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Minimum Bandwidth Nyquist and Root-Nyquist Pulses for Optical Intensity Channels

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Abstract— **Indoor diffuse wireless optical intensity channels are bandwidth constrained channels in which all transmitted signals must be non-negative. In order to control the impact of inter-symbol interference (ISI) on this channel, pulse shaping is required. This paper derives the minimum bandwidth, ISI-free Nyquist pulse which satisfies the amplitude non-negativity constraint. The minimum bandwidth required is twice that of conventional electrical channels. Further, it is shown that there are no bandlimited root-Nyquist pulses satisfying the amplitude non-negativity constraint. In fact, all optical intensity root-Nyquist pulses are time-limited. Examples of optical intensity root-Nyquist pulses which have minimum fractional energy bandwidth are presented.**

I. Introduction

Unlike many fibre optical channels, diffuse indoor wireless optical intensity channels are *bandwidth constrained*. These links, pioneered by Gfeller and Bapst [1], provide a wireless channel at infrared wavelengths by transmitting a modulated optical intensity signal over a wide solid-angle. The transmitted optical radiation reflects off surfaces in the room until it is detected by a photodetector receiver which produces an output signal proportional to the received optical intensity. *Multipath distortion* arises due to the multiple reflected paths between transmitter and receiver and causes severe inter-symbol interference (ISI) in high-speed links [2]. This work considers the design of minimum bandwidth ISI-free Nyquist and root-Nyquist pulses for optical intensity channels.

Diffuse wireless optical links are attractive since they are inexpensive, do not require strict alignment and are free of spectral licensing issues [2], however, signalling design for such channels differs significantly from conventional electrical channels. The transmitter modulates only the instantaneous optical intensity of the optical carrier and, as a result, all transmitted waveforms must be *non-negative*. Multipath distortion leads to temporal dispersion of the transmitted pulses and can be modelled with a linear low pass frequency response [2–4]. Unlike radio channels, indoor wireless optical links do not exhibit multipath fading since the receiver photodiode integrates the optical intensity field over an area of millions of square wavelengths providing an inherent degree of spatial diversity [2, 3]. The frequency response of diffuse wireless optical links is further reduced due to the use of inexpensive, large area photodiodes which have large reverse bias depletion capacitances. Received signals are also corrupted by high-intensity shot noise to ambient illumination. This noise is conventionally modelled as being white, Gaussian distributed and signal independent [2].

The low-pass response of diffuse wireless optical channels can lead to ISI in pulse-amplitude modulated (PAM) systems. There exists a wealth of literature on signalling in electrical ISI channels, however, these results cannot be applied directly to optical intensity channels due to the amplitude non-negativity constraint on all transmitted signals. The area of pulse design for optical intensity channels has been of interest for some time [5]. Previous studies on spectrally efficient optical intensity signalling centered on the use of rectangular pulse sets for wireless optical channels [2, 6–8]. Wider classes of pulse sets have also been considered, however, all remain time-limited [9, 10]. Coding and equalization have been considered for rectangular on-off keying and pulse-position modulation to mitigate the effects of ISI on these channels [11, 12]. However, the problem of pulse design to minimize ISI in diffuse indoor wireless optical channels has not been extensively studied.

In this work, we consider the general question of minimum bandwidth, ISI free, optical intensity pulses which are suited to bandwidth constrained diffuse indoor wireless optical channels. Section II presents the derivation of the minimum bandwidth Nyquist pulse and shows that the non-negativity constraint doubles the minimum bandwidth required. Root-Nyquist optical intensity pulses are considered in Section III and are shown to be necessarily time-limited. The optical intensity root-Nyquist pulse of minimum fractional energy bandwidth is also presented. Conclusions and directions for future work are presented in Section IV.

