COMP ENG 4TL4:

Digital Signal Processing

Notes for Lecture #10

Tuesday, September 30, 2003

3.3 More on Sampling Theory

Relating the DTFT to the CTFT:

Recall from Lecture #2 that the impulse-train approximation $x_s(t)$ of a sampled continuous-time signal $x_c(t)$ has the Fourier transform:

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j\,(\Omega - k\Omega_s))\,,$$

where T is the sampling period, Ω is the continuous-time frequency in radians/s, and Ω_s $(= 2\pi f_s)$ is the angular sampling frequency in radians/s.

Consequently, $X_s(j\Omega)$ consists of copies of $X_c(j\Omega)$ scaled by $1/T$ and shifted by $k\Omega_s$.

Recall from the derivation of the DTFT in Lecture #8:

$$X_s(j\Omega) \;=\; \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n T} \qquad \text{and}$$

$$X\!\left(e^{j\omega}\right) \;=\; \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \qquad (\Leftarrow \text{DTFT})$$

It follows that:

$$X_s(j\Omega) = X\!\left(e^{j\omega}\right)\Big|_{\omega=\Omega T} = X\!\left(e^{j\Omega T}\right).$$

Consequently:

$$X\left(e^{j\Omega T}\right) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j\,(\Omega - k\Omega_s))\,,$$

or equivalently:

$$X\left(e^{j\omega}\right) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right).$$

From these equations we can see that the DTFT $X(e^{j\omega})$ is simply a frequency-scaled version of $X_s(j\Omega)$, with the frequency scaling specified by $\omega = \Omega T$.

This scaling can alternatively be viewed as a normalization of the frequency axis so that $\Omega = \Omega_s$ in $X_s(j\Omega)$ is normalized to $\omega = 2\pi$ radians in $X(e^{j\omega})$.

Resampling of discrete-time sequences:

Consider a discrete-time sequence $x[n]$ obtained by sampling a continuous time sequence $x_c(t)$ with sampling period T, i.e.:

$$x[n] = x_c(nT) .$$

It is often necessary to change the sampling rate of a discrete-time signal, such that:

$$x'[n] = x_c\left(nT'\right) ,$$

where $T' \neq T$.

One approach: reconstruct the continuous-time signal and resample it with period T'.

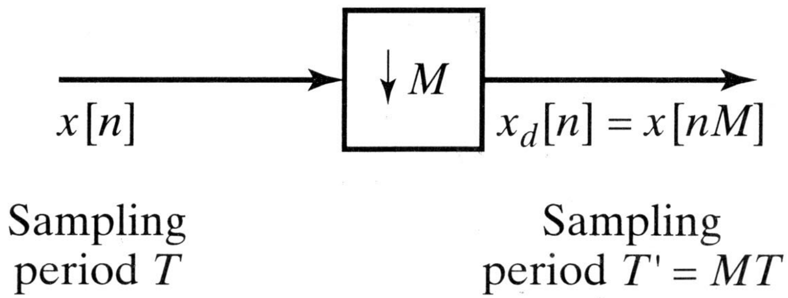
Problem: nonideal reconstruction filters, D/A converters and A/D converters.

Thus: need methods of changing the sampling rate that involve only discrete-time operations.

Sampling rate reduction by an integer factor:

$$x_d[n] = x[nM] = x_c(nMT)$$

decimation or downsampling



(Opppenheim and Schafer)

Figure 4.20 Representation of a compressor or discrete-time sampler.

To avoid aliasing, the signal $x[n]$ should be bandlimited to $\Omega_N < \pi/T$ radians/s ($\equiv \omega_N < \pi$ radians) $\Rightarrow$

For the decimated signal, an M-times lower cutoff frequency $\Omega_{\mathrm{d},N} < \pi/MT$ radians/s ($\equiv \omega_{\mathrm{d},N} < \pi/M$ radians) is required.

That is, aliasing can be avoided if:

- the original sampling rate was $\geq M$ times the Nyquist rate, or
- the bandwidth of the sequence is reduced by a factor of M by a discrete-time filter before downsampling.

