A]_{(p-1)n+q,(r-1)n+s}).\end{aligned}$$

An application of (A.2) then gives

$$\begin{aligned}[A\,\hat{\oplus}\,A]_{x,y}&=a_{pr}\delta_{qs}+a_{qs}\delta_{pr}-a_{ps}\delta_{qr}-a_{qr}\delta_{ps}\\ &=\det\begin{bmatrix}a_{pr} & a_{ps}\\ \delta_{qr} & \delta_{qs}\end{bmatrix}+\det\begin{bmatrix}\delta_{pr} & \delta_{ps}\\ a_{qr} & a_{qs}\end{bmatrix},\end{aligned}$$

and hence the (x,y)th element $(1\le x,y\le\frac{1}{2}n(n-1))$ of $A\,\hat{\oplus}\,A$ is equal to the scalar element in Fuller's definition of the bialternate sum.

Proof of Proposition 1

The proof of Proposition 1 closely follows those of the corresponding results in Mustafa (1994a, b, 1995) and Mustafa and Davidson (1994), but for completeness it is provided in some detail. The closed-loop dynamics matrix for the negative feedback connection of kI_m/s to $G(s)$ is

$$\bar{A} := \begin{bmatrix} A & kB \\ -C & -kD \end{bmatrix}.$$

Define the function

$$v(k) := \det(\bar{A}) \det(\bar{A} \tilde{\oplus} \bar{A}) = \left(\prod_{i=1}^{n+m} \lambda_i\right)\left[\prod_{j=1}^{n+m-1} \prod_{i=j+1}^{n+m} (\lambda_i + \lambda_j)\right],$$

where $\lambda_i (1 \le i \le n+m)$ are the eigenvalues of $\bar{A}$. Observe that if $\bar{A}$ has all its eigenvalues in the closed left half-plane then $v(k) = 0$ if and only if $\bar{A}$ has an eigenvalue on the imaginary axis. Hence $v(k)$ *guards* the open left half plane in the sense of Saydy *et al.* (1990). Since $G(0)$ has all its eigenvalues in the open right half-plane, $G(s)$ is integral controllable (Morari, 1985) and hence, for all sufficiently small positive k, all the eigenvalues of $\bar{A}$ are in the open left half-plane. Thus k^*_{max} is the smallest positive real root of $v(k) = 0$ (or $+\infty$ if there are no positive real roots).

The remainder of the proof involves showing that $v(k) = 0$ can be transformed to an eigenvalue problem. Firstly, since $G(s)$ is stable, A is invertible and, on applying Schur's determinantal formula,

$$\det(\bar{A}) = (-k)^m \det(A) \det[G(0)],$$

which is non-zero for all non-zero k. From Section 3.3,

$$\bar{A} \tilde{\oplus} \bar{A} = \begin{bmatrix} A \tilde{\oplus} A & k\tilde{D}_n^T(I_n \otimes B) & 0 \\ -(I_n \otimes C)\tilde{D}_n & A \oplus (-kD) & k(B \otimes I_m)\tilde{D}_m \\ 0 & -\tilde{D}_m^T(C \otimes I_m) & -k(D \tilde{\oplus} D) \end{bmatrix}.$$

Since $G(s)$ is stable, $A \tilde{\oplus} A$ is invertible, and applying Schur's determinantal formula gives

$$\det(\bar{A} \tilde{\oplus} \bar{A}) = \det(\bar{A} \tilde{\oplus} A) \det \begin{bmatrix} A \otimes I_m + kM & k(B \otimes I_M)\tilde{D}_m \\ -\tilde{D}_n(C \otimes I_m) & -k(D \tilde{\oplus} D) \end{bmatrix}, \quad (A.9)$$

where $M = -(I_n \otimes D) + (I_n \otimes C)\tilde{D}_n(A \tilde{\oplus} A)^{-1}\tilde{D}_n^T(I_n \otimes B)$.

We now give a Schur-type formula to simplify the second determinant on the right-hand side of (A.9). The proof involves some straightforward matrix and determinantal manipulations, and is given in Mustafa and Davidson (1994).

Lemma 2. Let $U, V \in \mathbb{R}^{p \times p}$, $W \in \mathbb{R}^{p \times q}$, $X \in \mathbb{R}^{q \times p}$ and $Y \in \mathbb{R}^{q \times q}$. If U and $Y - XU^{-1}W$ are non-singular then

$$\det \begin{bmatrix} U + V & W \\ X & Y \end{bmatrix} = \det(U) \det(Y - XU^{-1}W) \times \det\{I_q + [U^{-1} + U^{-1}W \times (Y - XU^{-1}W)^{-1}XU^{-1}]V\}.$$

