Computer Engineering 4TL4: Digital Signal Processing

Day Class | Instructor: Dr. I. C. BRUCE
Duration of Examination: 3 Hours
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This examination paper includes eleven (11) pages and eight (8) questions. You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancy to the attention of your invigilator.

Special Instructions: Use of Casio *fx*-991 calculator <u>only</u> is allowed.
All major questions (numbered 1 to 8) are worth equal marks.
Some equations and tables that may assist you are provided on pages 4–11.

1. The *step response* $y[n]$ of a causal, stable LTI system is:

$$y[n]=\left(\tfrac{1}{2}\right)^n u[n],$$

where $u[n]$ is the unit step function.

a. Find the z-domain transfer function $H(z)$ of this system.

b. Find the impulse response $h[n]$ of this system.

c. Find the linear constant coefficient difference (LCCD) equation that describes this system.

2. Consider the causal stable LTI system with the transfer function:

$$H(z)=\frac{1}{\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}.$$

Draw the block diagram for this system implemented in *parallel form* with 1^{st}-order subsystems.

3. A causal LTI system has the transfer function:

$$H(z)=\frac{\left(1+0.2z^{-1}\right)\left(1-9z^{-2}\right)}{1+0.81z^{-2}}.$$

a. Find transfer functions for a *minimum-phase* system $H_1(z)$ and an *all-pass* system $H_{\text{ap}}(z)$ such that:

$$H(z)=H_1(z)H_{\text{ap}}(z).$$

b. Sketch the pole-zero plots of $H(z)$, $H_1(z)$ and $H_{\text{ap}}(z)$, clearly show their ROCs by shading them on the pole-zero plots.

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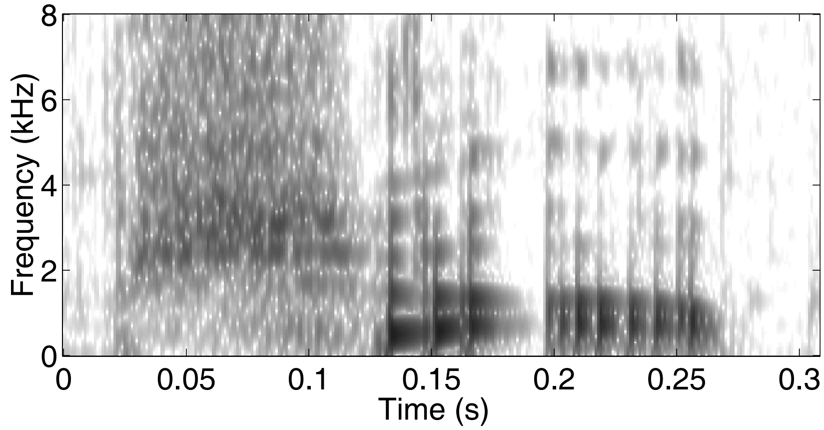
4. The spectrogram of a speech signal is shown below.
   a. Is this a *narrowband* spectrogram or a *wideband* spectrogram? Explain your reasoning.
   b. This signal is a single word, either "shop" or "posh". From the spectrogram, which of these two words does it appear to be? Explain your reasoning.



5. Two alternative methods of digital IIR filter design make use of the:
   i. impulse invariance transformation and
   ii. bilinear transformation.

   Explain:
   a. which of these two methods is applicable to a wider range of filter frequency response types (i.e., lowpass, highpass, bandpass, bandstop, multiband) and why, and
   b. which of these two methods is simpler and why.

6. Four values of a stationary random process $x[n]$ have been observed, giving the discrete-time sequence $v[n] = \{0.38, 0.66, 1.17, 0.86\}$.

   a. Use the *autocorrelation-based method* of spectral estimation to estimate the PSD of $x[n]$, using a triangular correlation window $w_c[m] = \{0.5, 1, 0.5\}$.

   b. Does $x[n]$ appear to be lowpass, bandpass or highpass?

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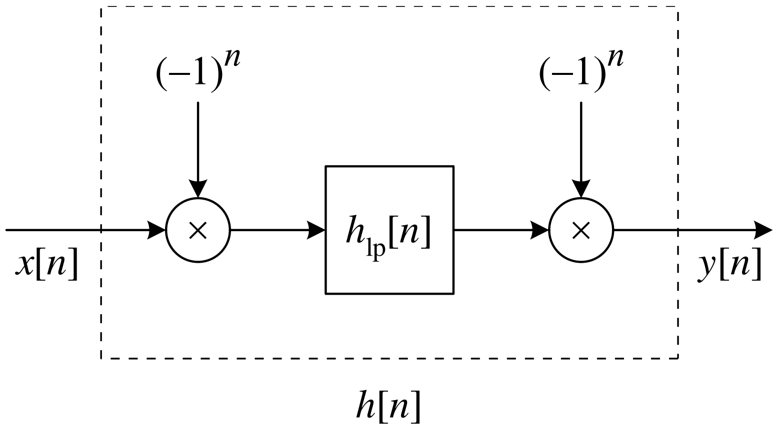
7. Let $h_{\mathrm{lp}}[n]$ denote the impulse response of an ideal lowpass filter with unity passband gain and cutoff frequency $\omega_c = \pi/4$. The figure below shows an ideal LTI frequency-selective filter $h[n]$ that incorporates $h_{\mathrm{lp}}[n]$ as a subsystem. Sketch $\left|H\left(e^{j\omega}\right)\right|$, the magnitude frequency response of $h[n]$, indicating explicitly the band-edge frequencies in terms of ω_c and specify whether the system is a lowpass, highpass, bandpass, bandstop or multiband filter.



