Minimum-Bandwidth Optical Intensity Nyquist Pulses

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***Abstract*—The indoor diffuse wireless optical intensity channel is bandwidth-limited due to multipath distortion, and all transmitted signal amplitudes are constrained to be nonnegative. In order to control the impact of intersymbol interference (ISI) on this channel, pulse shaping is required. This paper derives the minimum bandwidth, ISI-free Nyquist pulse which satisfies the amplitude nonnegativity constraint. The minimum bandwidth required is twice that of conventional electrical channels. With the addition of excess bandwidth, the optimal bandlimited optical intensity pulse, in the sense of minimizing average optical power, is shown to be a squared double-jump pulse. Thus, a bandwidth versus optical power efficiency tradeoff in pulse design is quantified. The impact of timing jitter on the probability of symbol error for various excess bandwidths is quantified via simulation. Further, it is shown that there are no bandlimited root-Nyquist pulses satisfying the amplitude nonnegativity constraint. In fact, all practical optical intensity root-Nyquist pulses are shown to be necessarily time-limited to a single symbol interval.**

***Index Terms*—Indoor diffuse infrared communication, Nyquist pulses, optical intensity modulation, wireless infrared channel.**

I. Introduction

UNLIKE MANY fiber optical channels, diffuse indoor wireless optical intensity channels are *bandwidth constrained*. These links, pioneered by Gfeller and Bapst [1], provide a wireless channel at infrared wavelengths by transmitting a modulated optical intensity signal over a wide solid angle. Since the physical quantity being modulated is the instantaneous optical intensity, all transmitted signals must be *nonnegative*. The transmitted optical radiation reflects off surfaces in the room until it is detected by a photodetector receiver which produces an output signal proportional to the received optical intensity. *Multipath distortion* arises due to the multiple reflected paths between transmitter and receiver and causes severe intersymbol interference (ISI) in high-speed links [2]. This paper considers the design of minimum-bandwidth ISI-free Nyquist and root-Nyquist pulses for optical intensity channels.

There exists a wealth of literature on signaling in electrical ISI channels [3]–[5], however, these results cannot be applied directly to optical intensity channels due to the amplitude nonnegativity constraint on all transmitted signals. Receiver design for fiber optical intensity pulse-amplitude modulated (PAM) systems has been of interest for some time. Personick [6] was among the first to consider receiver design for fiber optical channels. However, only rectangular, Gaussian, and exponential pulse shapes were considered with equalized raised-cosine pulse shapes. Subsequent generalizations of this work to different receiver structures also considered a limited set of transmit pulse shapes [7], [8]. In electrical channels, Halpern [9] investigated optimum finite-duration Nyquist pulses, but not root-Nyquist pulses subject to amplitude constraints.

For indoor wireless optical channels, previous studies on spectrally efficient signaling centered on the use of rectangular pulse sets [2], [10]–[12]. Wider classes of pulse sets have also been considered, however, all remain time-limited [13], [14]. Coding and equalization have been considered for rectangular on–off keying and pulse-position modulation to mitigate the effects of ISI on these channels [15], [16]. However, the problem of pulse design to minimize ISI in diffuse indoor wireless optical channels has not been extensively studied.

In this paper, we consider the general question of minimum-bandwidth ISI-free optical intensity pulses which are suited to bandwidth-constrained PAM signaling over diffuse indoor wireless optical channels. Section II presents an overview of the channel model and the amplitude constraints imposed by optical intensity channels. The minimum-bandwidth optical intensity Nyquist pulse is derived in Section III, and it is shown that the nonnegativity constraint doubles the minimum bandwidth required. In Section IV, we consider optical intensity Nyquist pulses with excess bandwidth, and find those which minimize the optical power. The tradeoff between excess bandwidth and optical power efficiency is thus quantified. The impact of timing jitter on the probability of symbol error is quantified. Root-Nyquist optical intensity pulses are considered in Section V and are shown to be necessarily time-limited. Conclusions and directions for future work are presented in Section VI.

II. Channel Model

Diffuse wireless optical links are attractive since they are inexpensive, do not require strict alignment, and are free of spectral licensing issues [2]; however, signaling design for such channels differs significantly from conventional electrical channels. The transmitter modulates only the instantaneous optical intensity of the optical carrier and, as a result, all transmitted waveforms must be nonnegative. Additionally, the average amplitude of all transmitted signals is proportional to the average optical power. To ensure eye and skin safety, the average amplitude of all transmissions is limited, unlike electrical channels, where the mean squared amplitude value is typically constrained. The transmitted optical intensity signal is emitted over

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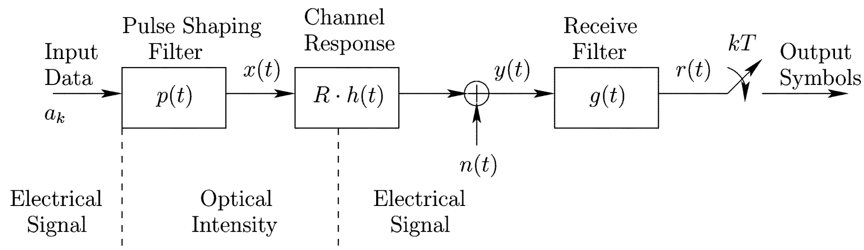

Fig. 1 Channel model of an optical intensity PAM system.

a wide solid angle and allowed to reflect diffusely from surfaces in a room.

The receiver is typically implemented as a photodiode which outputs a current in proportion to the optical power impinging on the device. Multipath distortion leads to temporal dispersion of the transmitted pulses, and can be modeled with a linear lowpass frequency response [2], [17], [18]. Unlike radio channels, indoor wireless optical links do not exhibit multipath fading, since the receiver photodiode integrates the optical intensity field over an area of millions of square wavelengths, providing an inherent degree of spatial diversity [2], [17]. The frequency response of diffuse wireless optical links is further reduced due to the use of inexpensive, large-area photodiodes which have large reverse bias depletion capacitances. Received signals are also corrupted by high-intensity shot noise due to ambient illumination. This noise is conventionally modeled as being white, Gaussian distributed, and signal independent [2].

As shown in Fig. 1, an accurate model from transmitted optical intensity $x(t)$ to received photocurrent $y(t)$ for indoor diffuse channels is [2]

$$y(t) = Rx(t) \otimes h(t) + n(t) \tag{1}$$

where R is the detector sensitivity in A/W, $\otimes$ is the convolution operator, $h(t)$ models the combined impact of multipath distortion and photodiode capacitance, and $n(t)$ is a white Gaussian noise process. Without loss of generality, let $R = 1$ for the balance of this paper. Since $x(t)$ is an optical intensity signal, $x(t) \geq 0$ and the average amplitude of $x(t)$ must be bounded. Thus, the indoor wireless optical channel can be modeled as an electrical baseband channel with nonnegativity and average amplitude constraints on $x(t)$.

