COMP ENG 4TL4:

Digital Signal Processing

Notes for Lecture #14

Wednesday, October 8, 2003

5. THE DISCRETE FOURIER TRANSFORM AND FAST FOURIER TRANSFORM

5.1 The Discrete Fourier Transform (DFT)

Recall the DTFT:

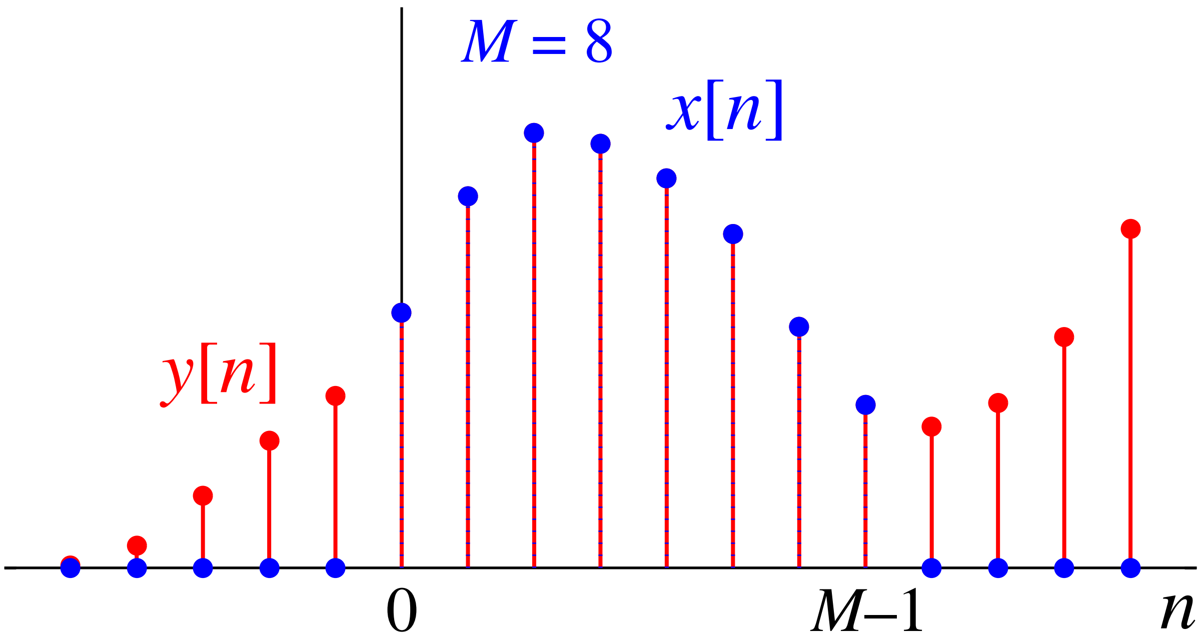
$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}.$$

The DTFT is not suitable for a practical DSP because:

- in any DSP application, we are able to store only a finite number of samples, and
- we are able to compute the spectrum only at specific discrete values of ω.

A finite sequence $x[n]$ that is M samples long can be obtained from a longer sequence $y[n]$ by applying a rectangular window of length M:

$$x[n] = \begin{cases} 0\,, & n < 0, \\ y[n]\,, & 0 \le n \le (M-1)\,, \\ 0\,, & n \ge M. \end{cases}$$

