COMP ENG 4TL4:

Digital Signal Processing

Notes for Lecture #16

Tuesday, October 14, 2003

5.3 Matrix Formulation of the DFT

Introduce the $N \times 1$ vectors:

$$\mathbf{x} = \left[x[0], x[1], \ldots, x[N-1]\right]^T,$$
$$\mathbf{X} = \left[X[0], X[1], \ldots, X[N-1]\right]^T,$$

where $X[k]$ is the DFT of the sequence $x[n]$,

and the $N \times N$ matrix:

$$\mathbf{W} = \begin{bmatrix} W^0 & W^0 & W^0 & \ldots & W^0 \\ W^0 & W^1 & W^2 & \ldots & W^{N-1} \\ W^0 & W^2 & W^4 & \ldots & W^{2(N-1)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ W^0 & W^{N-1} & W^{2(N-1)} & \ldots & W^{(N-1)^2} \end{bmatrix},$$

where $W = e^{-j2\pi/N}$.

<u>The DFT can be expressed in the matrix form:</u>

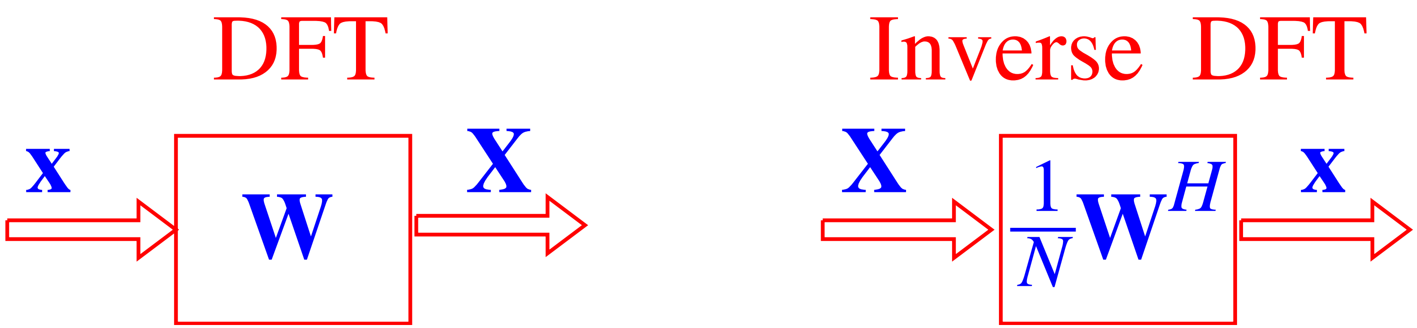
$$\mathbf{X} = \mathbf{W}\mathbf{x}.$$

Likewise, the inverse DFT can be given by:

$$\mathbf{x} = \frac{1}{N}\mathbf{W}^H\mathbf{X},$$

where the operator $\{\cdot\}^H$ indicates the Hermitian (or complex conjugate) transpose.

<u>Proof:</u> Elementary, using the fact that $\mathbf{W}^H\mathbf{W} = \mathbf{W}\mathbf{W}^H = N\,\mathbf{I}$, where $\mathbf{I}$ is the identity matrix.



5.4 Interpretation of the DFT via the Discrete Fourier Series (DFS)

Construct a periodic sequence by "repeating" the finite sequence $x[n]$, $n = 0,\ldots,N-1$:

$$\{\tilde{x}[n]\} = \{\ldots, \underbrace{x[0], \ldots, x[N-1]}_{=\{x[n]\}}, \underbrace{x[0], \ldots, x[N-1]}_{=\{x[n]\}}, \ldots\}$$

The discrete version of the Fourier Series can be written as:

$$\begin{aligned} \tilde{x}[n] &= \sum_k X_k \, e^{j2\pi\frac{kn}{N}} \\ &= \frac{1}{N} \sum_k \tilde{X}[k] \; e^{j2\pi\frac{kn}{N}}, \end{aligned}$$

where $\tilde{X}[k] = N X_k$.

