COMP ENG 4TL4 – Digital Signal Processing

Solutions to Homework Assignment #5

1. Four values of a stationary random process $x[n]$ have been observed, giving the discrete-time sequence $v[n] = \{0.05, –0.3, 0.5, –1\}$.

a. Use the *autocorrelation-based method* of spectral estimation to estimate the PSD of $x[n]$, using a triangular correlation window $w_c[m] = \{0.5, 1, 0.5\}$.

b. **Does $x[n]$ appear to be lowpass, bandpass or highpass? How confident should you be of this signal characterization, given the number of values observed and the length of the correlation window used? (20 pts)**

a. First we compute the estimate of autocorrelation function of the random process. Note that we will be applying a correlation window with nonzero values only at lags of $m = -1$, 0 and 1, so we only need to evaluate the autocorrelation estimator at those lags:

$$c_{vv}[m] = v[m] * v[-m] \Rightarrow$$

$$c_{vv}[-1] = 0.05\times 0 + (-0.3)\times 0.05 + 0.5\times(-0.3) + (-1)\times 0.5 + 0\times(-1) = -0.665$$
$$c_{vv}[0] = 0.05\times 0.05 + (-0.3)\times(-0.3) + 0.5\times 0.5 + (-1)\times(-1) = 1.3425$$
$$c_{vv}[1] = 0\times 0.05 + 0.05\times(-0.3) + (-0.3)\times 0.5 + 0.5\times(-1) + (-1)\times 0 = -0.665$$

$$\hat{\phi}_{vv}[m] = \frac{1}{Q}c_{vv}[m] = \frac{1}{4}c_{vv}[m].$$

Applying the triangular correlation window gives:

$$\hat{\phi}_{vv}[m]w_c[m] = \frac{1}{4}\{-0.665\delta[m+1] + 1.3425\delta[m] - 0.665\delta[m-1]\}\times\{0.5\delta[m+1] + \delta[m] + 0.5\delta[m-1]\}$$
$$= \{-0.0831\delta[m+1] + 0.3356\delta[m] - 0.0831\delta[m-1]\}.$$

Taking the Fourier transform of the windowed autocorrelation estimate gives the PSD estimator:

$$S(\omega) = \sum_{m=-1}^{1}\hat{\phi}_{vv}[m]w_c[m]e^{-j\omega m}$$
$$= \sum_{m=-1}^{1}\{-0.0831\delta[m+1] + 0.3356\delta[m] - 0.0831\delta[m-1]\}e^{-j\omega m}$$
$$= -0.0831e^{j\omega} + 0.3356 - 0.0831e^{-j\omega}$$
$$= 0.3356 - 0.1662\cos\omega.$$

b. From the final expression for $S(\omega)$ we would estimate that $x[n]$ is a highpass signal. However, for such a short observation sequence (only 4 values), we would not expect a very good characterization of the PSD. It is unlikely that the signal is lowpass, but it is possible that it is bandpass, and our observed sequence $v[n]$ happens to contain more high frequency energy for that finite observation period. Additionally, the correlation window is very short, which greatly smoothes the PSD estimate, so it would be hard to tell if the signal is bandpass with a fairly high center frequency. FYI, $x[n]$ in this case was in fact a highpass signal, but the estimated PSD is not very close to that of the actual PSD.

2. Consider an LTI system with the transfer function:

$$H(z) = \frac{z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-3z^{-1}\right)}.$$

a. Suppose that the system is known to be stable. Determine the output *y*[*n*] when the input *x*[*n*] is the unit step sequence.

b. Suppose that the ROC of *H*(*z*) includes *z* = ∞. Determine *y*[*n*] evaluated at *n* = 2 when *x*[*n*] is the sequence {1, 2, 3} for *n* = 0, 1 and 2, and is zero otherwise.

c. **Suppose we wish to recover *x*[*n*] from *y*[*n*] by processing *y*[*n*] with a LTI system whose impulse response is given by $h_i[n]$. Determine $h_i[n]$. Does $h_i[n]$ depend of the ROC of *H*(*z*)?**

