COMP ENG 4TL4:

Digital Signal Processing

Notes for Lecture #17

Wednesday, October 15, 2003

Frequency shift (modulation):

If $\quad X[k] = \mathcal{DFT}\{x[n]\}\,,$

then $\quad X[(k-m) \bmod N] = \mathcal{DFT}\left\{x[n]\, e^{j2\pi\frac{mn}{N}}\right\}.$

Also, if $\quad x[n] = \mathcal{DFT}^{-1}\{X[k]\}\,,$

then $x[n]\, e^{j2\pi\frac{mn}{N}} = \mathcal{DFT}^{-1}\{X[(k-m) \bmod N]\}\,.$

Proof: Similar to that for the circular shift property.

Parseval theorem:

$$\sum_{n=0}^{N-1} x[n]\, y^*[n] \;=\; \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, Y^*[k] \qquad \text{general form}$$

$$\sum_{n=0}^{N-1} |x[n]|^2 \;=\; \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2 \qquad \text{specific form}$$

Proof: Using the matrix formulation of the DFT, we obtain:

$$\begin{aligned} \mathbf{y}^H\mathbf{x} &= \left(\frac{1}{N}\mathbf{W}^H\mathbf{Y}\right)^H \left(\frac{1}{N}\mathbf{W}^H\mathbf{X}\right) \\ &= \frac{1}{N^2}\mathbf{Y}^H \underbrace{\mathbf{W}\mathbf{W}^H}_{=N\mathbf{I}}\mathbf{X} = \frac{1}{N}\mathbf{Y}^H\mathbf{X}. \end{aligned}$$

Conjugation:

If $\quad X[k] = \mathcal{DFT}\{x[n]\}$,

then $\quad X^*[(N-k) \bmod N] = \mathcal{DFT}\{x^*[n]\}$.

Also, if $\quad x[n] = \mathcal{DFT}^{-1}\{X[k]\}$,

then $\quad x^*[n] = \mathcal{DFT}^{-1}\{X^*[(N-k) \bmod N]\}$.

Proof:

$$\sum_{n=0}^{N-1} x^*[n]\, W^{kn}$$

$$= \left[\sum_{n=0}^{N-1} x[n]\, W^{-kn}\right]^* = \left[\sum_{n=0}^{N-1} x[n]\, \underbrace{W^{(-k \bmod N)n}}_{=W^{-kn}}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x[n]\, W^{[(N-k) \bmod N]n}\right]^* = X^*[(N-k) \bmod N]\,.$$

Circular convolution:

If $X[k] = \mathcal{DFT}\{x[n]\}$ and $Y[k] = \mathcal{DFT}\{y[n]\}$,

then $X[k]\,Y[k] = \mathcal{DFT}\{x[n] \circledast y[n]\}$.

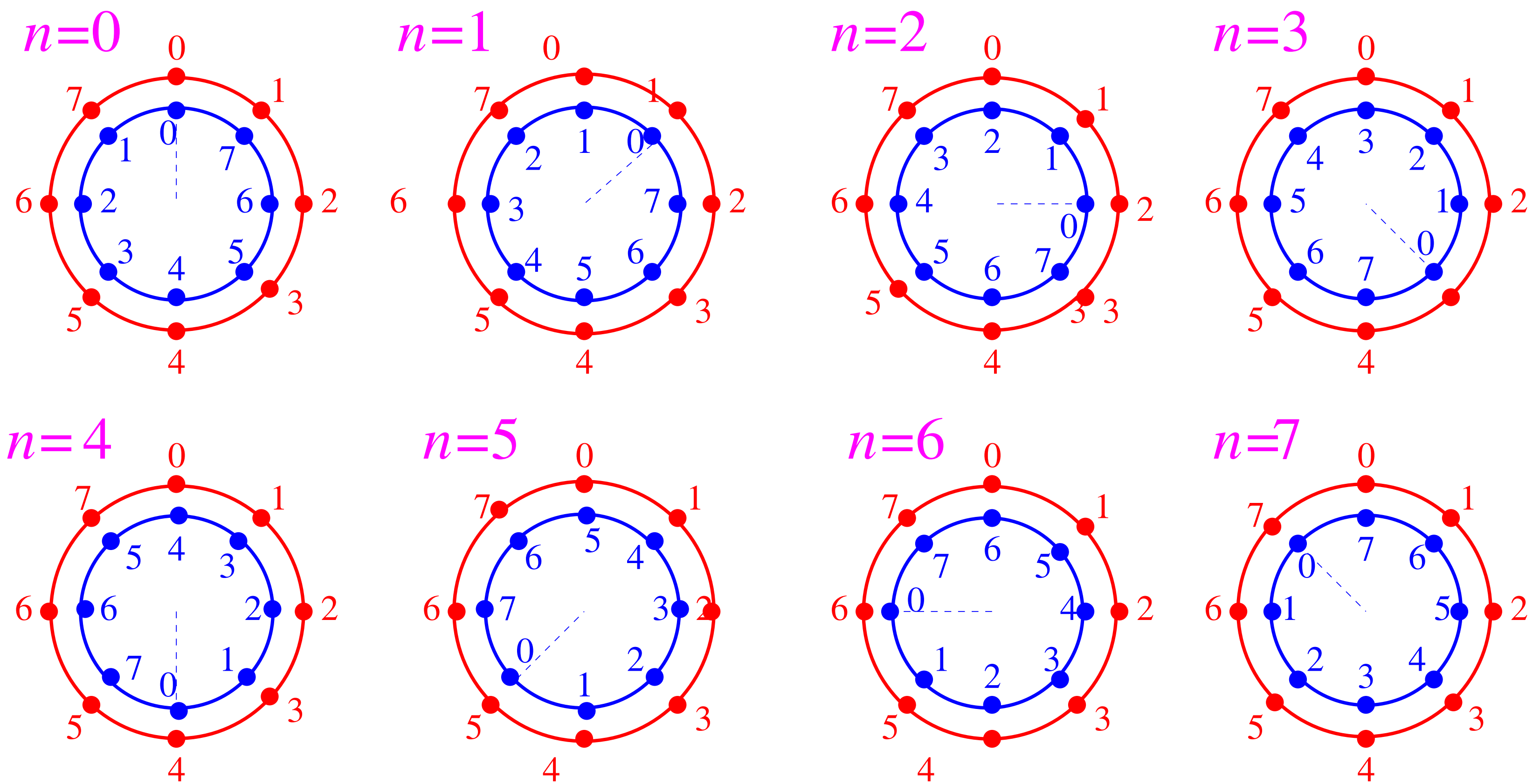
Also, if $x[n] = \mathcal{DFT}^{-1}\{X[k]\}$ and $y[n] = \mathcal{DFT}^{-1}\{Y[k]\}$,

then $x[n] \circledast y[n] = \mathcal{DFT}^{-1}\{X[k]\,Y[k]\}$.

Here $\circledast$ stands for circular convolution, defined by:

$$x[n] \circledast y[n] = \sum_{m=0}^{N-1} x[m]\, y[(n-m) \bmod N]\,.$$

Illustration of circular convolution for $N = 8$:



• - $x[n]$ spread clockwise

• - $y[n]$ spread counterclockwise

Example #1: Consider the circularly convolved sequences:

$$
\begin{aligned}
x[n] &= \{1, -1, -1, -1, 1, 0, 1, 2\}, \\
y[n] &= \{5, -4, 3, 2, -1, 1, 0, -1\},
\end{aligned}
$$

giving $z[n] = x[n] \circledast y[n]$. Then:

$$
\begin{aligned}
z[0] &= x[0]\,y[0]+x[1]\,y[7]+x[2]\,y[6]+x[3]\,y[5]+x[4]\,y[4] \\
&+ x[5]\,y[3]+x[6]\,y[2]+x[7]\,y[1]=-1
\end{aligned}
$$

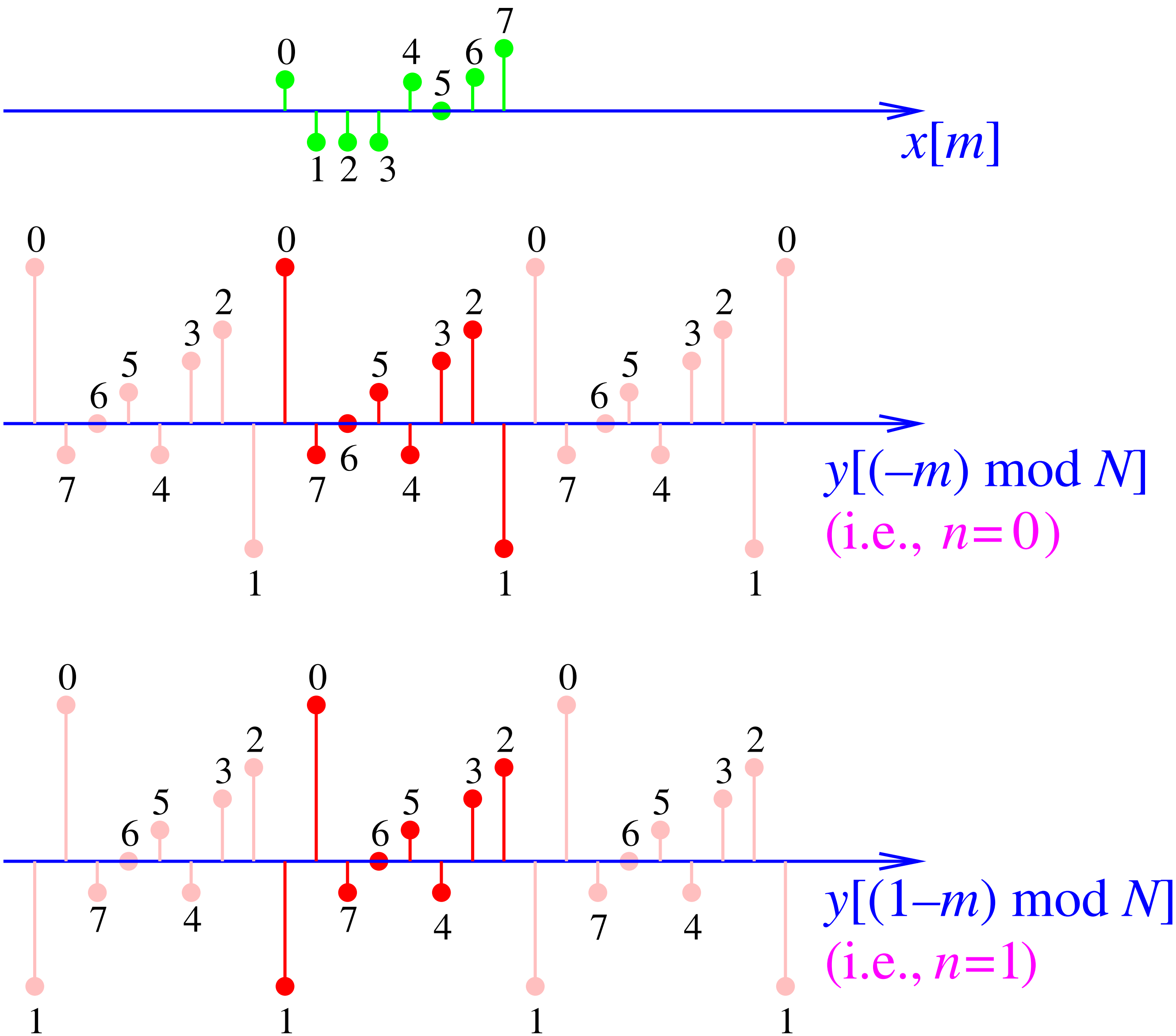
$$
\begin{aligned}
z[1] &= x[0]\,y[1]+x[1]\,y[0]+x[2]\,y[7]+x[3]\,y[6]+x[4]\,y[5] \\
&+ x[5]\,y[4]+x[6]\,y[3]+x[7]\,y[2]=1
\end{aligned}
$$

$$
\begin{aligned}
z[2] &= x[0]\,y[2]+x[1]\,y[1]+x[2]\,y[0]+x[3]\,y[7]+x[4]\,y[6] \\
&+ x[5]\,y[5]+x[6]\,y[4]+x[7]\,y[3]=6
\end{aligned}
$$

Example #1 (cont.):

$$\begin{aligned}
z[3] &= x[0]\,y[3]+x[1]\,y[2]+x[2]\,y[1]+x[3]\,y[0]+x[4]\,y[7] \\
&+ x[5]\,y[6]+x[6]\,y[5]+x[7]\,y[4] = -4 \\
z[4] &= x[0]\,y[4]+x[1]\,y[3]+x[2]\,y[2]+x[3]\,y[1]+x[4]\,y[0] \\
&+ x[5]\,y[7]+x[6]\,y[6]+x[7]\,y[5] = 5 \\
z[5] &= x[0]\,y[5]+x[1]\,y[4]+x[2]\,y[3]+x[3]\,y[2]+x[4]\,y[1] \\
&+ x[5]\,y[0]+x[6]\,y[7]+x[7]\,y[6] = -8 \\
z[6] &= x[0]\,y[6]+x[1]\,y[5]+x[2]\,y[4]+x[3]\,y[3]+x[4]\,y[2] \\
&+ x[5]\,y[1]+x[6]\,y[0]+x[7]\,y[7] = 4 \\
z[7] &= x[0]\,y[7]+x[1]\,y[6]+x[2]\,y[5]+x[3]\,y[4]+x[4]\,y[3] \\
&+ x[5]\,y[2]+x[6]\,y[1]+x[7]\,y[0] = 7
\end{aligned}$$

Example #1 (cont.): Illustration of the circular convolution process:



Circular convolution and linear convolution:

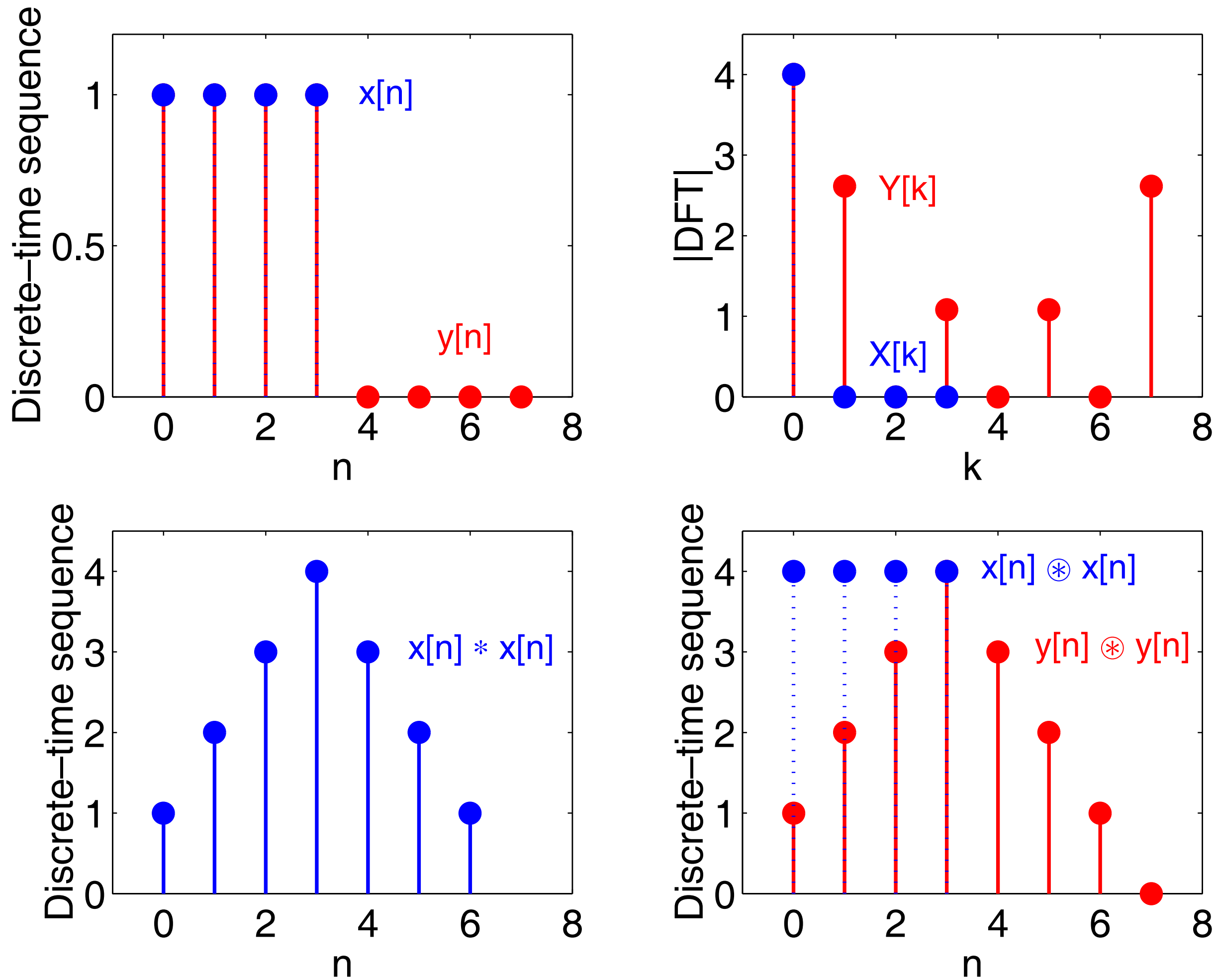
– A consequence of the circular convolution property is that circular convolution in the time domain can be computed efficiently via multiplication in the Fourier domain.

– If two discrete-time sequences of length L and P, respectively, are zero-padded to length N, such that $N \geq L + P - 1$, then the *circular convolution* of the sequences is equal to the *linear convolution* of the sequences.

– If $N < L + P - 1$, then the *circular convolution* of the sequences is a time-aliased version of the *linear convolution* of the sequences.
$\Rightarrow$ **Sampling in the time-domain produces aliasing in the frequency domain, and sampling in the frequency-domain produces aliasing in the time domain!**

– The upper two properties above allow efficient implementation of FIR filters in the Fourier domain on DSPs that have specialized hardware and/or software for computing DFTs.

Example #2: Consider the two discrete-time sequences, (i) $x[n]$: a rectangular pulse of length 4, and (ii) $y[n]$: the sequence $x[n]$ zero-padded to length 8. The circular convolution of $y[n]$ with itself is identical to the linear convolution of $x[n]$ with itself, while the circular convolution of $x[n]$ with itself is a time-aliased version of the linear convolution of $x[n]$ with itself.



Proof of the circular convolution property:

$$\mathcal{DFT}\{x[n] \circledast y[n]\}$$

$$= \sum_{n=0}^{N-1} \underbrace{\left[\sum_{m=0}^{N-1} x[m]\, y[(n-m) \bmod N]\right]}_{=x[n]\circledast y[n]} W^{kn}$$

$$= \sum_{m=0}^{N-1} \underbrace{\left[\sum_{n=0}^{N-1} y[(n-m) \bmod N]\, W^{kn}\right]}_{=Y[k]W^{km}} x[m]$$

$$= \; Y[k] \underbrace{\sum_{m=0}^{N-1} x[m]\, W^{km}}_{=X[k]} = X[k]\, Y[k]\,.$$

Multiplication:

If $X[k] = \mathcal{DFT}\{x[n]\}$ and $Y[k] = \mathcal{DFT}\{y[n]\}$,

$$\text{then} \quad \frac{1}{N} X[k] \circledast Y[k] = \mathcal{DFT}\{x[n]\, y[n]\}\,.$$

Also, if $x[n] = \mathcal{DFT}^{-1}\{X[k]\}$ and $y[n] = \mathcal{DFT}^{-1}\{Y[k]\}$,

$$\text{then} \quad x[n]\, y[n] = \mathcal{DFT}^{-1}\left\{\frac{1}{N} X[k] \circledast Y[k]\right\}.$$

Proof: Similar to that for the circular convolution property.