II. Minimum Bandwidth Optical Intensity Nyquist Pulse

Let $\mathcal{L}^2$ denote the space of real-valued functions defined over the real line with finite energy. Consider $p(t) \in \mathcal{L}^2$, $t \in \mathbb{R}$, with Fourier transform $P(\omega)$. Assume that $p(t)$ represents the optical intensity pulse shape due to the combined impact of the

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transmit and channel responses. The received signal, $r(t)$, in an optical intensity pulse amplitude modulated (PAM) system with symbol period $T > 0$ takes the form

$$r(t) = \sum_{k=-\infty}^{\infty} a_k p(t - kT) + n(t) \tag{1}$$

for some data sequence $a_k \in \mathbb{R}$ and white Gaussian noise process $n(t)$. In order to avoid ISI, $p(t)$ must satisfy the Nyquist criterion [13]

$$p(kT) = \delta_{k0}, \tag{2}$$

where δ_{kl} is the Kronecker delta,

$$\delta_{kl} = \begin{cases} 1 & : \text{ if } k = l \\ 0 & : \text{ otherwise} \end{cases}$$

and $k \in \mathbb{Z}$. Additionally, consider $p(t)$ which are bandlimited to bandwidth B such that

$$P(\omega) = 0, \qquad |\omega| \geq B. \tag{3}$$

Unlike electrical channels, $p(t)$ is an optical intensity signal and must also satisfy an amplitude non-negativity constraint, namely,

$$\forall t \in \mathbb{R} \qquad p(t) \geq 0. \tag{4}$$

The goal of this section is to find the $p(t)$ which minimizes B while satisfying constraints (2), (3) and (4).

Observe that in order to simultaneously satisfy both the amplitude non-negativity (4) and Nyquist (2) constraints, $p(t)$ must attain its minimum value of 0 at all non-zero integer sampling instants (i.e., at $t = kT, k \neq 0$). As a result, the derivative of $p(t)$, dp/dt, if it exists, must also be zero at all non-zero sampling instants. The existence of the derivative of $p(t)$ can be proved using a well known result from complex analysis. Consider extending $p(t)$ over the complex plane to yield $p(z)$, $z \in \mathbb{C}$. By the Paley-Wiener Theorem [14, Thm. 19.3], if $P(\omega)$ is bandlimited then $p(z)$ is analytic over the complex plane. As a result, dp/dt must exist for all $\mathbb{R}$. At $t = 0$, $p(0)$ is fixed to be unity by (2), however, the value of the first derivative is not specified explicitly. The observations on the derivative of $p(t)$ can be summarized as

$$\left.\frac{dp}{dt}\right|_{t=kT} = c \cdot \delta_{k0}, \tag{5}$$

for some $c \in \mathbb{R}$. Thus, for bandlimited optical intensity Nyquist pulses, the first derivative of $p(t)$ is a scaled Nyquist pulse. Figure 1 illustrates the constraints on $p(kT)$ and $dp/dt|_{t=kT}$ in time domain.

It is useful to cast the constraints on the amplitude of the time domain signal in frequency domain. The Nyquist constraint can be cast in frequency domain by taking the Fourier transform of (2), resulting in the well known expression

$$\sum_{m=-\infty}^{+\infty} P\left(\omega - m\frac{2\pi}{T}\right) = T. \tag{6}$$

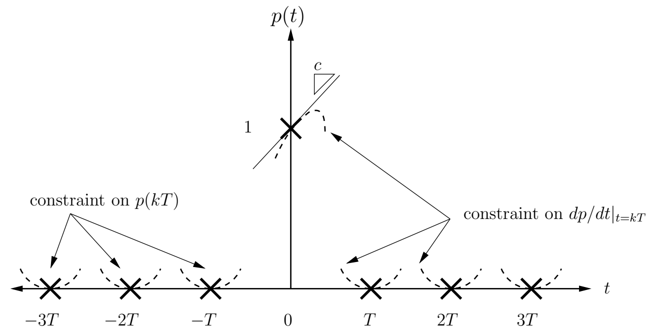

Fig. 1. Time Domain Representation of constraints on $p(kT)$ and $dp/dt|_{t=kT}$

Similarly, the constraint on dp/dt in (5) can be represented in the frequency domain as

$$\sum_{m=-\infty}^{+\infty} \jmath\left(\omega - m\frac{2\pi}{T}\right) P\left(\omega - m\frac{2\pi}{T}\right) = cT. \tag{7}$$

The problem now becomes one of determining the $P(\omega)$ of minimum bandwidth which satisfies both (6) and (7).