We simplify the second determinant on the right hand side of (A.9) by choosing (in the notation of Lemma 2) $U = A \otimes I_m$, which is invertible because A is stable by assumption. This choice of U gives (again in the notation of Lemma 2) $Y - XU^{-1}W = -k[G(0) \tilde{\oplus} G(0)]$, which is non-singular for $k \neq 0$. Thus, for $k \neq 0$, applying Lemma 2 gives

$$v(k) = (-k)^{m(m+1)/2} u \det(I_{nm} - kNM),$$

where

$$u = \det(A^{m+1}) \det(A \tilde{\oplus} A) \det[G(0)] \det[G(0) \tilde{\oplus} G(0)],$$

$$N = -(A^{-1} \otimes I_m) - (A^{-1}B \otimes I_m)\tilde{D}_m[G(0) \tilde{\oplus} G(0)]^{-1}\tilde{D}_m^T(CA^{-1} \otimes I_m).$$

Since u is non-zero and independent of k, the smallest positive real solution to $v(k) = 0$ is the smallest positive real solution to $\det(I_{nm} - kNM)$, that is, the reciprocal of the largest positive real eigenvalue of NM (or $+\infty$ if there are no positive real eigenvalues). The proposition follows by multiplying out NM, collecting terms and simplifying using standard properties of Kronecker algebra and Lemma 1 to show that $NM = Z$. This simplification is straightforward (and so is omitted)—the key step is to note that

$$\tilde{D}_m^T(CA^{-1} \otimes C)\tilde{D}_n(A \tilde{\oplus} A)^{-1}\tilde{D}_n^T(I_n \otimes B) = \tilde{D}_m^T(CA^{-1} \otimes CA^{-1}B).$$

Proof of results of Section 4.3

The proof of the results of Section 4.3 use similar techniques to the corresponding proofs in Mustafa (1994c). However, for completeness, they are provided in some detail.

Proof of Proposition 3. The closed-loop dynamics matrix is

$$\bar{A} = \begin{bmatrix} A - \alpha bc & (\beta - \alpha\gamma)b \\ -c & -\gamma \end{bmatrix}. \quad (A.10)$$

Consider the function

$$f(\alpha, \beta, \gamma) := \det(\bar{A}) \det(\bar{A} \tilde{\oplus} \bar{A}) \quad (A.11)$$

$$= \left(\prod_{i=1}^{n+m} \lambda_i\right)\left[\prod_{j=1}^{n+m-1} \prod_{i=j+1}^{n+m} (\lambda_i + \lambda_j)\right], \quad (A.12)$$

where $\lambda_i (1 \le i \le n+m)$ are the eigenvalues of $\bar{A}$. It is clear from (A.12) that the critical gains are the (finite) real solutions (if any) to $f(\alpha, \beta, \gamma) = 0$. Using the structure of $\bar{A}$ and the definitions in (13) and (14), $f(\alpha, \beta, \gamma)$ can be written as

$$f(\alpha, \beta, \gamma) = \prod_{i=1}^{2} \det(P_i - \alpha Q_i - \beta R_i - \gamma S_i + \alpha\gamma R_i) \quad (A.13)$$

The formulae for the critical gains follow by rearranging the terms within the determinants in (A.13) as is illustrated below for $\bar{\alpha}(\beta, \gamma)$.

The critical gains $\bar{\alpha}(\beta, \gamma)$ satisfy $f(\alpha, \beta, \gamma) = 0$ for given β and γ. Equivalently, they are the real values of α that satisfy

$$\det[(P_1 - \beta R_1 - \gamma S_1) - \alpha(Q_1 - \gamma R_1)] = 0, \quad (A.14)$$

or

$$\det[(P_2 - \beta R_2 - \gamma S_2) - \alpha(Q_2 - \gamma R_2)] = 0. \quad (A.15)$$

The real solutions to (A.14) and (A.15) are simply the real generalized eigenvalues given in the proposition.

Proof of Corollary 2. Using $\bar{A}$ in (A.10) and the assumptions and definitions of Corollary 2, an application of Schur's determinantal formula gives

$$\det(\bar{A} \tilde{\oplus} \bar{A}) = \det(\hat{A} \tilde{\oplus} \hat{A}) \det[\hat{A} + \beta V(\alpha) - \gamma I_n - \alpha\gamma V(\alpha)], \quad (A.16)$$

$$\det(\bar{A}) = (\alpha\gamma\hat{g}_0 - \beta\hat{g}_0 - \gamma) \det(\hat{A}). \quad (A.17)$$

Recall that the critical gains satisfy $f(\alpha, \beta, \gamma) = 0$, where $f(\alpha, \beta, \gamma)$ is defined in (A.11). Using the simplifications in (A.16) and (A.17) and the assumptions of the corollary, the critical gains satisfy

$$\det[\hat{A} + \beta V(\alpha) - \gamma I_n - \alpha\gamma V(\alpha)] = 0, \quad (A.18)$$

or

$$\alpha\gamma\hat{g}_0 - \beta\hat{g}_0 - \gamma = 0. \quad (A.19)$$

The formulae for the critical values of β in the corollary follow by fixing γ and writing (A.18) as a generalized eigenvalue problem for β and solving (A.19) for β. Similar manipulations give the formulae for the critical values of γ.