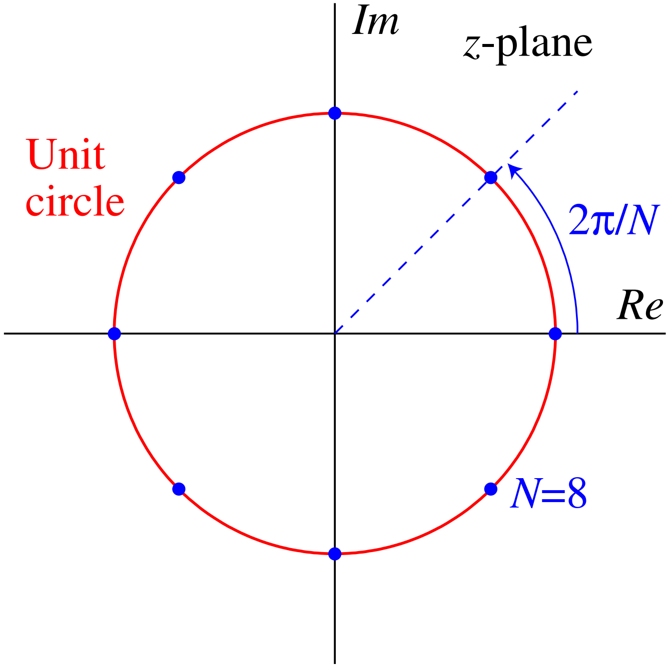


Let us sample the spectrum $X(e^{j\omega})$ in the frequency domain at N points:

$$X[k] = X\left(e^{jk\Delta\omega}\right), \qquad \Delta\omega = \frac{2\pi}{N}.$$

If $N = M$, then:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi\frac{kn}{N}} \qquad \text{DFT}$$



The inverse DFT is given by:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\ e^{j2\pi\frac{kn}{N}}.$$

Proof:

$$x[n] \;=\; \frac{1}{N} \sum_{k=0}^{N-1} \left\{ \sum_{m=0}^{N-1} x[m]\ e^{-j2\pi\frac{km}{N}} \right\} e^{j2\pi\frac{kn}{N}}$$

$$=\; \sum_{m=0}^{N-1} x[m] \underbrace{\left\{ \frac{1}{N} \sum_{k=0}^{N-1} e^{-j2\pi\frac{k(m-n)}{N}} \right\}}_{\delta[m-n]} \;=\; x[n]\,.$$

The DFT analysis and synthesis equations:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi\frac{kn}{N}} \qquad \text{analysis}$$

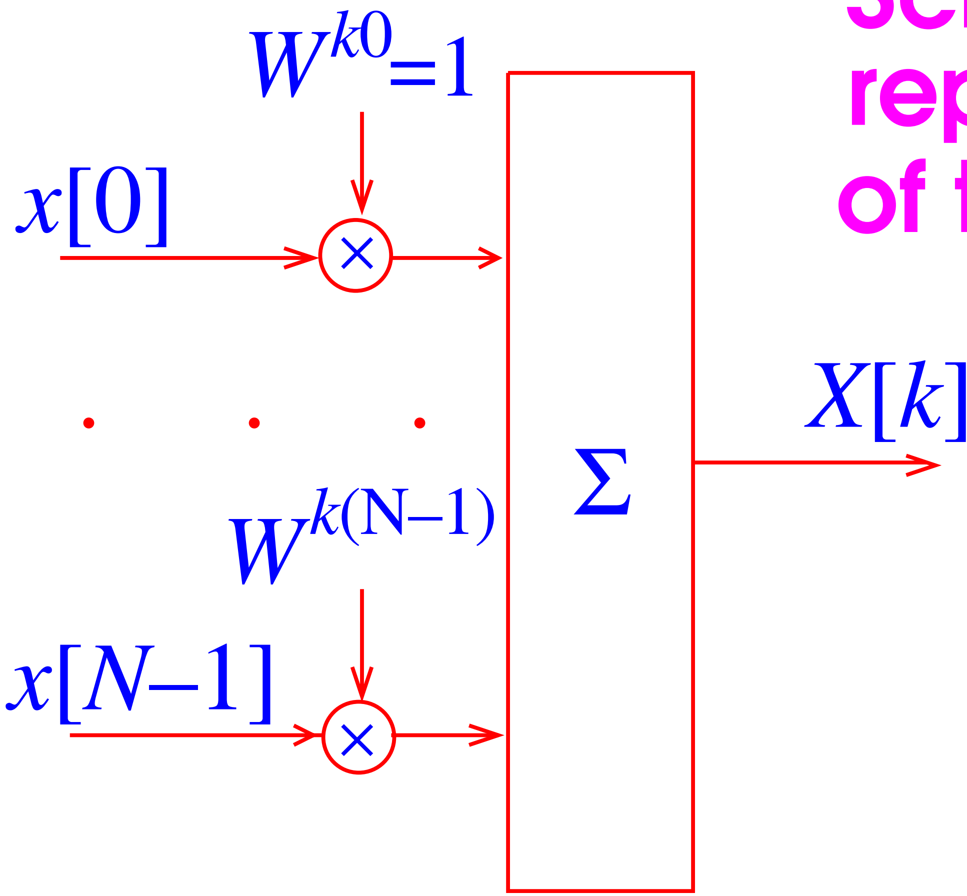
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j2\pi\frac{kn}{N}} \qquad \text{synthesis}$$

Alternative formulation:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W^{kn} \qquad \longleftarrow W = e^{-j\frac{2\pi}{N}}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, W^{-kn}$$

Schematic representation of the DFT



An important property of the DFT spectrum:

$$
\begin{aligned}
X[k+N] &= \sum_{n=0}^{N-1} x[n]\ e^{-j2\pi\frac{(k+N)n}{N}} \\
&= \left(\sum_{n=0}^{N-1} x[n]\ e^{-j2\pi\frac{kn}{N}}\right) e^{-j2\pi n} \\
&= X[k]\ e^{-j2\pi n} = X[k] \qquad \Rightarrow
\end{aligned}
$$

The DFT spectrum $X[k]$ is *periodic* with period N — recall that the DTFT spectrum is periodic as well, but with period 2π.

Example: For a rectangular pulse $x[n]$ of length M:

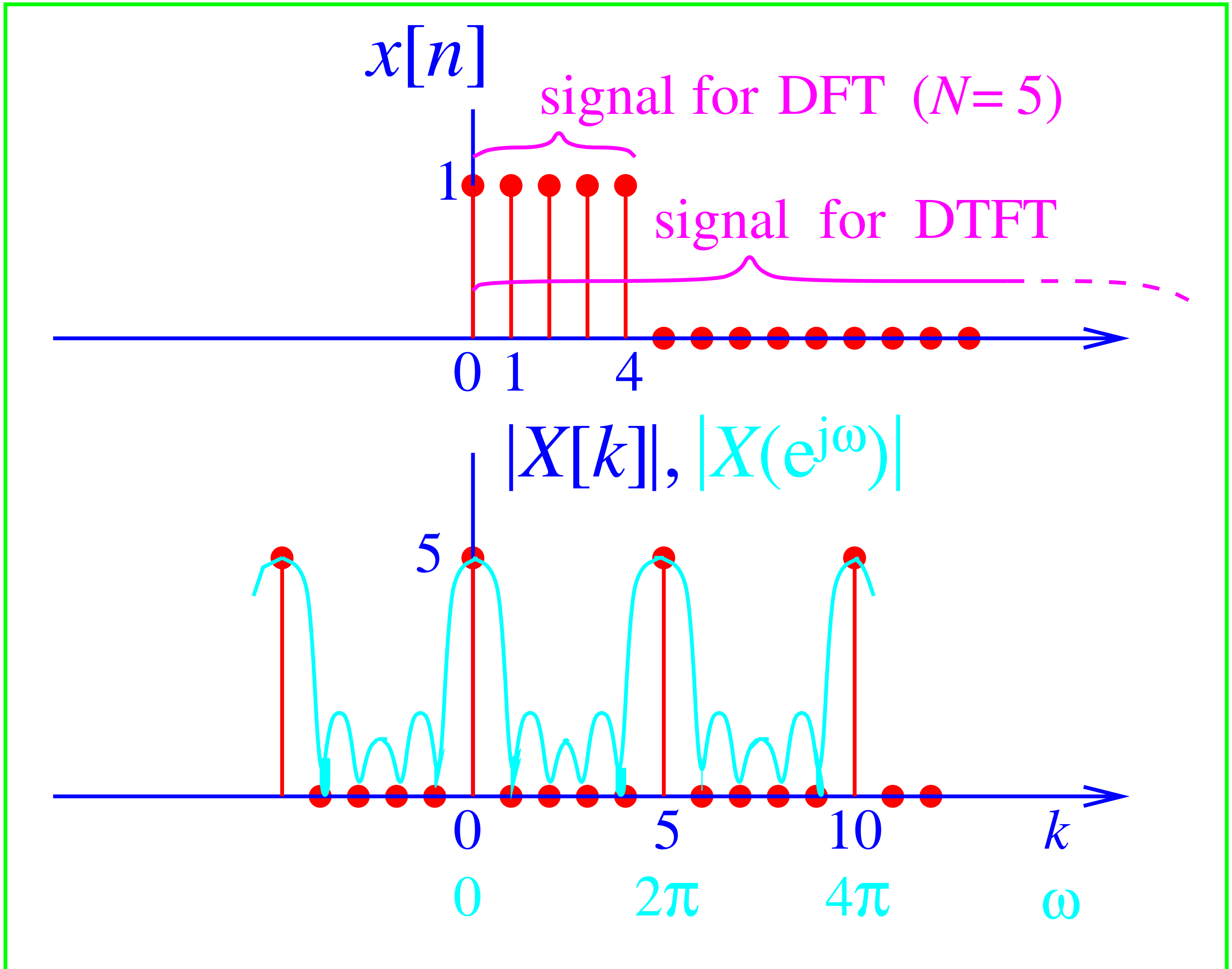
$$x[n] = \begin{cases} 1\,, & 0 \le n \le (M-1), \\ 0\,, & \text{otherwise,} \end{cases}$$

the N-point DFT, if $N = M$, is:

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]\; e^{-j2\pi\frac{kn}{N}} \\ &= N \sum_{i=-\infty}^{\infty} \delta[k+iN]\,. \end{aligned}$$

That is, the rectangular pulse is “interpreted” by the DFT as a spectral line at $k = 0$ ($\rightarrow \omega = 0$), and because of the periodicity of the DFT, spectral lines also appear at integer multiples of N.

The DFT and DTFT of a rectangular pulse: $(N = M = 5)$



Example (cont.): What happens with the DFT of this rectangular pulse if we increase N, i.e., add several extra zeros at the end of the windowed sequence? That is, let the DFT sequence be:

$$\{x_p[n]\} = \{x[0], \ldots, x[M-1], \underbrace{0, 0, \ldots, 0}_{N-M \text{ positions}} \},$$

where $x[0] = \ldots = x[M-1] = 1$.

This operation is referred to as zero-padding.

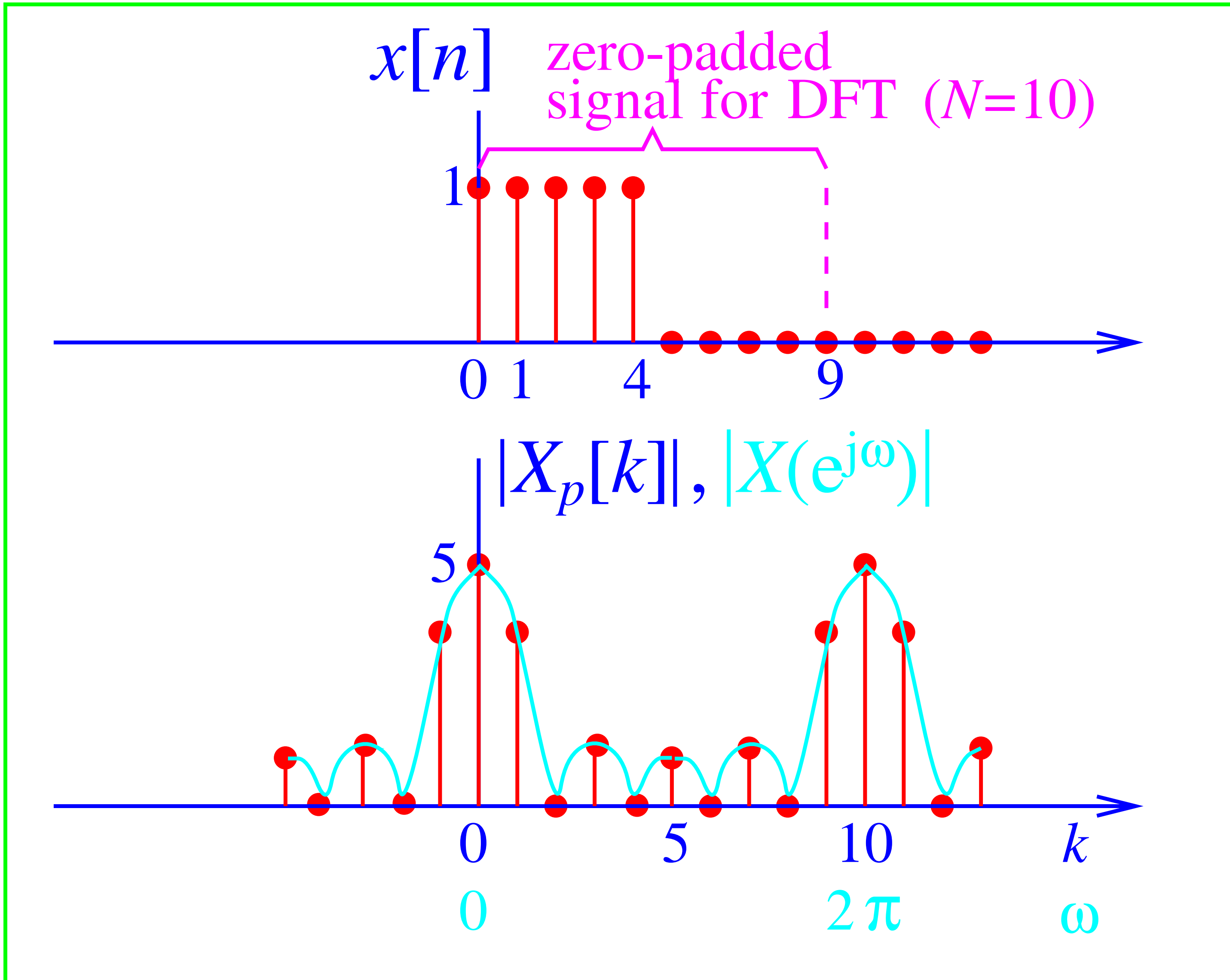
The zero-padded sequence $x_p[n]$ has the DFT:

$$X_p[k] = \sum_{n=0}^{N-1} x_p[n]\, e^{-j2\pi\frac{kn}{N}} = \sum_{n=0}^{M-1} e^{-j2\pi\frac{kn}{N}}$$

$$= \frac{\sin\left(\pi\frac{kM}{N}\right)}{\sin\left(\pi\frac{k}{N}\right)} e^{-j\pi\frac{k(M-1)}{N}}.$$

The DFT and DTFT of a rectangular pulse with zero-padding:

$(N = 10;\ M = 5)$



Properties of the DFT of zero-padded sequences:

Property 1: Using more and more zero-padding on a windowed sequence, we are able to “approximate” its DTFT better and better.

Property 2: Zero-padding cannot improve the resolution of spectral components because the resolution is proportional to the 1/M (where M is the length of the observation window) rather than 1/N (where N is the number of points in the DFT).

(We will return to the issue of spectral resolution in the next lecture.)

Remark: Zero-padding will be a very important tool for a fast implementation of the DFT.