Recall that the minimum bandwidth pulse satisfying the Nyquist constraint (2), $\sin(\pi t/T)/(\pi t/T)$, occupies a bandwidth of π/T rad/s but does not satisfy the non-negativity constraint (4). As a result, $B > \pi/T$. Without loss in generality, assume that the $B \in (\pi/T, 2\pi/T]$. We will now show that (6) and (7) are satisfied for only one value $B \in (\pi/T, 2\pi/T]$ which is the minimum bandwidth of any optical intensity Nyquist pulse.

Both (6) and (7) must be satisfied over every interval of ω. Due to the bounds on B, in the interval $\omega \in (-2\pi/T, 2\pi/T]$ (7) reduces to

$$\left(\omega + \frac{2\pi}{T}\right) P\left(\omega + \frac{2\pi}{T}\right) + \omega P(\omega) + \left(\omega - \frac{2\pi}{T}\right) P\left(\omega - \frac{2\pi}{T}\right) = -\jmath cT.$$

Expanding and collecting like terms gives

$$\omega \sum_{n=-1}^{1} P\left(\omega - n\frac{2\pi}{T}\right) + \frac{2\pi}{T}\left(P\left(\omega + \frac{2\pi}{T}\right) - P\left(\omega - \frac{2\pi}{T}\right)\right) = -\jmath cT. \tag{8}$$

The first term can be simplified by noting that for $B \in (\pi/T, 2\pi/T]$, over the interval $\omega \in (-2\pi/T, 2\pi/T]$, (6) reduces to

$$\sum_{n=-1}^{1} P\left(\omega - n\frac{2\pi}{T}\right) = T.$$

Applying this simplification to (8) yields

$$\omega T + \frac{2\pi}{T}\left(P\left(\omega + \frac{2\pi}{T}\right) - P\left(\omega - \frac{2\pi}{T}\right)\right) = -\jmath cT.$$

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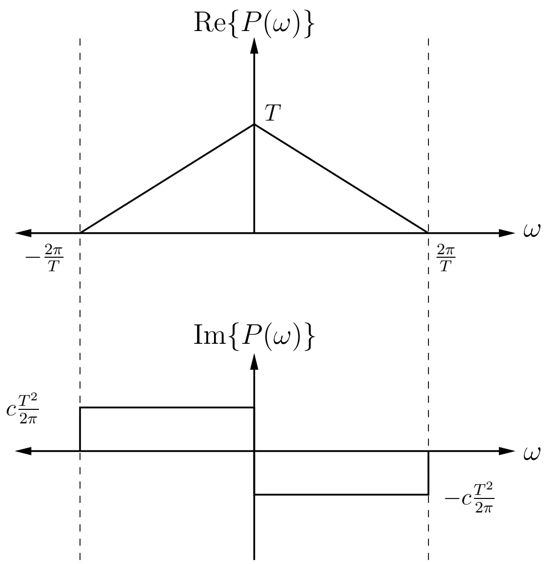

Fig. 2. Real and Imaginary Components of $P(\omega)$

The resulting sum of two shifted versions of $P(\omega)$ can be written over two distinct ranges of ω as

$$P\left(\omega + \frac{2\pi}{T}\right) = -\frac{T^2}{2\pi}\left(\omega + \jmath c\right), \quad \omega \in (-2\pi/T, 0] \tag{9}$$

and

$$P\left(\omega - \frac{2\pi}{T}\right) = \frac{T^2}{2\pi}\left(\omega + \jmath c\right), \quad \omega \in (0, 2\pi/T]. \tag{10}$$

The above expressions were derived under the assumption that $B \in (\pi/T, 2\pi/T]$. Notice that for $B < 2\pi/T$, (9) cannot be satisfied since the left-hand side goes to zero over the interval $(-2\pi/T + B, 0)$ while the right-hand side does not. Similarly, for $B < 2\pi/T$, the left-hand side of (10) goes to zero over $(0, 2\pi/T - B]$ while the right-hand side does not. Therefore, the minimum bandwidth under which the constraints are met is $B = 2\pi/T$.