Consider the wireless optical intensity PAM communications system depicted in Fig. 1. Let $\mathcal{L}^2$ denote the space of real-valued functions defined over the real line with finite energy. For pulse shape $p(t) \in \mathcal{L}^2$, the transmitted optical intensity signal can be written as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k p(t - kT)$$

where $a_k \in \mathbb{R}, a_k \geq 0$ is the PAM data sequence and T is the symbol period. Unlike electrical channels, $p(t)$ is an optical intensity signal and must satisfy an amplitude nonnegativity constraint, namely

$$\forall t \in \mathbb{R} \quad p(t) \geq 0. \tag{2}$$

Due to the lowpass characteristic of the multipath channel, in this paper we consider pulse shapes which are strictly bandlimited to bandwidth B, such that

$$P(\omega) = 0, \quad |\omega| \geq B \tag{3}$$

where $P(\omega)$ is the Fourier transform of $p(t)$. Additionally, it is assumed that the channel response $h(t)$ is flat in the band of interest, $|\omega| < B$. Thus, the received photocurrent signal is

$$y(t) = \sum_{k=-\infty}^{\infty} a_k p(t - kT) + n(t).$$

The average amplitude, i.e., average optical power, must also be constrained in order to ensure eye-safety regulations are met. The average optical power of the PAM system is

$$P = E\{a\}\bar{p}$$

where

$$\bar{p} = \frac{1}{T} \int_{-\infty}^{\infty} p(t)\, dt \tag{4}$$

and $E\{\cdot\}$ denotes expectation. Notice that $E\{a\}$ depends on the coding scheme and $\bar{p}$ depends on the selection of the pulse. Thus, the selection of pulses which satisfy the amplitude constraints and minimize $\bar{p}$ is an important factor determining system performance.

As shown in Fig. 1, the received photocurrent signal is filtered by a receive filter with impulse response $g(t)$ and sampled at the symbol rate. In the remainder of the paper, we consider two scenarios for the receive filter: 1) where the filter is flat and bandlimited to the channel bandwidth; and 2) where $g(t)$ is a matched filter. In both cases, optical intensity pulse shapes $p(t)$ which minimize the impact of ISI are explored.

III. Minimum-Bandwidth Optical Intensity Nyquist Pulse

Consider the case when the receive filter $g(t)$ is flat in the band of interest, i.e., $G(\omega) = 1$ for $|\omega| \leq B$, and 0, otherwise. In this case, the receive filter merely filters out-of-band noise. In order to avoid ISI at the output of the sampler, $p(t)$ must satisfy the Nyquist criterion [3]

$$p(kT) = \delta_{k0} \tag{5}$$

where δ_{kl} is the Kronecker delta

$$\delta_{kl} = \begin{cases} 1, & \text{if } k = l \\ 0, & \text{otherwise} \end{cases}$$

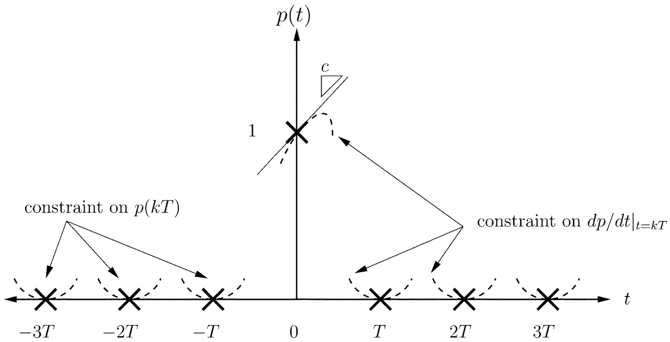

Fig. 2. Time-domain representation of the constraints on $p(kT)$ and $dp/dt|_{t=kT}$.

and $k \in \mathbb{Z}$. We define an *optical intensity Nyquist pulse* as any finite-energy pulse simultaneously satisfying the nonnegativity (2) and Nyquist (5) constraints. The goal of this section is to find the bandlimited optical intensity Nyquist pulse $p(t)$ which minimizes B while satisfying constraints (2), (3), and (5).

Observe that in order to simultaneously satisfy both the amplitude nonnegativity (2) and Nyquist (5) constraints, $p(t)$ must attain its minimum value of zero at all nonzero integer sampling instants (i.e., at $t = kT, k \neq 0$). As a result, the derivative of $p(t), dp/dt$, if it exists, must also be zero at all nonzero sampling instants. The existence of the derivative of $p(t)$ can be proved using a well-known result from complex analysis. Consider extending $p(t)$ over the complex plane to yield $p(z), z \in \mathbb{C}$. By the Paley–Wiener Theorem [19, Th. 19.3] [20, Th. 7.23], if $P(\omega)$ is bandlimited, then $p(z)$ is analytic over the complex plane. As a result, dp/dt must exist for all $\mathbb{R}$. At $t = 0, p(0)$ is fixed to be unity by (5); however, the value of the first derivative is not specified explicitly. Thus

$$\left.\frac{dp}{dt}\right|_{t=kT} = c \cdot \delta_{k0} \quad (6)$$

for some $c \in \mathbb{R}$. For the bandlimited optical intensity Nyquist pulse $p(t), dp/dt$ is also a scaled Nyquist pulse. Fig. 2 illustrates the constraints on $p(kT)$ and $dp/dt|_{t=kT}$ in the time domain.

It is useful to cast the constraints on the amplitude of the time-domain signal in the frequency domain. The Nyquist constraint can be cast in the frequency domain by taking the Fourier transform of (5), resulting in the well-known expression

$$\sum_{m=-\infty}^{+\infty} P\left(\omega - m\frac{2\pi}{T}\right) = T. \quad (7)$$

Similarly, the constraint on dp/dt in (6) can be represented in the frequency domain as

$$\sum_{m=-\infty}^{+\infty} \jmath\left(\omega - m\frac{2\pi}{T}\right) P\left(\omega - m\frac{2\pi}{T}\right) = cT. \quad (8)$$

The problem now becomes one of determining the $P(\omega)$ of minimum bandwidth which satisfies both (7) and (8).

Recall that the minimum bandwidth Nyquist pulse occupies a bandwidth of π/T rad/s, but does not satisfy the nonnegativity constraint. As a result, $B > \pi/T$. Without loss in generality, assume that the $B \in (\pi/T, 2\pi/T]$. We will now show that (7) and (8) are satisfied for only one value $B \in (\pi/T, 2\pi/T]$, which is the minimum bandwidth of any optical intensity Nyquist pulse.