The DTFT of $x_d[n] = x[nM] = x_c(nT')$ is:

$$X_d\left(e^{j\omega}\right) = \frac{1}{T'} \sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T'} - \frac{2\pi r}{T'}\right)\right).$$

Since $T' = MT$:

$$\begin{aligned} X_d\left(e^{j\omega}\right) &= \frac{1}{MT} \sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi r}{MT}\right)\right) \\ &= \frac{1}{M} \sum_{i=0}^{M-1} \left[\frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega - 2\pi i}{MT} - \frac{2\pi k}{T}\right)\right)\right] \\ &= \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j(\omega/M - 2\pi i/M)}\right). \end{aligned}$$

That is:

– $X(e^{j\omega})$ consists of copies of $X_c(j\Omega)$ scaled by $1/T$, and frequency scaled by $1/T$ and shifted by $2\pi k$, and

– $X_d(e^{j\omega})$ consists of copies of $X(e^{j\omega})$ scaled by $1/M$, and frequency scaled by $1/M$ and shifted by $2\pi i$.

Downsampling example #1: $(M = 2)$

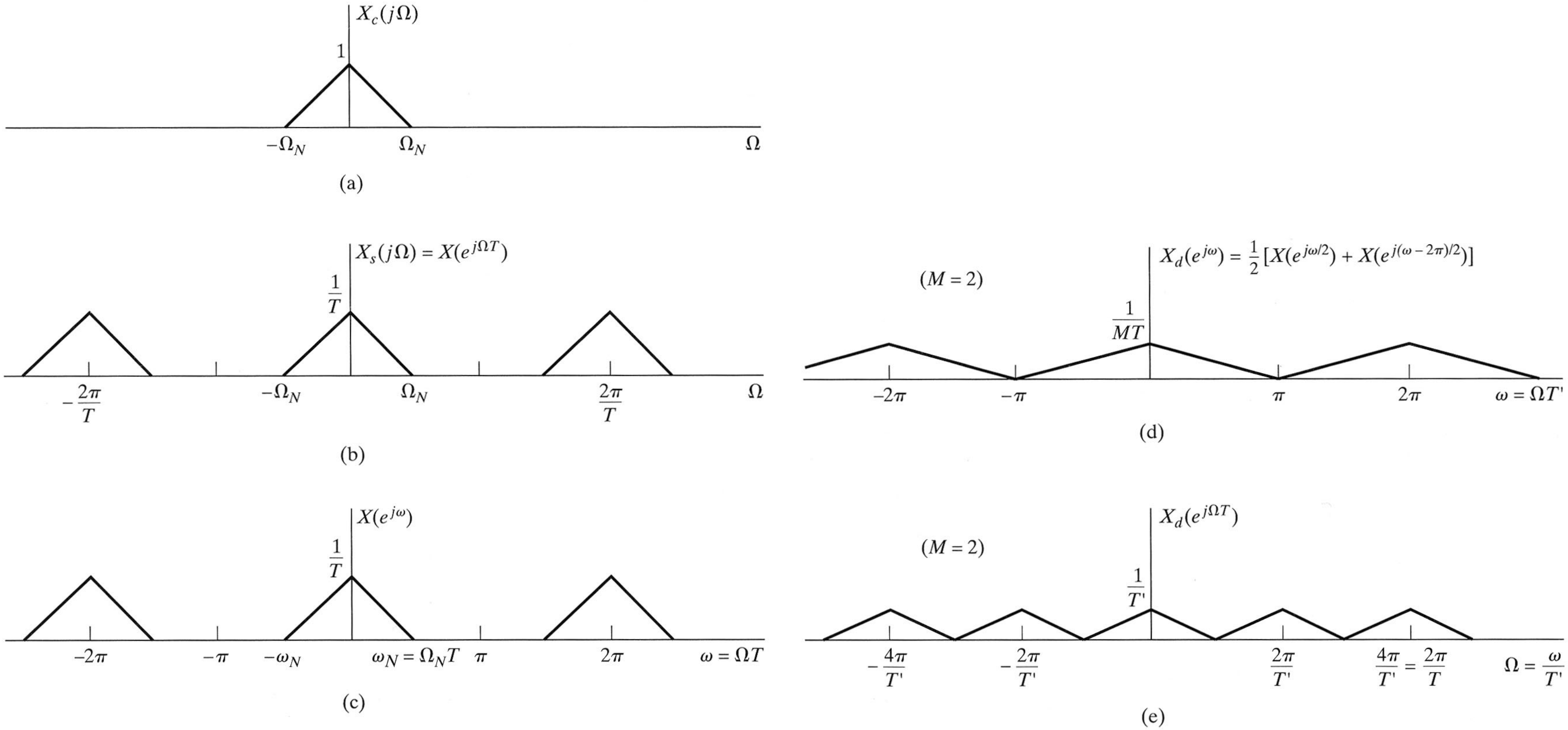


Figure 4.21 Frequency-domain illustration of downsampling.