Expressions (9) and (10) can be combined to define the real and imaginary components of $P(\omega)$ separately as,

$$\mathrm{Re}\left\{P(\omega)\right\} = \begin{cases} \frac{T^2}{2\pi}\left(\omega + \frac{2\pi}{T}\right) & : \ \frac{-2\pi}{T} < \omega \leq 0 \\ -\frac{T^2}{2\pi}\left(\omega - \frac{2\pi}{T}\right) & : \ 0 < \omega \leq \frac{2\pi}{T} \\ 0 & : \ \text{otherwise} \end{cases}$$

$$\mathrm{Im}\left\{P(\omega)\right\} = \begin{cases} c\frac{T^2}{2\pi} & : \ \frac{-2\pi}{T} < \omega \leq 0 \\ -c\frac{T^2}{2\pi} & : \ 0 < \omega \leq \frac{2\pi}{T} \\ 0 & : \ \text{otherwise} \end{cases}$$

The expressions for the real and imaginary components of $P(\omega)$ are plotted in Figure 2.

Since the Fourier transform is linear, the inverse transform of the real and imaginary components can be calculated separately and added in time domain. After some simplification, the resulting expression for $p(t)$ is

$$p(t) = (1 + ct) \cdot \frac{\sin^2(\pi t/T)}{(\pi t/T)^2}. \tag{11}$$

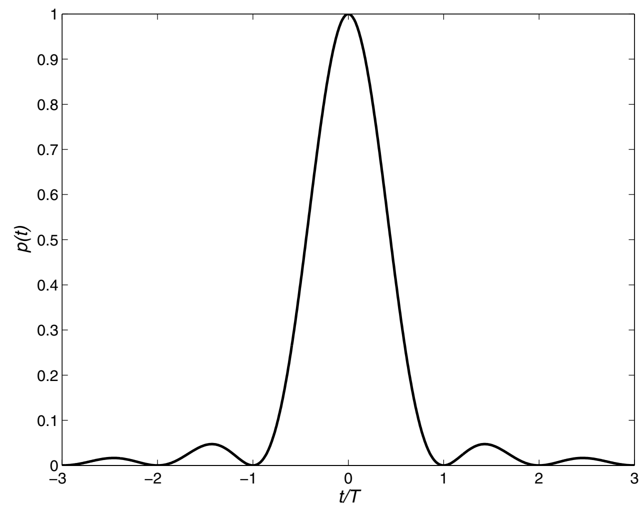

Fig. 3. Minimum bandwidth optical intensity Nyquist pulse in time domain.

The value of c can be constrained by noting that the Nyquist constraints on $p(t)$ (2) and on dp/dt (5) are necessary but not sufficient to guarantee that the non-negativity constraint in (4) holds. That is, the set of all non-zero Nyquist pulses are a subset of the set of functions described in (11). Applying the non-negativity constraint to (11) yields that for some $c \in \mathbb{R}$ and $\forall t \in \mathbb{R}$

$$(1 + ct) \cdot \frac{\sin^2(\pi t/T)}{(\pi t/T)^2} \geq 0$$

which in turn implies that $c = 0$. Thus, there is a unique minimum bandwidth optical intensity Nyquist pulse.