Both (7) and (8) must be satisfied over every interval of ω. Due to the bounds on B, in the interval $\omega \in (-2\pi/T, 2\pi/T]$, (8) reduces to

$$\left(\omega + \frac{2\pi}{T}\right) P\left(\omega + \frac{2\pi}{T}\right) + \omega P(\omega) + \left(\omega - \frac{2\pi}{T}\right) P\left(\omega - \frac{2\pi}{T}\right) = -\jmath cT.$$

Expanding and collecting like terms gives

$$\omega \sum_{n=-1}^{1} P\left(\omega - n\frac{2\pi}{T}\right) + \frac{2\pi}{T}\left(P\left(\omega + \frac{2\pi}{T}\right) - P\left(\omega - \frac{2\pi}{T}\right)\right) = -\jmath cT. \quad (9)$$

Since $B \in (\pi/T, 2\pi/T]$, the first term can be simplified over the interval $\omega \in (-2\pi/T, 2\pi/T]$ using (7) to give

$$\omega T + \frac{2\pi}{T}\left(P\left(\omega + \frac{2\pi}{T}\right) - P\left(\omega - \frac{2\pi}{T}\right)\right) = -\jmath cT.$$

The resulting sum of two shifted versions of $P(\omega)$ can be written over two ranges of ω as

$$P\left(\omega + \frac{2\pi}{T}\right) = -\frac{T^2}{2\pi}(\omega + \jmath c), \quad \omega \in (-2\pi/T, 0] \quad (10)$$

$$P\left(\omega - \frac{2\pi}{T}\right) = \frac{T^2}{2\pi}(\omega + \jmath c), \quad \omega \in (0, 2\pi/T]. \quad (11)$$

The above expressions were derived under the assumption that $B \in (\pi/T, 2\pi/T]$. Notice that for $B < 2\pi/T$, (10) cannot be satisfied, since the left-hand side goes to zero over the interval $(-2\pi/T + B, 0)$, while the right-hand side does not. Similarly, for $B < 2\pi/T$, the left-hand side of (11) goes to zero over $(0, 2\pi/T - B]$, while the right-hand side does not. Therefore, $B \geq 2\pi/T$.

Expressions (10) and (11) can be combined to define $P(\omega)$ as

$$P(\omega) = \begin{cases} T^2/(2\pi)(\omega + 2\pi/T) + jcT^2/(2\pi), & \frac{-2\pi}{T} < \omega \leq 0 \\ -T^2/(2\pi)(\omega - 2\pi/T) - jcT^2/(2\pi), l & 0 < \omega \leq \frac{2\pi}{T} \\ 0, & \text{otherwise.} \end{cases}$$

Computing the inverse transform of the real and imaginary components separately and simplifying the sum yields

$$p(t) = (1 + ct) \cdot \frac{\sin^2(\pi t/T)}{(\pi t/T)^2}. \quad (12)$$

The value of c can be constrained by noting that the Nyquist constraints on $p(t)$ (5) and on dp/dt (6) are necessary but not sufficient to guarantee that the nonnegativity constraint in (2) holds. That is, the set of optical intensity Nyquist pulses is a subset of the set of functions described in (12). Clearly, $c = 0$ in order for the pulse defined in (12) to satisfy the nonnegative amplitude constraint. Thus, there is a unique minimum-bandwidth optical intensity Nyquist pulse.

The minimum-bandwidth optical intensity Nyquist pulse occupies a bandwidth of $2\pi/T$ rad/s and takes the form

$$p_B^{\min}(t) = \operatorname{sinc}^2(\pi t/T).$$

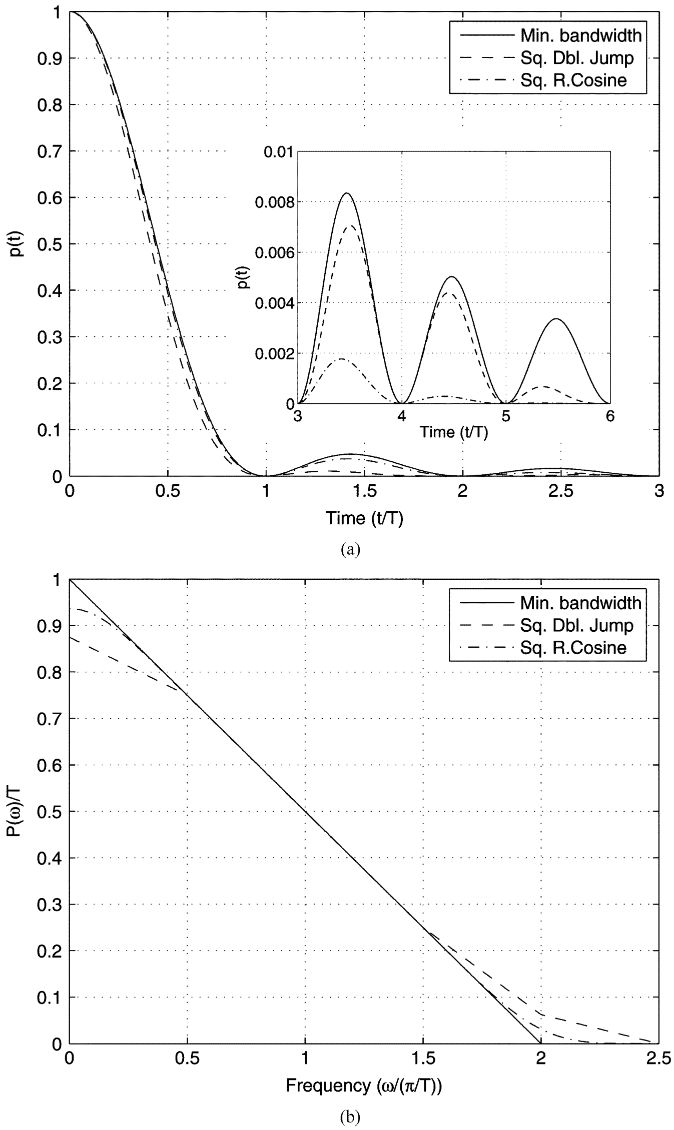

Fig. 3. Time (a) and frequency (b) domain plots of the minimum-bandwidth optical intensity Nyquist pulse, squared raised-cosine, and optimal squared double-jump pulse for $\alpha = 0.25$. Note that all time and frequency pulses are real, even functions.

Fig. 3 presents a plot of the minimum-bandwidth optical intensity Nyquist pulse. Notice that it satisfies the Nyquist criterion as well as the nonnegativity constraint. Additionally, the pulse decays as $1/|t|^2$, lessening the impact of timing errors as compared with the minimum-bandwidth solution for electrical channels. However, the minimum-bandwidth optical intensity Nyquist pulse requires twice the bandwidth over the electrical case. Notice also that the average amplitude of $p(t)$, as defined in (4), is $\bar{p}_B^{\min} = 1$.

IV. Excess Bandwidth Optical Intensity Nyquist Pulses

In the practical setting where timing jitter is present in the symbol rate sampler, the design of better pulses, in the sense of opening the eye of the received signal, may be possible. In conventional electrical channels, the minimum-bandwidth Nyquist pulse is impractical. Pulses with excess bandwidth, such the family of raised-cosine pulses [21], have widespread use due to their faster rolloff time and improved immunity to timing jitter.