(30 pts)

a. For the system to be stable, the ROC must include the unit circle of the *z*-plane. The only possible ROC that satisfies this requirement is $\frac{1}{2} < |z| < 3$, corresponding to a two-sided impulse response.
The *z*-transform of the input $x[n] = u[n]$, the unit step sequence is:

$$X(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1.$$

The output in response to the unit step sequence is then:

$$\begin{aligned} Y(z) &= H(z)X(z) \\ &= \frac{z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-3z^{-1}\right)\left(1-z^{-1}\right)} \\ &= \frac{\frac{4}{5}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{5}}{1-3z^{-1}} - \frac{1}{1-z^{-1}}, \quad 1 < |z| < 3. \end{aligned}$$

$$\Rightarrow y[n] = \frac{4}{5}\left(\frac{1}{2}\right)^n u[n] - \frac{1}{5}(3)^n u[-n-1] - u[n].$$

b. If the ROC includes $z = \infty$, then the system is causal, i.e., $h[n] = 0$ for $n < 0$. Since $x[n]$ is also zero for $n < 0$, we know that $y[n]$ is zero for $n < 0$, so one method to solve this problem is to calculate values of $y[n]$ using the forward recursion of the LCCD equation describing this system:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-3z^{-1}\right)} = \frac{z^{-2}}{1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}}$$

$$\Rightarrow Y(z) - \frac{7}{2}z^{-1}Y(z) + \frac{3}{2}z^{-2}Y(z) = z^{-2}X(z)$$

$$\Rightarrow y[n] - \frac{7}{2}y[n-1] + \frac{3}{2}y[n-2] = x[n-2]$$

$$\Rightarrow y[n] = \frac{7}{2}y[n-1] - \frac{3}{2}y[n-2] + x[n-2].$$

The forward recursion gives:

$$y[0] = \tfrac{7}{2}\times 0 - \tfrac{3}{2}\times 0 + 0 = 0,$$
$$y[1] = \tfrac{7}{2}\times 0 - \tfrac{3}{2}\times 0 + 0 = 0,$$
$$y[2] = \tfrac{7}{2}\times 0 - \tfrac{3}{2}\times 0 + 1 = 1.$$

A second method is to find the impulse response $h[n]$ and convolve it with the input $x[n]$:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-2}}{\left(1-\tfrac{1}{2}z^{-1}\right)\left(1-3z^{-1}\right)} = \frac{\tfrac{2}{5}z^{-1}}{1-3z^{-1}} - \frac{\tfrac{2}{5}z^{-1}}{1-\tfrac{1}{2}z^{-1}}$$
$$\Rightarrow h[n] = \tfrac{2}{5}(3)^{n-1}u[n-1] - \tfrac{2}{5}\left(\tfrac{1}{2}\right)^{n-1}u[n-1].$$

The first three values of $h[n]$ are:

$$h[0] = 0,$$
$$h[1] = \tfrac{2}{5} - \tfrac{2}{5} = 0,$$
$$h[2] = \tfrac{6}{5} - \tfrac{1}{5} = 1,$$

giving the first three output values of:

$$y[0] = 0,$$
$$y[1] = 0,$$
$$y[2] = 1.$$

A third method is to compute this convolution as a multiplication in the *z*-domain and take the inverse *z*-transform.

c. The inverse system has the transfer function:

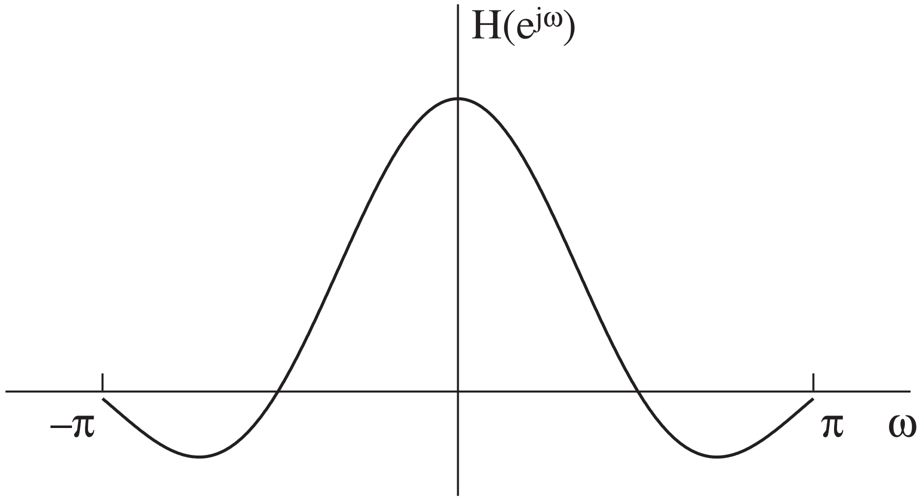
$$H_i(z) = \frac{1}{H(z)} = \frac{1-\tfrac{7}{2}z^{-1}+\tfrac{3}{2}z^{-2}}{z^{-2}} = z^2 - \tfrac{7}{2}z + \tfrac{3}{2},$$

with the ROC being the entire *z*-plane *except for* $z = \infty$. This ROC intersects with both the stable, noncausal ROC for $H(z)$, i.e., $\tfrac{1}{2} < |z| < 3$ and the unstable, causal ROC for $H(z)$, i.e., $|z| > 3$, so the ROC of the inverse system is *independent* of the ROC of $H(z)$.