(Opppenheim and Schafer)

Downsampling example #2: $(M = 3)$

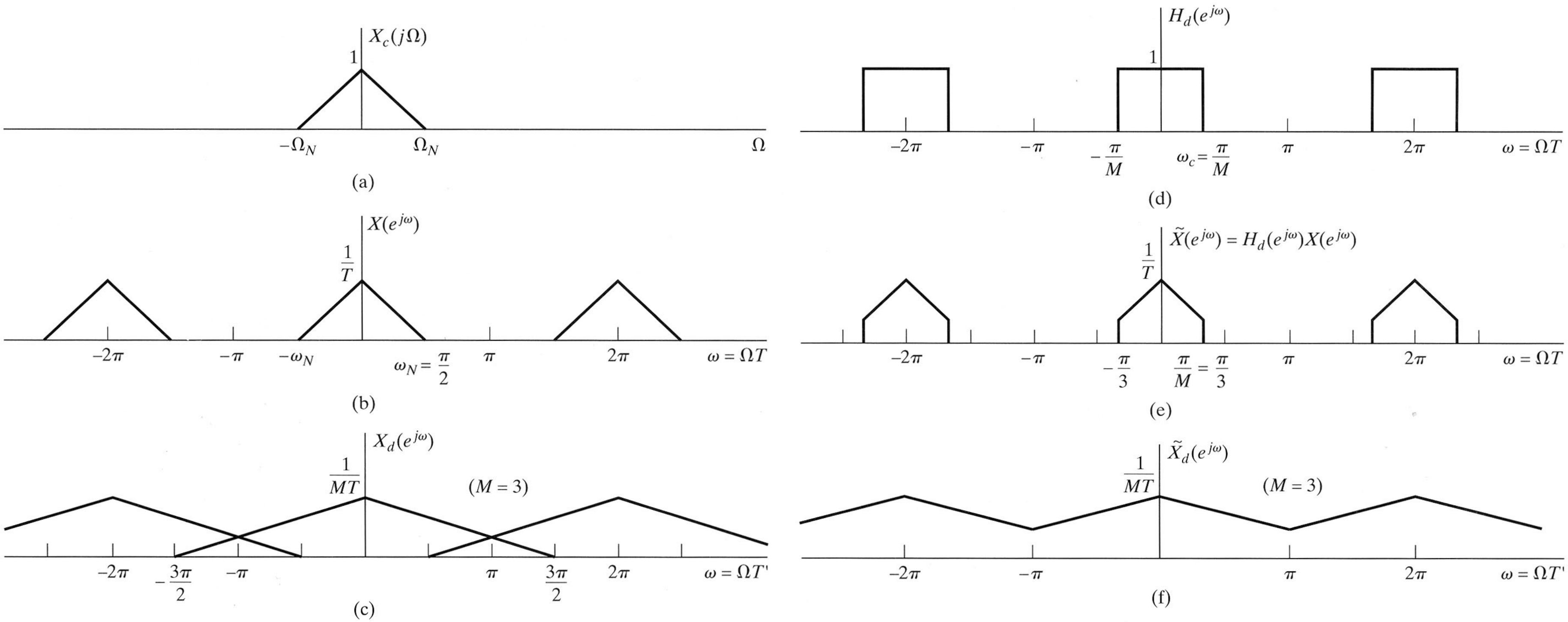


Figure 4.22 (a)–(c) Downsampling with aliasing. (d)–(f) Downsampling with prefiltering to avoid aliasing.