In this section, we generalize the results of the previous section to optical intensity Nyquist pulses with α excess bandwidth, that is, strictly bandlimited to $(1+\alpha)2\pi/T$ rad/s for $\alpha \in [0,1]$. In all cases, it is assumed that the receive filter $g(t)$ is flat in the band of interest. We demonstrate that the addition of excess bandwidth can be used to reduce the average optical power of a PAM system. For a given α, the optimal optical intensity Nyquist pulse, in the sense of minimizing the average optical power, is then derived.

A. Generalized Nyquist Pulses

We define a *generalized Nyquist pulse* as a complex-valued function $q(t)$, satisfying

$$q(kT) = e^{j\phi}\delta_{k0}$$

for some $\phi \in \mathbb{R}$ and for all $k \in \mathbb{Z}$. Let $\mathcal{Q}_\alpha$ denote the set of all generalized Nyquist pulses with α excess bandwidth, i.e., of bandwidth $(1+\alpha)\pi/T$ rad/s.

The Fourier transform of a generalized Nyquist pulse $Q(\omega)$ must satisfy a modified form of (7)

$$\sum_{m=-\infty}^{+\infty} Q\left(\omega - m\frac{2\pi}{T}\right) = Te^{j\phi}.$$

Unlike real Nyquist pulses, generalized Nyquist pulses $q(t) \in \mathcal{Q}_\alpha$ need not possess odd symmetry about the π/T frequency. Let $\mathbb{C}(\omega)$ be the set of all complex-valued functions in $\omega \in \mathbb{R}$ and define the set

$$\mathcal{U}_\alpha \triangleq \{U(\omega) \in \mathbb{C}(\omega) : U(0) = 1, U(\omega) = 0 \text{ for } \omega \notin [0, \alpha\pi/T]\}.$$

Extending the expression for real Nyquist pulses [3], [5] yields a general form for the Fourier transform of every $q(t) \in \mathcal{Q}_\alpha$ as shown in (13) at the bottom of the page, where $B = \pi/T$

$$Q(\omega) = \begin{cases} 0, & \omega < -(1+\alpha)B \\ Te^{j\phi}(1 - U((1+\alpha)B + \omega)), & -(1+\alpha)B \le \omega < -B \\ Te^{j\phi}V(-\omega - (1-\alpha)B), & -B \le \omega < -(1-\alpha)B \\ Te^{j\phi}, & -(1-\alpha)B \le \omega < (1-\alpha)B \\ Te^{j\phi}U(\omega - (1-\alpha)B), & (1-\alpha)B \le \omega < B \\ Te^{j\phi}(1 - V((1+\alpha)B - \omega)), & B \le \omega < (1+\alpha)B \\ 0, & \omega \ge (1+\alpha)B \end{cases} \tag{13}$$

and $U(\omega), V(\omega) \in \mathcal{U}_\alpha$. The following theorem relates the set of bandlimited optical intensity Nyquist pulses to the set of bandlimited generalized Nyquist pulses.

Theorem 1: A real-valued pulse $p(t) \in \mathcal{L}^2$ which is bandlimited to B rad/s is an optical intensity Nyquist pulse if and only if there exists a complex-valued generalized Nyquist pulse $q(t)$ of bandwidth $B/2$ rad/s such that $p(t) = |q(t)|^2$.

Proof: (Necessity): Let $\mathcal{F}\{\cdot\}$ denote the Fourier transform operator. Suppose we have a complex-valued function $q(t)$ which is a generalized Nyquist pulse and is bandlimited to $B/2$ rad/s. Clearly, the pulse $|q(t)|^2$ is real and nonnegative and at each sample instant, $|q(kT)|^2 = \delta_{k0}$, i.e., it is a Nyquist pulse. The Fourier transform $\mathcal{F}\{|q(t)|^2\}$ is the autocorrelation of $\mathcal{F}\{q(t)\}$ and thus, the bandwidth of $|q(t)|^2$ must be twice that of $q(t)$, namely, B rad/s. Therefore, $|q(t)|^2$ is an optical intensity Nyquist pulse of bandwidth B rad/s.

(Sufficiency): Consider a $p(t)$ which is an optical intensity Nyquist pulse of bandwidth B rad/s. Notice that $\sqrt{p(t)}$ is also a real-valued nonnegative function of time. Define the complex-valued time function

$$q(t) = \sqrt{p(t)}e^{j\theta(t)} \tag{14}$$

where $\theta(t)$ is a real function of time. Clearly, $|q(t)|^2 = p(t)$ and $q(t)$ is a generalized Nyquist pulse. Notice that the definition of the phase function $\theta(t)$ does not impact the squared modulus of $q(t)$. The problem of characterizing the set of complex-valued functions which have a given square modulus is well-known in optics, and is termed the *phase retrieval problem* [22], [23]. This problem considers the reconstruction of complex optical wavefronts given only their intensity, i.e., their squared modulus. In the case of bandlimited, nonnegative $p(t) \in \mathcal{L}^2$ as in (14), it can be shown that there exists a phase function $\theta(t)$ such that $q(t)$ is bandlimited to $B/2$ rad/s [23, Th. III], [22, Th. 2]. Thus, there exists a complex-valued $q(t)$ bandlimited to $B/2$ rad/s such that $q(kT) = e^{j\theta(0)}\delta_{k0}$ and $|q(t)|^2 = p(t)$. □

Theorem 1 implies that any bandlimited optical intensity Nyquist pulse is the square modulus of a generalized Nyquist pulse. Therefore, every optical intensity Nyquist pulse with α excess bandwidth, i.e., with bandwidth $(1+\alpha)2\pi/T$, is the square modulus of some $q(t) \in \mathcal{Q}_\alpha$. For example, consider the *squared raised cosine* pulse with α excess bandwidth, which can be constructed as

$$p_\alpha^{\mathrm{rc}}(t) = \mathrm{sinc}^2(\pi t/T)\left(\frac{\cos(\alpha t/T)}{1-(2\alpha t/T)^2}\right)^2. \tag{15}$$

Notice that this pulse occupies a bandwidth of $(1+\alpha)2\pi/T$ rad/s and for $\alpha > 0$ decays as $|t|^{-6}$. Fig. 3 presents a plot of the pulse in the time and frequency domains for $\alpha = 0.25$.

B. Optimal Optical Intensity Nyquist Pulses

Consider an optical intensity PAM system with pulse shape $p(t)$ with α excess bandwidth constructed as $p(t) = |q(t)|^2$, $q(t) \in \mathcal{Q}_\alpha$. Using *Theorem 1*, it is possible to construct any number of optical intensity pulses with α excess bandwidth satisfying Nyquist's constraints. However, the average optical power depends on the average amplitude of $p(t)$, $\bar{p}$ defined in (4). For a given excess bandwidth α, we define an *optimal optical intensity Nyquist pulse* as one which minimizes $\bar{p}$ while satisfying all amplitude constraints.

