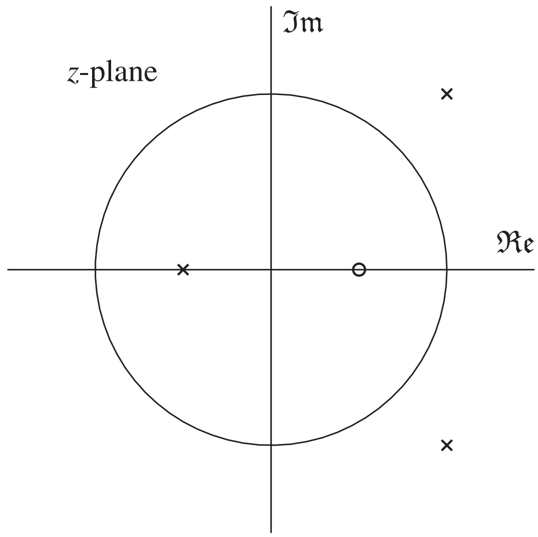
COMP ENG 4TL4 – Digital Signal Processing

Solutions to Homework Assignment #4

1. The figure below shows the pole-zero plot for a causal LTI system with a real-valued impulse response. Indicate which of the following properties apply to this system, justifying each answer:

- **i. stable**
- **ii. FIR**
- **iii. minimum phase**
- **iv. all-pass**
- **v. generalized linear phase** **(20 pts)**



i. stable? No. For a causal system to be stable, all the poles must lie within the unit circle.

ii. FIR? No. FIR systems can only have implicit poles at $z = 0$ or $z = \infty$.

iii. minimum phase? No. Minimum-phase systems have all poles and zeros inside the unit circle.

iv. all-pass? No. All-pass systems have poles and zeros in conjugate reciprocal pairs.

v. generalized linear phase? No. IIR filters with rational system functions cannot be generalized linear phase. **(20 pts)**

2. A causal LTI system has the transfer function:

$$H(z) = \frac{\left(1-0.5z^{-1}\right)\left(1+4z^{-2}\right)}{1-0.64z^{-2}}.$$

a. Find transfer functions for a *minimum-phase* system $H_1(z)$ and an *all-pass* system $H_{\mathrm{ap}}(z)$ such that:

$$H(z) = H_1(z)H_{\mathrm{ap}}(z).$$

b. Sketch the pole-zero plots of $H(z)$, $H_1(z)$ and $H_{\mathrm{ap}}(z)$, indicating their ROCs. (20 pts)

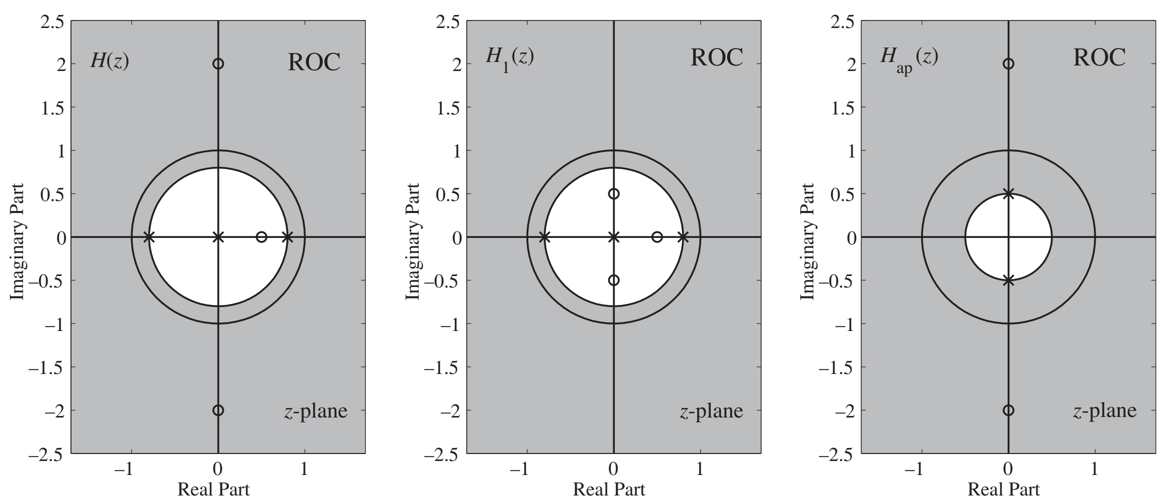
a. The system $H(z)$ has a pair of conjugate nonminimum-phase zeros at $z = \pm 2j$. To form the minimum-phase system $H_1(z)$, these zeros must be reflected to their conjugate reciprocal locations at $z = \pm 0.5j$, giving:

$$H_1(z) = \frac{\left(1-0.5z^{-1}\right)\left(1+0.25z^{-2}\right)}{1-0.64z^{-2}}, \qquad |z| > 0.8.$$

In order for $H_1(z)H_{\mathrm{ap}}(z)$ to equal $H(z)$, the all-pass system $H_{\mathrm{ap}}(z)$ must have the pair of conjugate nonminimum-phase zeros at $z = \pm 2j$ from $H(z)$ and a pair of poles at the conjugate reciprocal locations $z = \pm 0.5j$, giving:

$$H_{\mathrm{ap}}(z) = \frac{1+4z^{-2}}{1+0.25z^{-2}}, \qquad |z| > 0.5.$$

b. The pole-zero plots of $H(z)$, $H_1(z)$ and $H_{\mathrm{ap}}(z)$, with their ROCs, are given below.



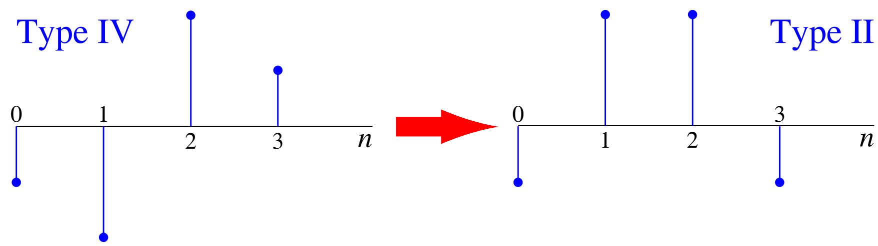
3. Let $h_{\mathrm{lp}}[n]$ denote the impulse response of an FIR generalized linear-phase *lowpass* filter. The impulse response $h_{\mathrm{hp}}[n]$ of an FIR generalized linear-phase *highpass* filter can be obtained by the transformation:

$$h_{hp}[n] = (-1)^n h_{lp}[n].$$

If we wish $h_{\mathrm{hp}}[n]$ to be symmetric or antisymmetric, could we use a Type IV FIR generalized linear-phase filter for $h_{\mathrm{lp}}[n]$? Justify your answer. (20 pts)

No. A Type IV FIR filter transformed according to the equation given above would produce a Type II FIR filter, which cannot be highpass because of its zero at $z = -1 \rightarrow \omega = \pi$.

For example:



We can also note that a Type IV FIR filter cannot be lowpass because of its zero at $z = 1 \rightarrow \omega = 0$, so it would be impossible to design a lowpass Type IV FIR filter in the first place.

4. Use the *bilinear transformation* IIR filter design method to design a discrete-time 2nd-order lowpass Butterworth filter with cutoff frequency $\omega_c = \pi/4$ radians, assuming a sampling frequency $f_s = 4$ kHz.

a. Give details of each step of the design procedure and give the analog and digital filter transfer functions $H_c(s)$ and $H(z)$, respectively, making sure that you simplify your expression for $H(z)$ so that its numerator and denominator are either (i) products of factors in terms of the explicit poles and zeros of $H(z)$ or (ii) polynomials in descending negative powers of *z*.

b. Does the assumed sampling frequency of $f_s = 4$ kHz have any effect on your expression for $H(z)$? Why or why not? (20 pts)

a. The first step is to find the prewarped analog cutoff frequency Ω_c corresponding to ω_c:

$$\begin{aligned}\Omega_c &= \frac{2}{T}\tan\left(\omega_c/2\right)\\ &= \frac{2}{T}\tan\left(\pi/8\right) = 8000\tan\left(\pi/8\right) = 3.3137\times 10^3 \text{ radians/s.}\end{aligned}$$

The second step is to design a 2nd-order (N = 2) analog Butterworth filter $H_c(s)$ with cutoff frequency Ω_c:

$$s_k = \Omega_c \exp\left[j\left(\frac{\pi}{2}+\frac{(2k+1)\pi}{2N}\right)\right],\quad 0\le k\le N-1 \qquad \Rightarrow$$

$$\begin{aligned}s_0 &= \Omega_c\, \mathrm{e}^{j3\pi/4}\\ s_1 &= \Omega_c\, \mathrm{e}^{j5\pi/4} = \Omega_c\, \mathrm{e}^{-j3\pi/4}\end{aligned} \qquad \Rightarrow$$

$$\begin{aligned}H_c(s) &= \frac{-\Omega_c\, \mathrm{e}^{j3\pi/4}}{s-\Omega_c\, \mathrm{e}^{j3\pi/4}}\times\frac{-\Omega_c\, \mathrm{e}^{-j3\pi/4}}{s-\Omega_c\, \mathrm{e}^{-j3\pi/4}}\\ &= \frac{\Omega_c^2}{\left(s-\Omega_c\, \mathrm{e}^{j3\pi/4}\right)\left(s-\Omega_c\, \mathrm{e}^{-j3\pi/4}\right)} = \frac{1.0981\times 10^7}{\left(s-3.3137\times 10^3\times \mathrm{e}^{j3\pi/4}\right)\left(s-3.3137\times 10^3\times \mathrm{e}^{-j3\pi/4}\right)}.\end{aligned}$$