Therefore, the minimum bandwidth optical intensity Nyquist pulse occupies a bandwidth of $2\pi/T$ rad/s and takes the form

$$p(t) = \frac{\sin^2(\pi t/T)}{(\pi t/T)^2}.$$

Figure 3 presents a plot of the minimum bandwidth optical intensity Nyquist pulse. Notice that it satisfies the Nyquist criterion as well as the non-negativity constraint. Additionally, the pulse decays as $1/t^2$ lessening the impact of timing errors as compared to the minimum bandwidth solution for electrical channels. However, the minimum bandwidth optical Nyquist pulse requires twice the bandwidth over the electrical channel.

III. Optical Intensity Root-Nyquist Pulses

In the previous section, the problem of designing an optical intensity signal with zero-ISI was considered. This section extends the previous analysis by considering the use of an optimal receive filter which is matched to the received pulse shape. This matched filter maximizes the output signal-to-noise ratio and provides sufficient statistics for detection in additive white Gaussian noise channels, such as the indoor diffuse wireless optical channels [15].

As in Section II, consider an optical intensity PAM system with combined pulse shape $p(t)$. However, in this case the received signal (1) is filtered by a matched filter with impulse

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response $p(-t)$ prior to sampling. In this case, to ensure zero ISI at the output of the sampler, the signal at the output of the matched filter must satisfy Nyquist's criterion $p(t) * p(-t)|_{t=kT} = \delta_{k0}$ or equivalently,

$$\int_{-\infty}^{\infty} p(\tau)p(\tau - kT)d\tau = \delta_{k0}. \quad (12)$$

Pulses $p(t)$ satisfying (12) are termed *root-Nyquist pulses*.

In this section, we show that (i) there are no bandlimited optical intensity root-Nyquist pulses and (ii) that all practical optical intensity root-Nyquist pulses are time-limited to a single symbol interval.

A. Existence of Bandlimited Root-Nyquist Pulses

Diffuse indoor wireless optical channels are bandwidth constrained due to multipath distortion. The pulse shape chosen for diffuse optical PAM systems needs to take the bandwidth constraint into account in order to minimize ISI. This section demonstrates that bandlimited root-Nyquist pulses and corresponding matched filters do not exist. In order to aid in the proof of theorem, we define the *support* of a function $f(x)$ over domain Ω as $\mathcal{S}(f;\Omega) \triangleq \{x \in \Omega : f(x) \neq 0\}$. Define also $\mathcal{S}^c(f)$ as the complement of the support set of $f(x)$, i.e., the set of values for which $f(x)$ is zero. The consequence of the non-negativity constraint on the bandwidth of the pulses is investigated in the following theorem.

Theorem 1: There exist no bandlimited optical intensity root-Nyquist pulses.

Proof: Suppose $p(t) \in \mathcal{L}^2$ is a bandlimited optical intensity root-Nyquist pulse. By the Paley-Wiener Theorem [14], the extension of $p(t)$ to the complex plane, $p(z)$, must be analytic. Furthermore, by the Unique Continuation Theorem [14, Thm. 10.18], for $p(z)$ analytic either $\mathcal{S}^c(p;\mathbb{C}) = \mathbb{C}$ or is at most countable. That is, either $p(t) = 0$ for all $t \in \mathbb{R}$ or $p(t) > 0$ almost everywhere in $\mathbb{R}$ due to the non-negativity of $p(t)$.

Since $p(t)$ is an optical intensity signal, $p(t)p(t-kT) \geq 0$. For $k \in \mathbb{Z}\setminus\{0\}$, the root-Nyquist constraint (12) together with the non-negativity constraint (4) imply that $p(t)p(t-kT) = 0$ almost everywhere on $\mathbb{R}$. However, we have already shown that the bandlimitedness of $p(t)$ implies that $p(t) > 0$ almost everywhere and by extension that $p(t)p(t-kT) > 0$ almost everywhere. This is a contradiction, proving the theorem. □

Theorem 1 demonstrates that the amplitude non-negativity constraint of optical intensity channels eliminates the possibility of finding bandlimited root-Nyquist pulses. This seems to run in contrast to the requirement of bandwidth constrained diffuse optical systems. Although strictly bandlimited solutions do not exist, the potential still exists to find bandwidth efficient pulses for this channel. In Section III-C a family of bandwidth efficient optical intensity root-Nyquist pulses is described.