Remarking that:

$$W^{-kn} = e^{j2\pi\frac{kn}{N}} = e^{j2\pi\frac{(k+mN)n}{N}} = W^{-(k+mN)n},$$

for integer values of m, we obtain that the summation in the Discrete Fourier Series (DFS) should contain only N terms:

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]\; e^{j2\pi\frac{kn}{N}} \qquad \text{DFS}$$

The DFS coefficients are given by:

$$\tilde{X}[k] = \sum_{k=0}^{N-1} \tilde{x}[n]\; e^{-j2\pi\frac{kn}{N}} \qquad \text{inverse DFS}$$

Proof:

$$\sum_{n=0}^{N-1} \tilde{x}[n]\ e^{-j2\pi\frac{kn}{N}}$$

$$= \sum_{n=0}^{N-1} \left\{ \frac{1}{N} \sum_{p=0}^{N-1} \tilde{X}[p]\ e^{j2\pi\frac{pn}{N}} \right\} e^{-j2\pi\frac{kn}{N}}$$

$$= \sum_{p=0}^{N-1} \tilde{X}[p] \underbrace{\left\{ \frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi\frac{(p-k)n}{N}} \right\}}_{\delta[p-k]} = \tilde{X}[k]$$

The DFS analysis and synthesis equations are:

$$\tilde{X}[k] = \sum_{k=0}^{N-1} \tilde{x}[n]\; e^{-j2\pi \frac{kn}{N}} \qquad \text{analysis}$$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k]\; e^{j2\pi \frac{kn}{N}} \qquad \text{synthesis}$$

Remarks:

- The DFS and DFT analysis and synthesis equations are identical except for the fact that the DFT is applied to a finite (nonperiodic) sequence $x[n]$, whereas the DFS is applied to a periodic sequence $\tilde{x}[n]$.
- The conventional (continuous-time) FS represent a periodic signal using an *infinite number* of complex exponentials, whereas the DFS represent such a signal using a *finite number* of complex exponentials.

5.5 Properties of the DFT

Linearity:

If $X[k] = \mathcal{DFT}\{x[n]\}$ and $Y[k] = \mathcal{DFT}\{y[n]\}$,

then $a\,X[k] + b\,Y[k] = \mathcal{DFT}\{a\,x[n] + b\,y[n]\}$,

where the lengths of both sequences should be equalized by means of zero-padding.

Also, if $x[n] = \mathcal{DFT}^{-1}\{X[k]\}$ and $y[n] = \mathcal{DFT}^{-1}\{Y[k]\}$,

then $a\,x[n]+b\,y[n] = \mathcal{DFT}^{-1}\{a\,X[k] + b\,Y[k]\}$.

Circular shift of a sequence:

If $X[k] = \mathcal{DFT}\{x[n]\}$,

then $X[k]\, e^{-j2\pi\frac{km}{N}} = \mathcal{DFT}\{x[(n-m) \bmod N]\}$.

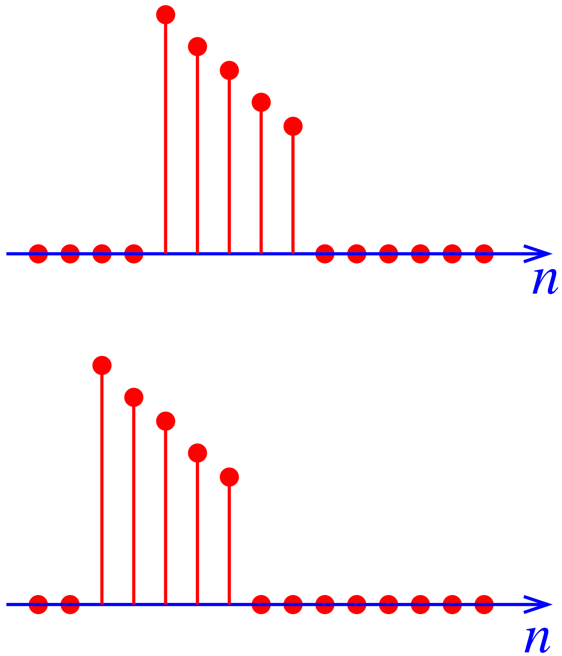
Also, if $x[n] = \mathcal{DFT}^{-1}\{X[k]\}$,

then $x[(n-m) \bmod N] = \mathcal{DFT}^{-1}\left\{X[k]\, e^{-j2\pi\frac{km}{N}}\right\}$,

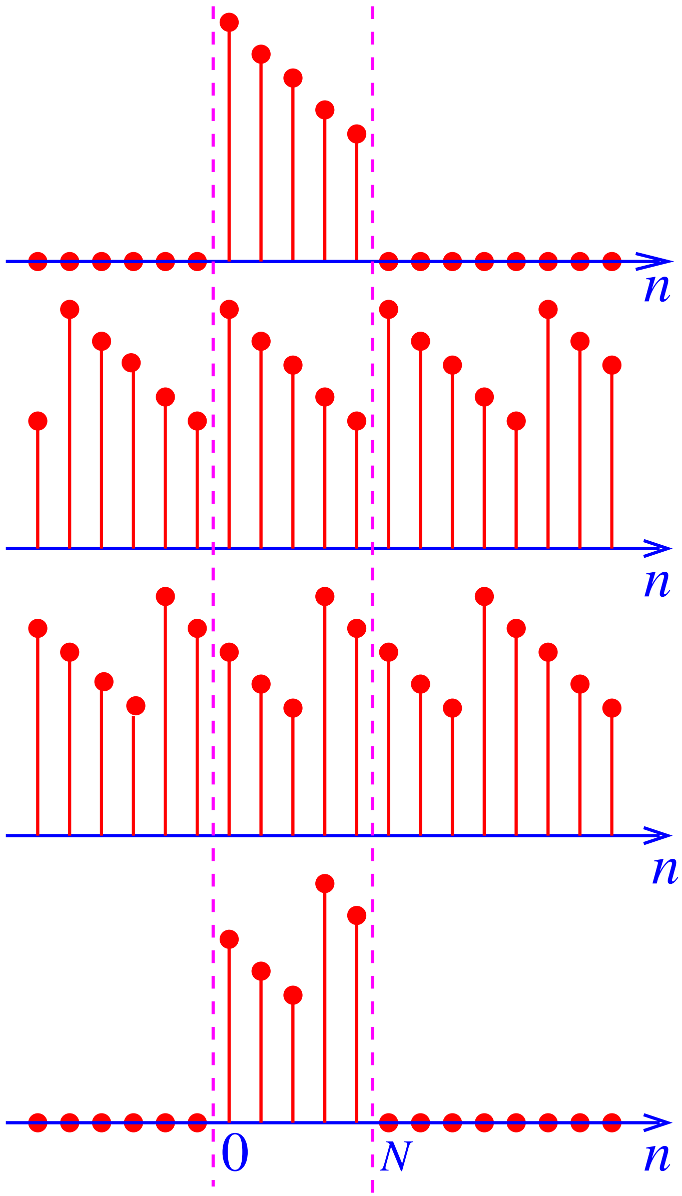
where the operation $\mathrm{mod}\ N$ is exploited for denoting the periodic extension $\tilde{x}[n]$ of the signal $x[n]$:

$$\tilde{x}[n] = x[n \bmod N].$$

conventional shift



circular shift



Proof of the circular shift property:

$$\sum_{n=0}^{N-1} x[(n-m) \bmod N]\, W^{kn}$$

$$= \sum_{n=0}^{N-1} x[(n-m) \bmod N]\, W^{k(n-m+m)}$$

$$= W^{km} \sum_{n=0}^{N-1} x[(n-m) \bmod N]\, W^{k(n-m)}$$

$$= W^{km} \sum_{n=0}^{N-1} x[(n-m) \bmod N]$$

$$\cdot\, W^{k((n-m) \bmod N)} = W^{km} X[k]\,,$$

where we use the facts that $W^{k(l \bmod N)} = W^{kl}$ and that the order of summation in the DFT does not change its result.

Example of the DFT circular time-shifting property: $m = 1$