The third step is to apply the bilinear transformation and simplify the expression for the z-domain transfer function $H(z)$:

$$H(z) = H_c(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{\left(\frac{2}{T}\tan(\pi/8)\right)^2}{\left(\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)-\frac{2}{T}\tan(\pi/8)\mathrm{e}^{j3\pi/4}\right)\left(\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)-\frac{2}{T}\tan(\pi/8)\mathrm{e}^{-j3\pi/4}\right)}$$

$$= \frac{\left(\tan(\pi/8)\right)^2}{\left(\left(\frac{1-z^{-1}}{1+z^{-1}}\right)-\tan(\pi/8)\mathrm{e}^{j3\pi/4}\right)\left(\left(\frac{1-z^{-1}}{1+z^{-1}}\right)-\tan(\pi/8)\mathrm{e}^{-j3\pi/4}\right)}$$

$$= \frac{\left(\tan(\pi/8)\right)^2\left(1+z^{-1}\right)^2}{\left(1-z^{-1}-\tan(\pi/8)\mathrm{e}^{j3\pi/4}\left(1+z^{-1}\right)\right)\left(1-z^{-1}-\tan(\pi/8)\mathrm{e}^{-j3\pi/4}\left(1+z^{-1}\right)\right)}$$

$$= \frac{0.1716\left(1+2z^{-1}+z^{-2}\right)}{\left(1-\tan(\pi/8)\mathrm{e}^{j3\pi/4}-\left(1+\tan(\pi/8)\mathrm{e}^{j3\pi/4}\right)z^{-1}\right)\left(1-\tan(\pi/8)\mathrm{e}^{-j3\pi/4}-\left(1+\tan(\pi/8)\mathrm{e}^{-j3\pi/4}\right)z^{-1}\right)}$$

$$= \frac{0.1716\left(1+2z^{-1}+z^{-2}\right)}{\left(1.2929-j0.2929-\left(0.7071+j0.2929\right)z^{-1}\right)\left(1.2929+j0.2929-\left(0.7071-j0.2929\right)z^{-1}\right)}$$

$$= \frac{0.1716+0.3431z^{-1}+0.1716z^{-2}}{1.7574-1.6568z^{-1}+0.5858z^{-2}},$$

or normalizing the transfer function such that $a[0] = 1$ gives:

$$H(z) = \frac{0.0976+0.1953z^{-1}+0.0976z^{-2}}{1-0.9428z^{-1}+0.3333z^{-2}},$$

or expressing the transfer function in terms of its explicit poles and zeros gives:

$$H(z) = \frac{0.0976\left(1+z^{-1}\right)^2}{\left(1-\left(0.4714+j0.3333\right)z^{-1}\right)\left(1-\left(0.4714-j0.3333\right)z^{-1}\right)}.$$

b. No. We see in the derivation above (in the step indicated by the arrow) that the sampling period T cancels out of the expression for $H(z)$. This will always be the case when we start with the filter specifications for the digital filter. The frequency axis is prewarped in step 1 and then warped back again with the inverse warping in step 3, so the sampling period T has no effect.

5. A causal LTI system has the transfer function:

$$H(z) = \frac{1 - \frac{1}{4}z^{-1}}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}.$$

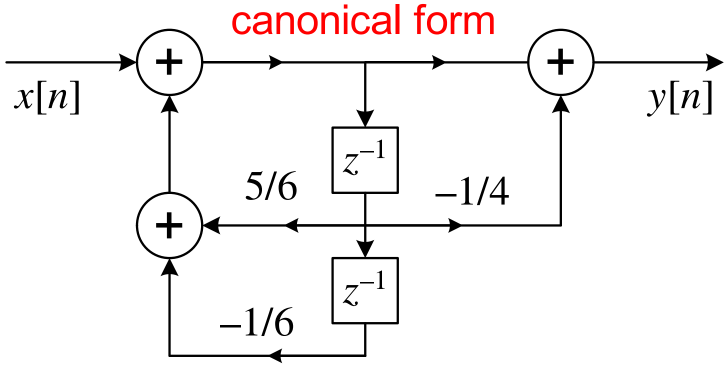
Draw the block diagrams for this filter implemented in:

a. direct form II (canonical form), and

b. parallel form with 1st-order subsystems. **(20 pts)**

a. Direct form II can be obtained directly from the transfer function.

direct form II/
canonical form



b. The parallel form can be obtained by partial fraction expansion of the transfer function:

$$\begin{aligned} H(z) &= \frac{1 - \frac{1}{4}z^{-1}}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}} \\ &= \frac{1 - \frac{1}{4}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)} \\ &= \frac{\frac{3}{2}}{1 - \frac{1}{2}z^{-1}} + \frac{-\frac{1}{2}}{1 - \frac{1}{3}z^{-1}}. \end{aligned}$$

parallel form using
1st-order subsystems