8. Consider the finite sequence $x[n] = \{1,\ \ 0.75,\ \ 0.09,\ \ -0.06,\ \ -1,\ \ -0.85\}$. The DFT of this sequence $X[k]$ has been calculated for $0 \le k \le 5$ and lossy compression has been applied to the sequence of DFT coefficients $X[k]$ by ignoring (i.e., setting to zero) the coefficients for k = 2, 3 and 4, to obtain the sequence of compressed DFT coefficients:

$$\tilde{X}[k] = \{-0.07,\ \ 1.465 - j2.3296,\ \ 0,\ \ 0,\ \ 0,\ \ 1.465 + j2.3296\}.$$

a. Compute the reconstructed sequence $\tilde{x}[n]$ by taking the inverse DFT of $\tilde{X}[k]$.

b. Calculate the root mean squared (RMS) error between $\tilde{x}[n]$ and $x[n]$. Express this RMS error as a percentage of the RMS of $x[n]$.

THE END

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Equations, Tables and Other Information

Unit step function:

$$u[n] = \begin{cases} 0\,, & n < 0 \\ 1\,, & n \geq 0 \end{cases}$$

Unit impulse function:

$$\delta[n] = \begin{cases} 1\,, & n = 0 \\ 0\,, & n \neq 0 \end{cases}$$

Trigonometric identities:

$$\begin{aligned} e^{j(\omega n + \phi)} &= \cos(\omega n + \phi) + j\sin(\omega n + \phi) \\ \cos(\omega n + \phi) &= \frac{1}{2}\{e^{j(\omega n+\phi)} + e^{-j(\omega n+\phi)}\} \\ \sin(\omega n + \phi) &= \frac{1}{2j}\{e^{j(\omega n+\phi)} - e^{-j(\omega n+\phi)}\} \end{aligned}$$

Linear convolution:

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

***N*-point circular convolution:**

$$x[n] \circledast y[n] = \sum_{m=0}^{N-1} x[m]\, y[(n-m) \bmod N]$$

Autoregressive Moving Average (ARMA) difference equation:

$$\sum_{k=0}^{N} a[k]\, y[n-k] = \sum_{k=0}^{M} b[k]\, x[n-k]$$

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Discrete-Time Fourier Transform (DTFT):

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \qquad \text{DTFT}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega \qquad \text{inverse DTFT}$$

***z*–Transform:**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \qquad z\text{–transform}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z)\, z^{n-1}\, dz \qquad \text{inverse } z\text{–transform}$$

Discrete Fourier Transform (DFT):

$$X[k] = \sum_{n=0}^{N-1} x[n]\; e^{-j2\pi \frac{kn}{N}} \qquad \text{DFT}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\; e^{j2\pi \frac{kn}{N}} \qquad \text{inverse DFT}$$

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Phase delay:

$$\Theta(\omega) = -\frac{\angle H(e^{j\omega})}{\omega}$$

Group delay:

$$\tau(\omega) = -\frac{d}{d\omega}\left\{\angle H(e^{j\omega})\right\}$$

Generalized linear phase system:

$$H\left(e^{j\omega}\right) = A\left(e^{j\omega}\right) e^{j(\beta-\omega\alpha)}$$

Impulse invariance transformation:

$$H\left(e^{j\omega}\right) \;=\; H_c\left(j\frac{\omega}{T}\right), \qquad |\omega| \leq \pi$$

$$\Rightarrow h[n] \;=\; T\, h_c(nT)$$

Bilinear transformation:

$$H(z) = H_c(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$\Omega = \frac{2}{T}\tan\left(\omega/2\right) \;\leftrightarrow\; \omega = 2\arctan\left(\Omega T/2\right)$$

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Short-Time Fourier Transform (STFT):

$$X[n,\omega) = \sum_{m=-\infty}^{\infty} x[n+m]\, w[m]\, e^{-j\omega m} \qquad \text{STFT}$$

$$x[n] = \frac{1}{2\pi w[0]} \int_0^{2\pi} X[n,\omega)\, d\omega \qquad \text{inverse STFT}$$