The impulse response of the inverse system is then:

$$h_i[n] = \delta[n+2] - \tfrac{7}{2}\delta[n+1] + \tfrac{3}{2}\delta[n].$$

3. **The Fourier transform of a stable LTI system is purely real-valued and is shown in the figure below. Determine whether this system has a stable inverse system. (10 pts)**



This system must have a zero on the unit circle of the z-plane, because its frequency response has a value of zero at ~ $\pm\pi/2$. Thus, the inverse of this system would have a pole on the unit circle, and consequently the inverse system *could not be stable*.

4. Consider the finite sequence $x[n] = \{0, 0.5878, 0.9511, 0.9511, 0.5878, 0\}$.

a. Calculate the DFT of this sequence $X[k],\quad 0 \le k \le N-1$. Apply lossy compression to the vector of DFT coefficients $X[k]$ by ignoring (i.e., setting to zero) the coefficients for $k = 2$, 3 and 4, to obtain the compressed vector $\tilde{X}[k]$. Reconstruct the sequence $\tilde{x}[n]$ by taking the inverse DFT of $\tilde{X}[k]$.

b. **Calculate the root mean squared (RMS) error between $\tilde{x}[n]$ and $x[n]$. Express this RMS error as a percentage of the RMS of $x[n]$. (20 pts)**

a. The DFT of the sequence $x[n]$ is:

$$X[k]=\sum_{n=0}^{5}x[n]e^{-j2\pi kn/6} \Rightarrow$$

$$\begin{aligned}
X[0]&=\sum_{n=0}^{5}x[n]e^{0}\\
&=0+0.5878+0.9511+0.9511+0.5878+0\\
&=3.0778
\end{aligned}$$

$$\begin{aligned}
X[1]&=\sum_{n=0}^{5}x[n]e^{-j2\pi n/6}\\
&=0e^{0}+0.5878e^{-j2\pi/6}+0.9511e^{-j2\pi 2/6}+0.9511e^{-j2\pi 3/6}+0.5878e^{-j2\pi 4/6}+0e^{-j2\pi 5/6}\\
&=-1.4267-j0.8237
\end{aligned}$$

$$\begin{aligned}
X[2]&=\sum_{n=0}^{5}x[n]e^{-j2\pi 2n/6}\\
&=0e^{0}+0.5878e^{-j2\pi 2/6}+0.9511e^{-j2\pi 4/6}+0.9511e^{-j2\pi 6/6}+0.5878e^{-j2\pi 8/6}+0e^{-j2\pi 10/6}\\
&=-0.1123-j0.1944
\end{aligned}$$

$$\begin{aligned}
X[3]&=\sum_{n=0}^{5}x[n]e^{-j2\pi 3n/6}\\
&=0e^{0}+0.5878e^{-j2\pi 3/6}+0.9511e^{-j2\pi 6/6}+0.9511e^{-j2\pi 9/6}+0.5878e^{-j2\pi 12/6}+0e^{-j2\pi 15/6}\\
&=0-j0
\end{aligned}$$

$$\begin{aligned}
X[4]&=X^{*}[2]\\
&=-0.1123+j0.1944
\end{aligned}$$

$$\begin{aligned}
X[5]&=X^{*}[1]\\
&=-1.4267+j0.8237
\end{aligned}$$

Note that for a real-valued sequence $x[n]$ we can make use of the symmetry properties of the DFT to calculate the DFT coefficients for frequency indices k above the Nyquist frequency (i.e., $k > N/2$), in this case $k > 3 \Rightarrow X[4]$ and $X[5]$.

The compressed DFT sequence $\tilde{X}[k]$ is then:

$$\begin{aligned}
\tilde{X}[0]&=3.0778\\
\tilde{X}[1]&=-1.4267-j0.8237\\
\tilde{X}[2]&=0\\
\tilde{X}[3]&=0\\
\tilde{X}[4]&=0\\
\tilde{X}[5]&=-1.4267+j0.8237,
\end{aligned}$$

and the reconstructed sequence $\tilde{x}[n]$ is obtained by taking the inverse DFT of $\tilde{X}[k]$:

$$\tilde{x}[n] = \frac{1}{6}\sum_{k=0}^{5}\tilde{X}[k]e^{j2\pi kn/6} \Rightarrow$$

$$\tilde{x}[0] = \frac{1}{6}\sum_{k=0}^{5}\tilde{X}[k]e^{0}$$

$$= \frac{1}{6}\{3.0778 - 1.4267 - j0.8237 - 1.4267 + j0.8237\} = 0.0374$$

$$\tilde{x}[1] = \frac{1}{6}\sum_{k=0}^{5}\tilde{X}[k]e^{j2\pi k/6}$$

$$= \frac{1}{6}\{3.0778 - (1.4267 + j0.8237)e^{j2\pi/6} - (1.4267 - j0.8237)e^{j2\pi 5/6}\} = 0.5130$$

$$\tilde{x}[2] = \frac{1}{6}\sum_{k=0}^{5}\tilde{X}[k]e^{j2\pi k2/6}$$

$$= \frac{1}{6}\{3.0778 - (1.4267 + j0.8237)e^{j2\pi 2/6} - (1.4267 - j0.8237)e^{j2\pi 10/6}\} = 0.9885$$

$$\tilde{x}[3] = \frac{1}{6}\sum_{k=0}^{5}\tilde{X}[k]e^{j2\pi k3/6}$$

$$= \frac{1}{6}\{3.0778 - (1.4267 + j0.8237)e^{j2\pi 3/6} - (1.4267 - j0.8237)e^{j2\pi 15/6}\} = 0.9885$$

$$\tilde{x}[4] = \frac{1}{6}\sum_{k=0}^{5}\tilde{X}[k]e^{j2\pi k4/6}$$

$$= \frac{1}{6}\{3.0778 - (1.4267 + j0.8237)e^{j2\pi 4/6} - (1.4267 - j0.8237)e^{j2\pi 20/6}\} = 0.5130$$

$$\tilde{x}[5] = \frac{1}{6}\sum_{k=0}^{5}\tilde{X}[k]e^{j2\pi k5/6}$$

$$= \frac{1}{6}\{3.0778 - (1.4267 + j0.8237)e^{j2\pi 5/6} - (1.4267 - j0.8237)e^{j2\pi 25/6}\} = 0.0374,$$

giving the sequence $\tilde{x}[n] = \{0.0374, 0.5130, 0.9885, 0.9885, 0.5130, 0.0374\}$.

b. The RMS error is:

$$\begin{aligned}\text{RMSE} &= \sqrt{E\left\{\left(x[n]-\tilde{x}[n]\right)^2\right\}}\\ &= \sqrt{\frac{1}{6}\left\{2\times(0-0.0374)^2+2\times(0.5878-0.5130)^2+2\times(0.9511-0.9885)^2\right\}}\\ &= 0.0529.\end{aligned}$$

The RMS of $x[n]$ is:

$$\begin{aligned}\text{RMS} &= \sqrt{E\left\{\left(x[n]\right)^2\right\}}\\ &= \sqrt{\frac{1}{6}\left\{2\times 0.5878^2+2\times 0.9511^2\right\}}\\ &= 0.6455,\end{aligned}$$

giving a percentage RMS error of:

$$\frac{\text{RMSE}}{\text{RMS}}\times 100 = \frac{0.0529}{0.6455}\times 100 \approx 8.2\%$$

5. Consider the causal LTI system:

$$H(z) = 1 - z^{-1}.$$

a. Show that $H(z)$ has generalized linear phase by expressing the system's frequency response in the form $H(e^{j\omega}) = A(e^{j\omega})e^{j(\beta-\omega\alpha)}$, where $A(e^{j\omega})$ is a real-valued function and β and α are constants.

b. Determine which standard type of FIR generalized linear-phase filter $H(z)$ is:

Type I, II, III, or IV. (20 pts)

a. The system's frequency response is obtained by evaluating the transfer function $H(z)$ on the unit circle, i.e., $z = e^{j\omega}$:

$$\begin{aligned}H(e^{j\omega}) &= H(z)\big|_{z=e^{j\omega}} = 1 - e^{-j\omega}\\ &= e^{-j\omega/2}\left(e^{j\omega/2}-e^{-j\omega/2}\right)\\ &= e^{-j\omega/2}2j\sin(\omega/2)\\ &= 2\sin(\omega/2)e^{-j\omega/2+j\pi/2}\\ &= A(e^{j\omega})e^{j(\beta-\omega\alpha)},\end{aligned}$$

where $A(e^{j\omega}) = 2\sin(\omega/2)$, $\beta = \pi/2$ and $\alpha = 1/2$.

b. This system's impulse response is $h[n] = \delta[n] - \delta[n-1] \Rightarrow$ it has order 1 (length 2) and is antisymmetrical, so it is a <u>Type IV</u> FIR generalized linear-phase filter.