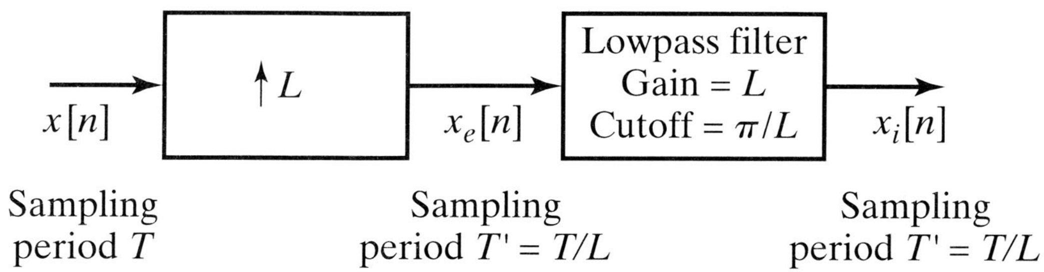
(Opppenheim and Schafer)

Increasing the sampling rate by an integer factor:

$$x_i[n] = x[n/L] = x_c(nT/L)$$

interpolation
or upsampling

where $n = 0,\ \pm L,\ \pm 2L, \ldots$



(Opppenheim and Schafer)

Figure 4.24 General system for sampling rate increase by L.

$$x_e[n] = \begin{cases} x[n/L]\,, & n = 0, \pm L, \pm 2L, \ldots, \\ 0, & \text{otherwise}, \end{cases}$$

expander

or equivalently:

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n - kL]\,.$$

Viewing the upsampling operation in the frequency domain:

$$X_e\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k]\, \delta[n-kL] \right) e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega Lk} = X\left(e^{j\omega L}\right),$$

we observe that the Fourier transform at the output of the expander is a frequency-scaled version of the input, i.e., ω is replaced by ωL so that ω is now normalized by $\omega = \Omega T'$.

As illustrated on the next slide, $X_i(e^{j\omega})$ can be obtained from $X_e(e^{j\omega})$ by correcting the amplitude scale from $1/T$ to $1/T'$ and by removing all the frequency-scaled images of $X_c(j\Omega)$ except at integer multiples of 2π, via a discrete-time lowpass filter.