Discrete Short-Time Fourier Transform (Discrete STFT):

$$X[n,k] = \sum_{m=0}^{L-1} x[n+m]\, w[m]\, e^{-j(2\pi/N)km}$$

Spectrogram:

$$S[n,k] = |X[n,k]|^2$$

$$\Rightarrow S_{\text{dB}}[n,k] = 20\log_{10}|X[n,k]| \quad \text{in dB}$$

Periodogram:

$$\widehat{P}_p(\omega) = \frac{1}{L}\left|\sum_{n=0}^{L-1} x[n]\, e^{-j\omega n}\right|^2$$

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Welch Periodogram:

$$\widehat{P}_W(\omega) = \frac{1}{K} \sum_{k=0}^{K-1} \widehat{P}_k(\omega)$$

$$\widehat{P}_k(\omega) = \frac{1}{M} \left| \sum_{n=0}^{M-1} x_k[n] \, e^{-j\omega n} \right|^2$$

Autocorrelation-based spectral estimator:

$$S(\omega) = \sum_{m=-(M-1)}^{M-1} \widehat{\phi}_{xx}[m] \, w_c[m] \, e^{-j\omega m}$$

$$\widehat{\phi}_{xx}[m] = \frac{1}{Q} c_{vv}[m]$$

$$c_{vv}[m] = v[m] * v[-m]$$

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TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Parseval's theorem:	
8. $\displaystyle\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	
9. $\displaystyle\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$	

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n - n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\displaystyle\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\dfrac{1}{1 - ae^{-j\omega}}$
5. $u[n]$	$\dfrac{1}{1 - e^{-j\omega}} + \displaystyle\sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
6. $(n + 1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\dfrac{1}{(1 - ae^{-j\omega})^2}$
7. $\dfrac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\dfrac{1}{1 - 2r\cos\omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\dfrac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\displaystyle\sum_{k=-\infty}^{\omega} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\displaystyle\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi}\delta(\omega + \omega_0 + 2\pi k)]$

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TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
9. $[\cos \omega_0 n]u[n]$	$\dfrac{1-[\cos \omega_0]z^{-1}}{1-[2\cos \omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $[\sin \omega_0 n]u[n]$	$\dfrac{[\sin \omega_0]z^{-1}}{1-[2\cos \omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $[r^n \cos \omega_0 n]u[n]$	$\dfrac{1-[r\cos \omega_0]z^{-1}}{1-[2r\cos \omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $[r^n \sin \omega_0 n]u[n]$	$\dfrac{[r\sin \omega_0]z^{-1}}{1-[2r\cos \omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
4	3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{Re}\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{Im}\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
10	3.4.8	Initial-value theorem:		
		$x[n]=0,\quad n<0$	$\lim_{z\to\infty} X(z)=x[0]$	

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DFT properties:

TABLE 8.2

	Finite-Length Sequence (Length N)	N-point DFT (Length N)
1.	$x[n]$	$X[k]$
2.	$x_1[n]$, $x_2[n]$	$X_1[k]$, $X_2[k]$
3.	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
4.	$X[n]$	$Nx[((-k))_N]$
5.	$x[((n-m))_N]$	$W_N^{km} X[k]$
6.	$W_N^{-\ell n} x[n]$	$X[((k-\ell))_N]$
7.	$\sum_{m=0}^{N-1} x_1(m) x_2[((n-m))_N]$	$X_1[k] X_2[k]$
8.	$x_1[n] x_2[n]$	$\frac{1}{N} \sum_{\ell=0}^{N-1} X_1(\ell) X_2[((k-\ell))_N]$
9.	$x^*[n]$	$X^*[((-k))_N]$
10.	$x^*[((-n))_N]$	$X^*[k]$
11.	$\mathcal{R}e\{x[n]\}$	$X_{\text{ep}}[k] = \frac{1}{2}\{X[((k))_N] + X^*[((-k))_N]\}$
12.	$j\mathcal{I}m\{x[n]\}$	$X_{\text{op}}[k] = \frac{1}{2}\{X[((k))_N] - X^*[((-k))_N]\}$
13.	$x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x^*[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
14.	$x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x^*[((-n))_N]\}$	$j\mathcal{I}m\{X[k]\}$
Properties 15–17 apply only when $x[n]$ is real.		
15.	Symmetry properties	$\begin{cases} X[k] = X^*[((-k))_N] \\ \mathcal{R}e\{X[k]\} = \mathcal{R}e\{X[((-k))_N]\} \\ \mathcal{I}m\{X[k]\} = -\mathcal{I}m\{X[((-k))_N]\} \\ \lvert X[k]\rvert = \lvert X[((-k))_N]\rvert \\ \sphericalangle\{X[k]\} = -\sphericalangle\{X[((-k))_N]\} \end{cases}$
16.	$x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
17.	$x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x[((-n))_N]\}$	$j\mathcal{I}m\{X[k]\}$