Let $p_\alpha^{\mathrm{opt}}(t) = |q_\alpha^{\mathrm{opt}}|^2$ denote an optimal optical intensity Nyquist pulse with α excess bandwidth with minimal average amplitude

$$\bar{p}_\alpha^{\mathrm{opt}} = \frac{1}{T}\int_{-\infty}^{\infty} p_\alpha^{\mathrm{opt}}(t)\,dt.$$

If $P_\alpha^{\mathrm{opt}}(\omega) = \mathcal{F}\{p_\alpha^{\mathrm{opt}}(t)\}$ and $Q_\alpha^{\mathrm{opt}}(\omega) = \mathcal{F}\{q_\alpha^{\mathrm{opt}}(t)\}$, notice that $\bar{p}_\alpha^{\mathrm{opt}} = P_\alpha^{\mathrm{opt}}(0)/T$, and that $P_\alpha^{\mathrm{opt}}(\omega)$ is the scaled autocorrelation of $Q_\alpha^{\mathrm{opt}}(\omega)$. Using these two facts yields

$$\bar{p}_\alpha^{\mathrm{opt}} = \min_{q(t)\in\mathcal{Q}_\alpha} \frac{1}{T}\frac{1}{2\pi}\int_{-(1+\alpha)\pi/T}^{(1+\alpha)\pi/T} |Q(\omega)|^2\,d\omega.$$

Substituting the general expression for $Q(\omega)$ in (13) and performing a suitable change of variables gives

$$\bar{p}_\alpha^{\mathrm{opt}} = (1-\alpha) + \frac{T}{2\pi}\min_{U,V\in\mathcal{U}_\alpha}\int_0^{\alpha\pi/T} |U(\omega)|^2 + |1-U(\omega)|^2 + |V(\omega)|^2 + |1-V(\omega)|^2\,d\omega.$$

The minimization over U and V can be done independently, and for some $W \in \mathcal{U}_\alpha$

$$\bar{p}_\alpha^{\mathrm{opt}} = (1-\alpha) + \frac{T}{\pi}\min_{W\in\mathcal{U}_\alpha}\int_0^{\alpha\pi/T} |W(\omega)|^2 + |1-W(\omega)|^2\,d\omega$$

which simplifies to

$$\bar{p}_\alpha^{\mathrm{opt}} = 1 + \frac{2T}{\pi}\min_{W\in\mathcal{U}_\alpha}\int_0^{\alpha\pi/T} |W(\omega)|^2 - \mathrm{Re}\{W(\omega)\}\,d\omega. \tag{16}$$

Since the amplitude of $W(\omega)$ can be arbitrarily specified at each ω, the integral is minimized when the integrand is minimized pointwise. It is clear that for a given ω, the integrand is minimized when $W(\omega)$ is real. Denote the $W(\omega)$ which minimizes (16) as $W^{\mathrm{opt}}(\omega)$. The minimization at each ω is trivial, and $W^{\mathrm{opt}}(\omega) = 1/2$ for $\omega \in (0, \alpha\pi/T]$ and $W^{\mathrm{opt}}(0) = 1$, since $W \in \mathcal{U}_\alpha$.

Substituting $W^{\mathrm{opt}}(\omega)$ into (16) gives the minimum average optical power for any optical intensity Nyquist pulse with α excess bandwidth as

$$\bar{p}_\alpha^{\mathrm{opt}} = 1 - \frac{\alpha}{2}. \tag{17}$$

Furthermore, substituting $W^{\mathrm{opt}}(\omega)$ into (13) demonstrates that the minimum average optical power cost occurs when

$$Q_\alpha^{\mathrm{opt}}(\omega) = \begin{cases} Te^{j\phi}, & |\omega| < (1-\alpha)\pi/T \\ Te^{j\phi}/2, & (1-\alpha)\pi/T \le |\omega| < (1+\alpha)\pi/T \\ 0, & |\omega| \ge (1+\alpha)\pi/T. \end{cases}$$

This $q_\alpha^{\rm opt}(t)$ is a generalization of Frank's "double-jump" pulse to complex-valued functions [4, Case B]. This pulse was derived for electrical up-converted PAM channels, and was demonstrated to be optimal in the sense of minimizing the mean-square error across carrier-phase variations.

Therefore, the optimal optical intensity Nyquist pulse with excess bandwidth α incurring the minimum average optical power cost is

$$\begin{aligned} p_\alpha^{\rm opt}(t) &= \left|q_\alpha^{\rm opt}(t)\right|^2 \\ &= \left(\frac{(1-\alpha)}{2}{\rm sinc}\left((1-\alpha)\frac{\pi t}{T}\right)\right. \\ &\quad \left. +\frac{(1+\alpha)}{2}{\rm sinc}\left((1+\alpha)\frac{\pi t}{T}\right)\right)^2 \end{aligned}$$

i.e., it is a *squared double-jump* pulse. Define the triangle function as

$$\triangle(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & \text{otherwise.} \end{cases}$$

The frequency-domain expression for $p_\alpha^{\rm opt}(t)$ is

$$\begin{aligned} P_\alpha^{\rm opt}(\omega) &= \frac{(1-\alpha)T}{4}\triangle\left(\frac{\omega}{2B(1-\alpha)}\right) \\ &\quad + \frac{(1+\alpha)T}{4}\triangle\left(\frac{\omega}{2B(1+\alpha)}\right) \\ &\quad + \frac{T}{2}\triangle\left(\frac{\omega}{2B}\right) - \frac{\alpha T}{2}\triangle\left(\frac{\omega}{2B\alpha}\right). \end{aligned}$$

Notice that this pulse decays as $|t|^{-2}$, just as the minimum-bandwidth pulse. Fig. 3 presents a plot of the optimal pulse in the time and frequency domains for $\alpha = 0.25$.

For the squared raised-cosine pulse (15), the average amplitude of the pulse can be computed in a similar fashion to give $\bar{p}_\alpha^{\rm rc} = 1 - \alpha/4$. Recall that the average amplitude, i.e., optical power, of the minimum-bandwidth optical Nyquist pulse is $\bar{p}_B^{\min} = 1$. Both squared raised-cosine and the optimal squared double-jump pulses both realize an optical power gain at the expense of bandwidth efficiency. Therefore, through the efficient design of pulses, the addition of excess bandwidth can be traded off for savings in average optical power. In the following section, we will quantify these gains and their impact on the probability of error performance of optical PAM systems.

C. Numerical Results and Comparison

In the previous section, it was demonstrated that the addition of excess bandwidth can be used to achieve a gain in average optical power over the minimum bandwidth case. Fig. 4 plots the optical power gain $\bar{p}_B^{\min}/\bar{p}_\alpha$ as a function of α for the optimal squared double-jump, using (17), and squared raised-cosine pulses. The maximum gain available occurs when $\alpha = 1$, using (17), the optimal pulse provides an optical power gain of 3 dB over the minimum bandwidth optical Nyquist pulse, while the squared raised-cosine pulse provides a gain of 1.25 dB at the expense of doubling bandwidth.

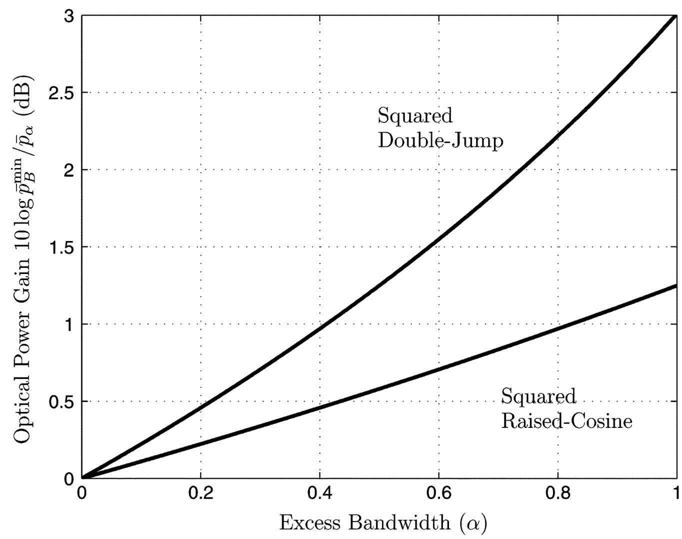

Fig. 4. Tradeoff between the optical power gain $\bar{p}_B^{\min}/\bar{p}_\alpha$ versus excess bandwidth for squared double-jump and squared raised-cosine pulses.

Fig. 5 presents the eye diagrams for the minimum-bandwidth and squared double-jump and squared raised-cosine pulse for $\alpha = 0.25$. In each case, on–off keying (OOK) was employed, and the average optical power of each scheme was set to unity. In order to permit simulation, all pulses were truncated to $\pm 6T$, and all binary pulse sequences were superimposed.

Recall that the vertical eye opening is a measure of the noise margin, while the horizontal opening at the midpoint measures sensitivity to errors in sampling phase. Note that in all cases the eye is open. Qualitatively, the minimum-bandwidth optical Nyquist pulse in Fig. 5(a) is significantly more sensitive to timing errors. Additionally, the gain inherent to squared double-jump pulse provides a significantly higher noise margin over the other pulses.

In order to quantify the gains of the optimal optical intensity Nyquist pulse, the impact of timing phase error on the probability of error of an OOK system is computed using the well-known technique of Beaulieu [24], and is presented in Table I. In all cases, the optical signal-to-noise ratio (SNR) was set to 9.5 dB and the probability of symbol error was estimated for various timing offsets and excess bandwidths. For the excess bandwidth pulses, the probability of error decreases with increasing α due to the inherent optical power gain. In all cases, the probability of error for OOK using the $p_\alpha^{\rm opt}(t)$ pulse shape outperforms the other pulses. This performance is consistent with the wide eye-opening of $p_\alpha^{\rm opt}(t)$ given in Fig. 5.

Interestingly, although the performance of $p_\alpha^{\rm opt}(t)$ in terms of timing mismatch error is superior to the squared raised-cosine, $p_\alpha^{\rm opt}(t)$ decays as $1/|t|^2$, while $p_\alpha^{\rm rc}(t)$ decays as $1/|t|^6$. As shown in Fig. 3(a), the magnitude of the two sidelobes of $p_\alpha^{\rm opt}(t)$ nearest to the cursor are smaller than those of $p_\alpha^{\rm rc}(t)$. It is interesting to note that $p_\alpha^{\rm opt}(t)$ initially decays faster than the squared raised-cosine pulse, however, asymptotically decays at a slower rate. A similar phenomena was noted in [5] and [25] for electrical Nyquist pulses.

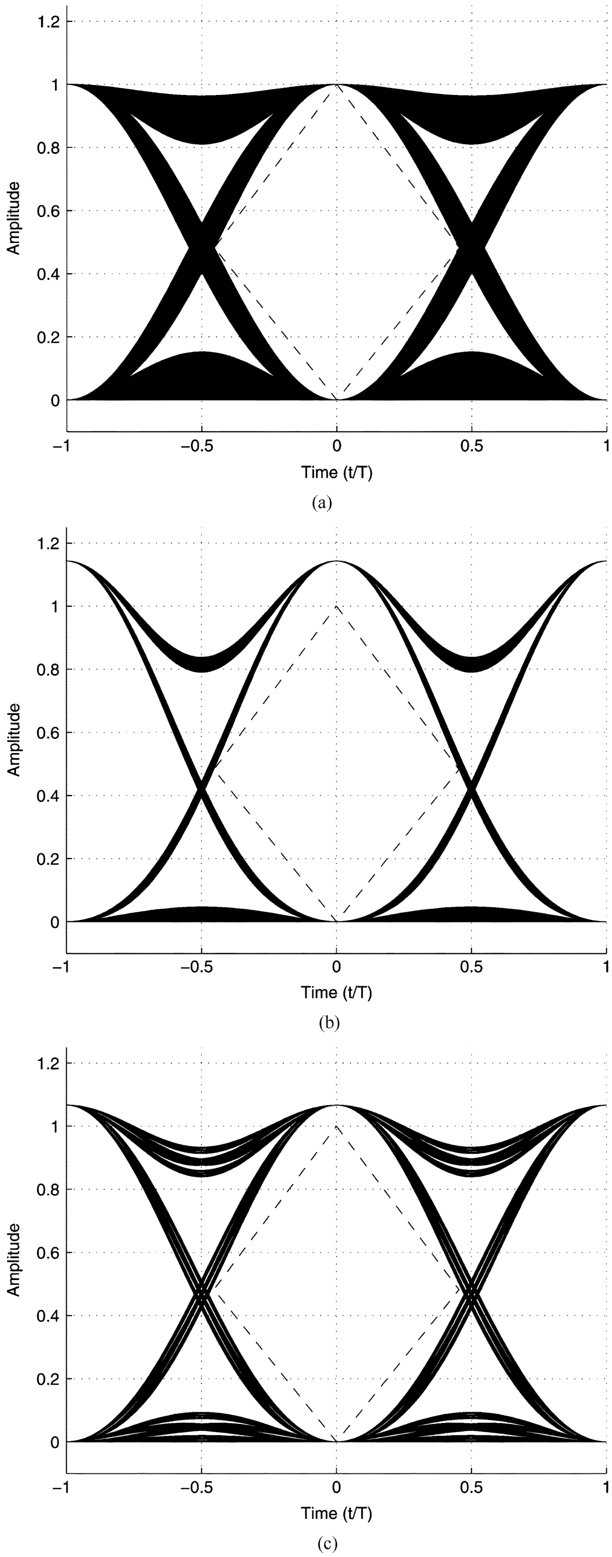

Fig. 5. Eye diagrams for binary sequences for (a) minimum-bandwidth optical intensity pulse, (b) squared double-jump pulse ($\alpha = 0.25$), and (c) squared raised-cosine pulse ($\alpha = 0.25$). The pulses are scaled so that the average optical power of each is unity. The inscribed diamond indicates the extent of the eye for the minimum-bandwidth case.