Upsampling example: $(L = 2)$

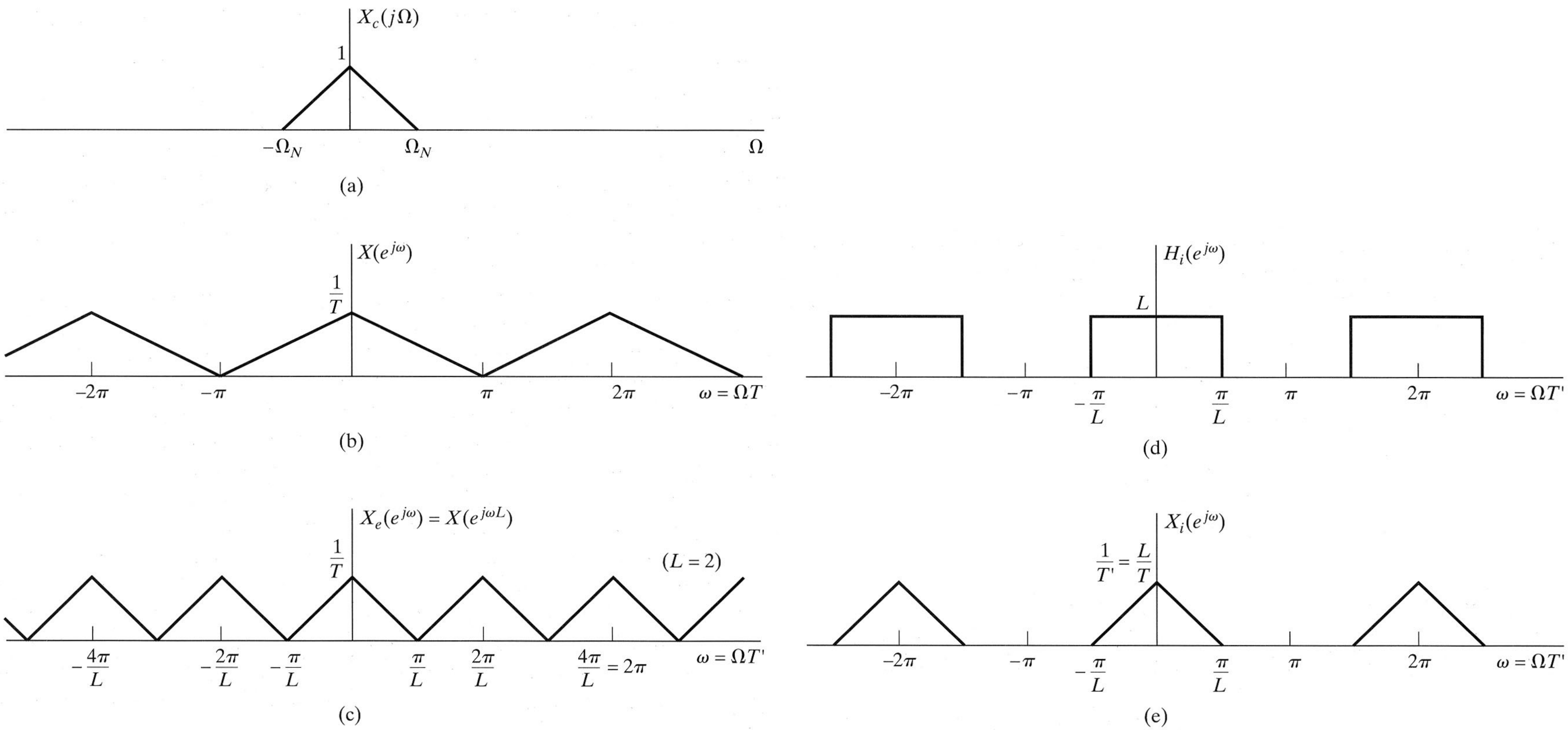


Figure 4.25 Frequency-domain illustration of interpolation.

(Opppenheim and Schafer)

As was the case for the lowpass reconstruction filter in D/A conversion, this discrete-time lowpass filter can be viewed as an *interpolator* in the time domain, with the impulse response:

$$h_i[n] = \frac{\sin(\pi n/L)}{\pi n/L}.$$

In practice, an ideal lowpass filter cannot be implemented exactly, but very good approximations can be designed.

In some cases, very simple interpolation processes are adequate, such as *linear interpolation*, which has the impulse response:

$$h_{\mathrm{lin}}[n] = \begin{cases} 1 - |n|/L, & |n| \le L, \\ 0, & \text{otherwise.} \end{cases}$$

Linear interpolation is illustrated on the next slide.

<u>Linear interpolation example:</u> $(L = 5)$

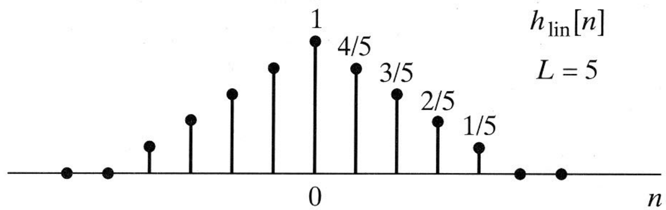
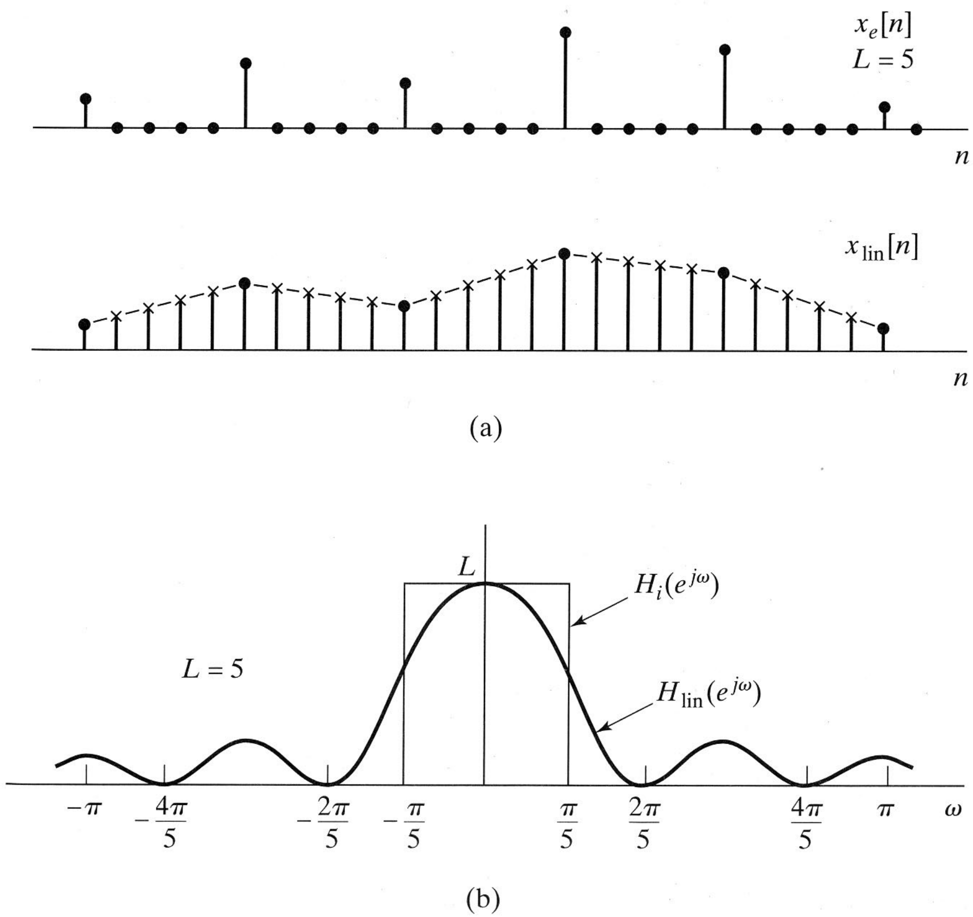