B. Time-Limitedness of Root-Nyquist Pulses

This section further studies the implications of the amplitude non-negativity constraint of optical intensity channels on the set of root-Nyquist pulses. Here, we demonstrate that not only are root-Nyquist optical intensity pluses not bandlimited but they must also be time-limited.

Theorem 2: All root-Nyquist optical intensity pulses have support of Lebesgue measure at most T.

Proof: Assume $p(t)$ is an optical intensity root-Nyquist pulse. Let $\mu(A)$ denote the *Lebesgue measure* of a set $A \subset \mathbb{R}$ [16]. In all cases of interest, $\mathcal{S}(p;\mathbb{R})$ is measurable. The support of $p(t)$, $\mathcal{S}(p;\mathbb{R}) \subset \mathbb{R}$ and can be expanded as,

$$\mathcal{S}(p;\mathbb{R}) = \bigcup_{k=-\infty}^{\infty} \mathcal{S}(p;[kT,(k+1)T)). \quad (13)$$

The measure of the support set is thus,

$$\mu(\mathcal{S}(p;\mathbb{R})) = \sum_{k=-\infty}^{\infty} \mu(\mathcal{S}(p;(kT,(k+1)T))). \quad (14)$$

In order to satisfy both (4) and (12), for all $k \in \mathbb{Z}\setminus\{0\}$, $p(t)p(t-kT) = 0$ almost everywhere. Therefore, if the interval $(a,b) \subseteq \mathcal{S}(p;\mathbb{R})$ then $(a-kT, b-kT) \subset \mathcal{S}^c(p;\mathbb{R})$ for all $k \in \mathbb{Z}\setminus\{0\}$. For the partition defined in (13), the intersection of the support in each T interval must result in a set of measure zero, i.e,

$$\mu(\ \mathcal{S}(p;(kT,(k+1)T))) - kT \cap \mathcal{S}(p;[lT,(l+1)T))) - lT\ ) = 0$$

for all $k \neq l$. Therefore, the support in each interval is disjoint of all other intervals (or is at most a countable) modulo T. At best the support in each T interval, modulo T, is a covering of the interval $(0,T)$. As a result, the measure (14) can be at most $\mu(\mathcal{S}(p;\mathbb{R})) \leq T$. □

Theorem 2 can be interpreted as requiring optical intensity root-Nyquist pulses to be non-zero only over a total time period of T seconds. In particular, in typical practical systems it is desirable to consider pulse shapes which have their energy maximized around the cursor at time $t = 0$. In this case, Theorem 2 implies that for pulses concentrated about the cursor $\mathcal{S}(p;\mathbb{R}) = [-T/2, T/2]$, i.e., practical optical intensity root-Nyquist pulses are *time-limited* to a single symbol interval.

C. Minimum Bandwidth Optical Intensity Root-Nyquist Pulses

In electrical channels, the minimum bandwidth Nyquist pulse is also a root-Nyquist pulse. However, in optical intensity channels, root-Nyquist pulses are necessarily time-limited. As a result, the definition of bandwidth for these pulses is not straight forward. Consider defining a *fractional energy bandwidth* of the function $x(t) \in \mathcal{L}^2$ as

$$W_\epsilon(x) = \inf\left\{ W \in [0,\infty)\ : \int_{-W}^{W} |X(f)|^2 df \geq (1-\epsilon)\int_{-\infty}^{\infty} |X(f)|^2 df \right\}.$$

As Theorem 2 asserts, practical optical intensity root-Nyquist pulses are time-limited to the interval $[-T/2, T/2)$. In this