V. Optical Intensity Root-Nyquist Pulses

The previous section considered the problem of designing optical intensity pulses with zero-ISI when the receive filter $g(t)$ is flat in the band of interest. Here we consider the case when $g(t)$ is matched to the received pulse shape. This matched filter maximizes the output SNR and provides sufficient statistics for detection in additive white Gaussian noise channels, such as the indoor diffuse wireless optical channels [26].

As in Section III, consider an optical intensity PAM system with pulse shape $p(t)$. In this case, let the receive filter response be set to $g(t) = p(-t)$. To ensure zero ISI at the output of the sampler, the pulse shape at the output of the matched filter must satisfy Nyquist's criterion $p(t) \otimes p(-t)|_{t=kT} = \delta_{k0}$, or equivalently

$$\int_{-\infty}^{\infty} p(\tau)p(\tau - kT)\, d\tau = \delta_{k0}. \tag{18}$$

Pulses $p(t)$ satisfying (18) are termed *root-Nyquist pulses*. An optical intensity root-Nyquist pulse is one which simultaneously satisfies (2) and (18).

In this section, we show that there are no bandlimited optical intensity root-Nyquist pulses, and that all practical optical intensity root-Nyquist pulses are time-limited to a single symbol interval.

A. Existence of Bandlimited Root-Nyquist Pulses

A bandlimited optical intensity root-Nyquist pulse satisfies (3), in addition to (18) and the nonnegativity amplitude constraint. Define the *support* of a function $f(x)$ over domain Ω as

$$\mathcal{S}(f;\Omega) \stackrel{\triangle}{=} \{x \in \Omega : f(x) \neq 0\}.$$

Define also $\mathcal{S}^c(f)$ as the complement of the support set of $f(x)$, i.e., the set of values for which $f(x)$ is zero. The consequence of the nonnegativity and bandlimited constraints on set of root-Nyquist pulses is summarized in the following theorem.

Theorem 2: There exist no bandlimited optical intensity root-Nyquist pulses.

Proof: Suppose the opposite, that $p(t) \in \mathcal{L}^2$ is a bandlimited optical intensity root-Nyquist pulse. By the Paley–Wiener Theorem [19], [20], the extension of $p(t)$ to the complex plane $p(z)$ must be analytic. Furthermore, by the Unique Continuation Theorem [19, Th. 10.18], for $p(z)$ analytic either $\mathcal{S}^c(p;\mathbb{C}) = \mathbb{C}$ or is at most countable. That is, either $p(t) = 0$ for all $t \in \mathbb{R}$ or $p(t) > 0$ almost everywhere in $\mathbb{R}$, due to the nonnegativity of $p(t)$.

Since $p(t)$ is an optical intensity signal, $p(t)p(t - kT) \geq 0$. For $k \in \mathbb{Z} \setminus \{0\}$, the root-Nyquist constraint (18) together with the nonnegativity constraint (2) imply that $p(t)p(t - kT) = 0$ almost everywhere on $\mathbb{R}$. However, we have already shown that the bandlimitedness of $p(t)$ implies that $p(t) > 0$ almost everywhere, and by extension, that $p(t)p(t - kT) > 0$ almost everywhere. This leads to a contradiction, proving the theorem. □

Theorem 2 demonstrates that the amplitude nonnegativity constraint of optical intensity channels eliminates the possibility of finding bandlimited root-Nyquist pulses. This seems to run in contrast to the requirement of bandwidth-constrained diffuse optical systems. Although strictly bandlimited solutions do not

TABLE I
PROBABILITY OF ERROR FOR OPTICAL SNR = 9.5 dB

	$t/T=0$	$t/T=0.05$	$t/T=0.1$	$t/T=0.2$	$t/T=0.3$
Min. Bandwidth	4.170e-6	4.996e-6	9.567e-6	2.367e-4	1.422e-2
Squared Double-Jump					
$\alpha=0.1$	1.361e-6	1.668e-6	3.399e-6	1.055e-4	9.178e-3
0.25	1.764e-7	2.298e-7	5.249e-7	1.902e-5	2.919e-3
0.35	3.304e-8	4.627e-8	1.249e-7	5.943e-6	1.373e-3
0.5	1.411e-9	2.366e-9	1.052e-8	1.696e-6	6.587e-4
Squared Raised-Cosine					
$\alpha=0.1$	2.433e-6	2.944e-6	5.839e-6	1.689e-4	1.227e-2
0.25	1.000e-6	1.234e-6	2.565e-6	8.644e-5	8.358e-3
0.35	5.209e-7	6.528e-7	1.403e-6	5.102e-5	5.996e-3
0.5	1.764e-7	2.280e-7	5.215e-7	2.134e-5	3.418e-3

exist, the potential still exists to find bandwidth-efficient pulses for this channel. In Section V-3, a family of bandwidth-efficient optical intensity root-Nyquist pulses is described.

B. Time-Limitedness of Root-Nyquist Pulses

This section further studies the implications of the amplitude nonnegativity constraint of optical intensity channels on the set of root-Nyquist pulses. Here, we demonstrate that not only are root-Nyquist optical intensity pluses not bandlimited, but they must also be time-limited.

Theorem 3: All root-Nyquist optical intensity pulses have support of Lebesgue measure at most T.