Figure 4.26 Impulse response for linear interpolation.



(Opppenheim and Schafer)

Figure 4.27 (a) Illustration of linear interpolation by filtering. (b) Frequency response of linear interpolator compared with ideal lowpass interpolation filter.

Changing the sampling rate by a noninteger factor:

It is possible to combine decimation and interpolation to change the sampling rate by a noninteger, rational factor L/M.

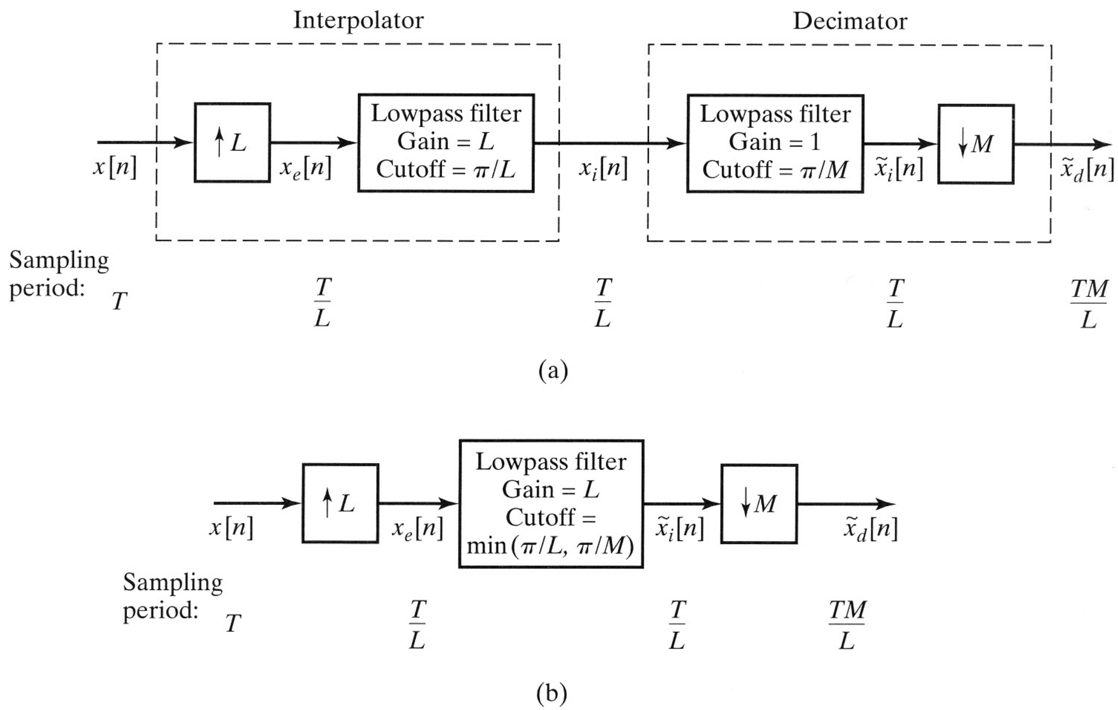


Figure 4.28 (a) System for changing the sampling rate by a noninteger factor. (b) Simplified system in which the decimation and interpolation filters are combined.

(Opppenheim and Schafer)

For example, if $L = 101$ and $M = 100$, then the sampling rate will be increased by a factor of 1.01.