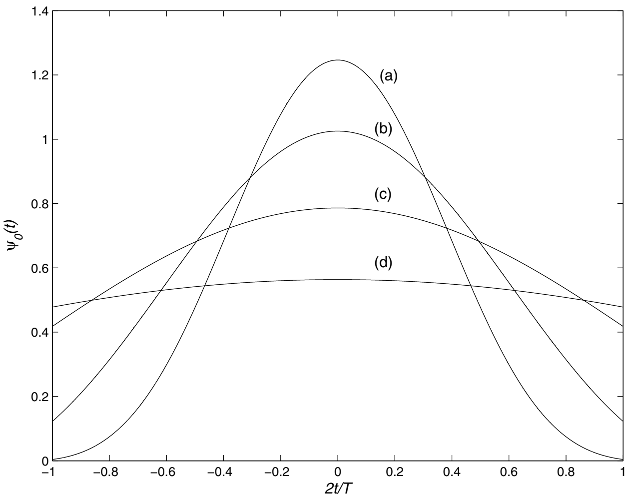

Fig. 4. Minimum fractional-energy bandwidth optical intensity root-Nyquist pulses for $2W_\epsilon T$ products and minimal ϵ value (a) 32 ($\epsilon = 2.1 \times 10^{-6}$) (b) 16 ($\epsilon = 4.1 \times 10^{-3}$) (c) 8 ($\epsilon = 0.12$) (d) 4 ($\epsilon = 0.43$).

section, we consider the time-limited root-Nyquist optical intensity pulses which, for a given W_ϵ, minimize ϵ.

For a given W_ϵ and T, the family of *prolate spheroidal wave functions*, $\psi_n(t)$, provide the highest spectral concentration of all time-limited functions limited to $[-T/2, T/2)$ [17]. Equivalently, for a given T and ϵ the $\psi_n(t)$ minimize the W_ϵ. This orthonormal family of functions arise as the eigenfunctions of the system

$$\lambda_i \psi(f) = \int_{-W_\epsilon}^{W_\epsilon} \frac{\sin \pi T(f-\tau)}{\pi(f-\tau)} \psi(\tau) d\tau$$

for a given T and W_ϵ. The function $\psi_0(t)$, corresponding to the largest eigenvalue λ_0, maximizes the spectral concentration, i.e., minimizes ϵ over all time-limited functions for a given $2W_\epsilon T$ product [18].

Although $\psi_0(t)$ has the minimum fractional energy bandwidth of all time-limited signals, it is not clear that this pulse satisfies the amplitude non-negativity constraint of optical intensity signals. It can further be demonstrated that $\psi_0(t)$ is both a smooth function of time and has no zero crossings in $(-T/2, T/2)$ [19]. As a result, $\psi_0(t)$ is compatible with optical intensity signalling. Figure 4 has a plot of the minimum fractional energy bandwidth root-Nyquist pulse $\psi_0(t)$ for a variety of $2W_\epsilon T$ products.

IV. Conclusions

This paper presents fundamental results on pulse shaping for optical intensity channels. Minimum bandwidth ISI-free pulse shapes are derived under the amplitude constraints of optical intensity channels. This work is especially useful for indoor diffuse wireless optical channels which are bandlimited due to multipath distortion. Unlike previous work, this paper does not restrict attention to time-limited or rectangular sets but derives general solutions which arise due to the amplitude constraints of the channel.

The minimum bandwidth optical intensity Nyquist pulse requires twice the bandwidth of the solution in traditional electrical channels. In the case where a more complex matched filter is available, it was shown that bandlimited root-Nyquist pulses do not exist . Indeed, it was demonstrated that practical optical intensity root-Nyquist pulses are time-limited to a single baud interval. The minimum bandwidth optical intensity root-Nyquist pulses are shown to be prolate spheroidal functions.

The results of this work serve as a starting point and guide for further studies on bandwidth efficient pulse design for optical intensity channels. The minimum bandwidth pulses developed here, although not immediately practical, serve as useful fundamental baselines for comparison with newly derived pulse shapes.

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