Proof: Assume $p(t)$ is an optical intensity root-Nyquist pulse. Let $\mu(A)$ denote the *Lebesgue measure* of a set $A \subset \mathbb{R}$ [27]. In all cases of interest, $\mathcal{S}(p;\mathbb{R})$ is measurable. The support of $p(t), \mathcal{S}(p;\mathbb{R}) \subset \mathbb{R}$ and can be expanded as the union of disjoint sets

$$\mathcal{S}(p;\mathbb{R}) = \bigcup_{k=-\infty}^{\infty} \mathcal{S}(p;[kT,(k+1)T)). \tag{19}$$

The measure of the support set is thus

$$\mu(\mathcal{S}(p;\mathbb{R})) = \sum_{k=-\infty}^{\infty} \mu(\mathcal{S}(p;(kT,(k+1)T))). \tag{20}$$

In order to satisfy both (2) and (18), for all $k \in \mathbb{Z}\backslash\{0\}, p(t)p(t-kT)=0$ almost everywhere. Therefore, if the interval $(a,b) \subseteq \mathcal{S}(p;\mathbb{R})$, then $(a-kT, b-kT) \subset \mathcal{S}^c(p;\mathbb{R})$ for all $k \in \mathbb{Z}\setminus\{0\}$. In other words, for the partition defined in (19), the measure of the intersection of the support in each T interval must be zero, i.e.,

$$\mu(\mathcal{S}(p;(kT,(k+1)T))) - kT \cap \mathcal{S}(p;[lT,(l+1)T))) - lT) = 0$$

for all $k \neq l$. Therefore, the support in each symbol interval is disjoint of all other intervals (or is at most a countable). At best, the support in each T interval, modulo T, is a covering of the interval $(0,T)$. As a result, the measure (20) can be at most

$$\mu(\mathcal{S}(p;\mathbb{R})) \leq T.$$

□

Theorem 3 can be interpreted as requiring optical intensity root-Nyquist pulses to be nonzero only over a total time period of T seconds. In particular, in practical systems, it is desirable to consider pulse shapes which have their energy maximized around the cursor at time $t=0$. In this case, *Theorem 3* implies that for pulses concentrated about the cursor $\mathcal{S}(p;\mathbb{R}) = [-T/2, T/2]$, i.e., practical optical intensity root-Nyquist pulses are *time-limited* to a single symbol interval.

C. Minimum-Bandwidth Optical Intensity Root-Nyquist Pulses

In electrical channels, the minimum bandwidth Nyquist pulse is also a root-Nyquist pulse. However, in optical intensity channels, practical root-Nyquist pulses are necessarily time-limited. As a result, the definition of bandwidth for these pulses is not straightforward. Consider defining a *fractional energy bandwidth* of the function $x(t) \in \mathcal{L}^2$ as

$$B_\epsilon(x) = \inf\left\{B \in [0,\infty) : \int_{-B}^{B} |X(f)|^2\, df \geq (1-\epsilon)\int_{-\infty}^{\infty} |X(f)|^2\, df\right\}.$$

As *Theorem 3* asserts, practical optical intensity root-Nyquist pulses are time-limited to the interval $[-T/2, T/2)$. In this section, we consider the time-limited root-Nyquist optical intensity pulses which, for a given B_ϵ, minimize ϵ.

For a given B_ϵ and T, the family of *prolate spheroidal wave functions*, $\psi_n(t)$, provide the highest spectral concentration of all time-limited functions limited to $[-T/2, T/2)$ [28]. Equivalently, for a given T and ϵ, the $\psi_n(t)$ minimize the B_ϵ. This orthonormal family of functions arise as the eigenfunctions of the system

$$\lambda_i \Psi_n(f) = \int_{-B_\epsilon}^{B_\epsilon} \frac{\sin \pi T(f-\tau)}{\pi(f-\tau)} \Psi_n(\tau)\, d\tau$$

where $\Psi_n(f) = \mathcal{F}\{\psi_n(t)\}$ for a given T and B_ϵ. The function $\psi_0(t)$, corresponding to the largest eigenvalue λ_0, maximizes the spectral concentration, i.e., minimizes ϵ over all time-limited functions for a given $2B_\epsilon T$ product [29].

Although $\psi_0(t)$ has the minimum fractional energy bandwidth of all time-limited signals, it is not clear that this pulse

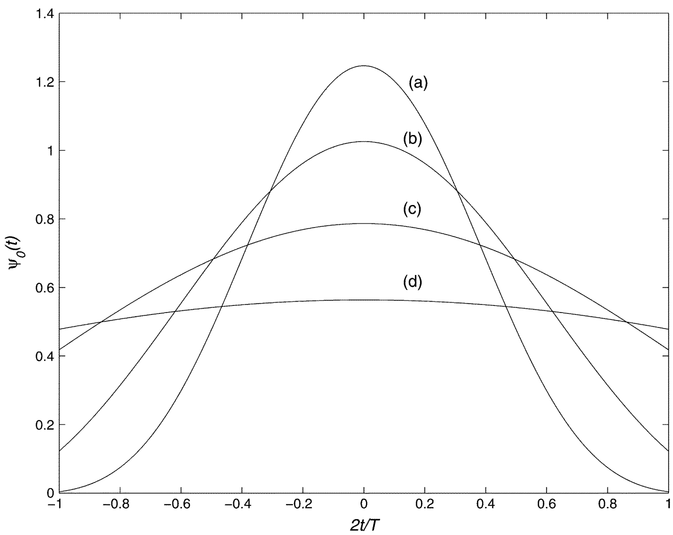

Fig. 6. Minimum fractional-energy bandwidth optical intensity root-Nyquist pulses for $2B_\epsilon T$ products and minimal ϵ value. (a) 32 ($\epsilon = 2.1 \times 10^{-6}$). (b) 16 ($\epsilon = 4.1 \times 10^{-3}$). (c) 8 ($\epsilon = 0.12$). (d) 4 ($\epsilon = 0.43$).

satisfies the amplitude nonnegativity constraint of optical intensity signals. It can further be demonstrated that $\psi_0(t)$ is both a smooth function of time and has no zero crossings in $(-T/2, T/2)$ [30]. As a result, $\psi_0(t)$ is compatible with optical intensity signaling. Fig. 6 has a plot of the minimum fractional energy bandwidth root-Nyquist pulse $\psi_0(t)$ for a variety of $2B_\epsilon T$ products.

VI. Conclusion

This paper presents fundamental results on pulse shaping for PAM optical intensity channels. Minimum-bandwidth ISI-free pulse shapes are derived under the amplitude constraints of optical intensity channels. This work is especially useful for indoor diffuse wireless optical channels which are bandlimited due to multipath distortion. Unlike previous work, this paper does not restrict attention to time-limited or rectangular sets, but derives general solutions which arise due to the fundamental amplitude constraints of the channel.

The minimum-bandwidth optical intensity Nyquist pulse requires twice the bandwidth of the solution in traditional electrical channels. If excess bandwidth is available, a squared double-jump optical intensity pulse minimizes the average optical power. In this manner, the tradeoff between the bandwidth of the optical intensity pulse shape and optical power cost is quantified. In the case where a more complex matched filter is available, it was shown that bandlimited root-Nyquist pulses do not exist, and that practical optical intensity root-Nyquist pulses are time-limited to a single baud interval.

The results of this work serve as a starting point and guide for further studies on bandwidth-efficient pulse design for optical intensity channels. Notice that these pulse designs can be applied to both PAM and pulse-position modulated systems. In the case of optical intensity Nyquist pulses, bandwidth-efficient pulses can be designed to approximate the optimal pulses derived here. Additionally, time-limited optical intensity root-Nyquist pulses can be derived which jointly minimize fractional energy bandwidth and optical power cost. The pulses derived in this work serve as useful fundamental baselines for comparison with newly derived pulse shapes.

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