COMP ENG 4TL4:

Digital Signal Processing

Notes for Lecture #12

Friday, October 3, 2003

4.3 The Inverse z-Transform

We found previously that:

$$X(z)\Big|_{z=re^{j\omega}} = \mathcal{F}\left\{x[n]\,r^{-n}\right\}.$$

Applying the inverse-DTFT gives:

$$\begin{aligned} x[n] &= r^n \mathcal{F}^{-1}\left\{X\left(re^{j\omega}\right)\right\} \\ &= r^n \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(re^{j\omega}\right) e^{j\omega n}\, d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\underbrace{re^{j\omega}}_{=z})(\underbrace{re^{j\omega}}_{=z})^n\, d\omega \\ &= \frac{1}{2\pi j}\oint X(z)\, z^{n-1}\, dz \qquad \leftarrow \qquad dz = jre^{j\omega}\, d\omega. \end{aligned}$$

This equation describes the <u>inverse</u> z-transform.

Remarks on the inverse z-transform:

- $\oint \cdots \, dz$ denotes integration around a closed circular contour centered at the origin and having the radius r.

- The value of r must be chosen so that the contour of integration $|z| = r$ belongs to the ROC.

- Contour integration in the complex plane may be a complicated task.

- Simpler alternative procedures exist for obtaining the sequence from its z-transform.

4.4 Alternative Methods for the Inverse z-Transform

Inspection method: Consists simply of becoming familiar with (or recognizing "by inspection") certain transform pairs.

Examples:

$$a^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

$$\delta[n-m] \leftrightarrow z^{-m}, \quad \begin{array}{l} z \neq 0 \ \text{ if } \ m > 0, \\ z \neq \infty \ \text{ if } \ m < 0 \end{array}$$

$$\sin(\omega n)\, u[n] \leftrightarrow \frac{[\sin \omega] z^{-1}}{1 - [2\cos\omega] z^{-1} + z^{-2}}, \quad |z| > 1$$

Extended inspection method: consists of expressing a complicated $X(z)$ as a *sum of simpler terms* and then applying the inspection method to each term.

Example:

$$
\begin{aligned}
X(z) &= z^2\left(1-0.5z^{-1}\right)\left(1+z^{-1}\right)\left(1-z^{-1}\right) \\
&= z^2-0.5z-1+0.5z^{-1} \qquad \Rightarrow
\end{aligned}
$$

$$
x[n] = \delta[n+2]-0.5\delta[n+1]-\delta[n]+0.5\delta[n-1]
$$

4.5 Properties of the z-Transform

Linearity:

$$\text{If} \quad X(z) = \mathcal{Z}\{x[n]\} \quad \text{and} \quad Y(z) = \mathcal{Z}\{y[n]\}$$

$$\text{then} \quad a\,X(z) + b\,Y(z) = \mathcal{Z}\{a\,x[n] + b\,y[n]\}\,,$$

$$\text{with ROC} = \text{ROC}_x \;\cap\; \text{ROC}_y.$$

$$\text{Also, if} \quad x[n] = \mathcal{Z}^{-1}\{X(z)\} \text{ and } y[n] = \mathcal{Z}^{-1}\{Y(z)\}$$

$$\text{then} \quad a\,x[n] + b\,y[n] = \mathcal{Z}^{-1}\{a\,X(z) + b\,Y(z)\}\,.$$

Time shifting:

If $\quad X(z) = \mathcal{Z}\{x[n]\} \quad$ then $\quad X(z)\,z^{-m} = \mathcal{Z}\{x[n-m]\}\,,$
with $\mathrm{ROC} = \mathrm{ROC}_x\,,$ except for possible addition/deletion of $z = 0$ or $z = \infty$.

Also, if $\quad x[n] = \mathcal{Z}^{-1}\{X(z)\}\,,$
then $x[n-m] = \mathcal{Z}^{-1}\Big\{X(z)\,z^{-m}\Big\}\,.$

Example: for $|z| > 0.25$, consider:

$$
\begin{aligned}
X(z) &= \frac{1}{z - 0.25} = z^{-1}\left(\frac{1}{1 - 0.25z^{-1}}\right) \\
&= z^{-1}\mathcal{Z}\{0.25^n u[n]\}
\end{aligned}
$$

$$
\Rightarrow \quad x[n] = 0.25^{n-1}u[n-1]\,.
$$

Multiplication by an exponential sequence:

If $\quad X(z) = \mathcal{Z}\{x[n]\} \quad$ then $\quad X(z/z_0) = \mathcal{Z}\{x[n]\, z_0^n\}\,,$

with $\;\mathrm{ROC} = |z_0|\; \mathrm{ROC}_x.$

Also, if $\quad x[n] = \mathcal{Z}^{-1}\{X(z)\}$ then,

$$x[n]\, z_0^n = \mathcal{Z}^{-1}\{X(z/z_0)\}\,.$$

Differentiation of $X(z)$:

If $\quad X(z) = \mathcal{Z}\{x[n]\}$ then $\; -\, z\, \dfrac{dX(z)}{dz} = \mathcal{Z}\{n\, x[n]\}\,,$

with $\;\mathrm{ROC} = \mathrm{ROC}_x.$

Also, if $\quad x[n] = \mathcal{Z}^{-1}\{X(z)\}$ then,

$$n\, x[n] = \mathcal{Z}^{-1}\left\{-z\, \frac{dX(z)}{dz}\right\}.$$

Example: Starting with the known transform pair:

$$u[n] \leftrightarrow \frac{1}{1-z^{-1}}\,, \qquad |z| > |1|\,,$$

determine $X(z)$ of:

$$\begin{aligned} x[n] &= 2r^n \cos(\omega n) u[n] \\ &= \left(re^{j\omega}\right)^n u[n] + \left(re^{-j\omega}\right)^n u[n]\,. \end{aligned}$$

Using the exponential multiplication property, we have:

$$\begin{aligned} \left(re^{j\omega}\right)^n u[n] &\leftrightarrow \frac{1}{1-z^{-1}re^{j\omega}}\,, \qquad |z| > |r|\,, \\ \left(re^{-j\omega}\right)^n u[n] &\leftrightarrow \frac{1}{1-z^{-1}re^{-j\omega}}\,, \qquad |z| > |r|\,. \end{aligned}$$

Using the linearity property, we obtain:

$$X(z) = \frac{1}{1-z^{-1}re^{j\omega}} + \frac{1}{1-z^{-1}re^{-j\omega}}\,, \qquad |z| > |r|\,.$$

Conjugation of a complex sequence:

If $\quad X(z) = \mathcal{Z}\{x[n]\} \quad$ then $\quad X^*(z^*) = \mathcal{Z}\{x^*[n]\}\,,$

with $\ \mathrm{ROC} = \mathrm{ROC}_x.$

Also, if $\quad x[n] = \mathcal{Z}^{-1}\{X(z)\}$ then,

$$x^*[n] = \mathcal{Z}^{-1}\{X^*(z^*)\}\,.$$

Time reversal:

If $\quad X(z) = \mathcal{Z}\{x[n]\} \quad$ then $\quad X(1/z) = \mathcal{Z}\{x[-n]\}\,,$

with $\ \mathrm{ROC} = 1/\mathrm{ROC}_x.$

Also, if $\quad x[n] = \mathcal{Z}^{-1}\{X(z)\}$ then,

$$x[-n] = \mathcal{Z}^{-1}\{X(1/z)\}\,.$$

Example: Consider the sequence:

$$x[n] = a^{-n}u[-n]\,,$$

which is a time-reversed version of:

$$y[n] = a^{n}u[n] \quad \leftrightarrow \quad Y(z) = \frac{1}{1-az^{-1}}\,, \quad |z| > |a|\,.$$

From the time reversal property:

$$\begin{aligned} X(z) &= Y(1/z) \\ &= \frac{1}{1-az}\,, \qquad |z| < \left|a^{-1}\right|. \end{aligned}$$

Convolution of sequences:

If $\quad X(z) = \mathcal{Z}\{x[n]\} \quad$ and $\quad Y(z) = \mathcal{Z}\{y[n]\}\,,$

then $\quad X(z)\,Y(z) = \mathcal{Z}\{x[n] * y[n]\}\,,$

with $\text{ROC} = \text{ROC}_x \ \cap \ \text{ROC}_y$.

Also, if $\quad x[n] = \mathcal{Z}^{-1}\{X(z)\}$ and $y[n] = \mathcal{Z}^{-1}\{Y(z)\}\,,$

then $\quad x[n] * y[n] = \mathcal{Z}^{-1}\{X(z)\,Y(z)\}\,.$

Example: Evaluate the convolution of:

$$x[n] = a^n u[n] \quad \text{and} \quad y[n] = u[n]\,,$$

for $|a| < 1$. The z-transforms are:

$$X(z) = \frac{1}{1 - az^{-1}}\,, \qquad |z| > a$$

$$Y(z) = \frac{1}{1 - z^{-1}}\,, \qquad |z| > 1 \qquad \Rightarrow$$

$$\begin{aligned} \mathcal{Z}\{x[n] * y[n]\} &= X(z)\,Y(z) = \frac{z^2}{(z-a)\,(z-1)} \\ &= \frac{1}{1-a}\left(\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right), \qquad |z| > 1. \end{aligned}$$

Using the linearity property and the standard z-transform pairs, we find that:

$$x[n]*y[n] = \frac{1}{1-a}\left(u[n] - a^{n+1}u[n]\right), \qquad |z| > 1.$$

Pole-zero ROC plot for the z-transform of the convolution of the sequences $u[n]$ and $a^{n+1}u[n]$: