COMP ENG 4TL4:

Digital Signal Processing

Notes for Lecture #20

Wednesday, October 22, 2003

6.4 The Phase Response and Distortionless Transmission

In most filter applications, the *magnitude response* $|H(e^{j\omega})|$ is of primary concern. However, the *phase response* may also be important:-

$$H\left(e^{j\omega}\right) = \left|H\left(e^{j\omega}\right)\right| e^{j\angle H\left(e^{j\omega}\right)},$$

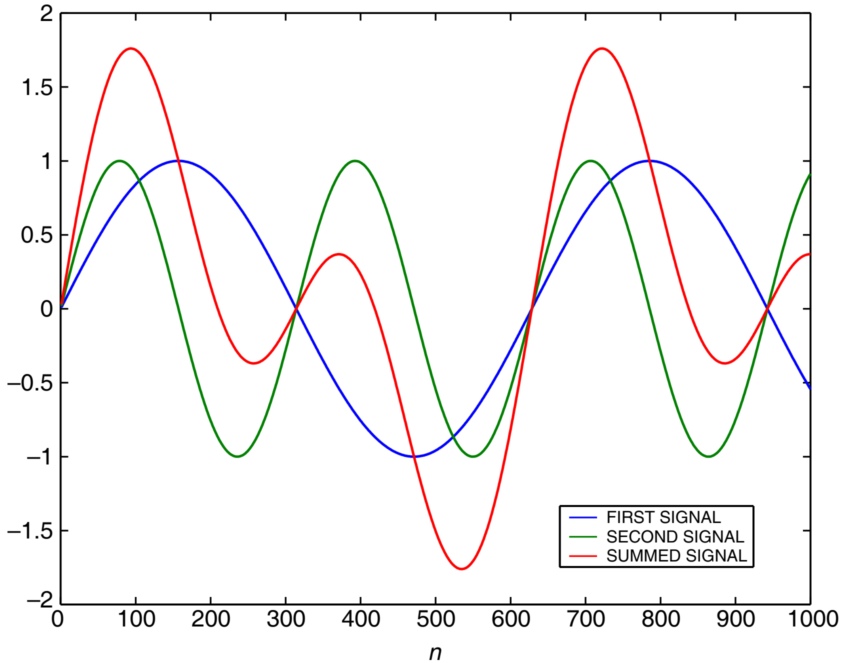
where $\angle H(e^{j\omega})$ is the phase response.

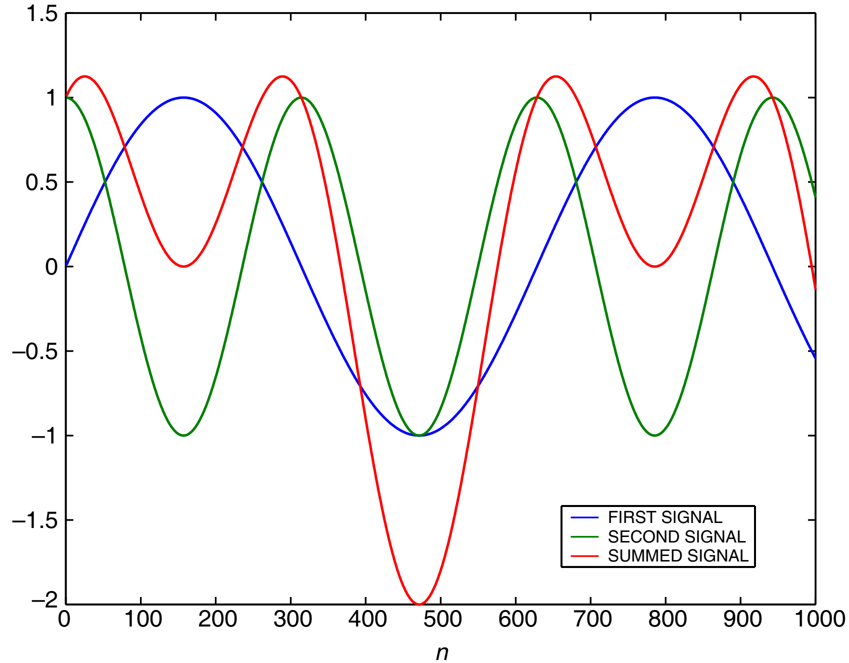
Definition: If a signal is transmitted through a system (filter) then, this system is said to provide a distortionless transmission if the signal form remains unaffected, i.e., if the output signal is a *delayed and scaled replica* of the input signal.

Two conditions of distortionless transmission:

- the system must amplify (or attenuate) each frequency component uniformly, i.e., the magnitude response must be uniform within the signal frequency band; and
- the system must delay each frequency component by the same discrete-time value (i.e., number of samples).

DISTORTED TRANSMISSION





Definition: The phase delay $\Theta(\omega)$ of a filter is the relative delay imposed on a *particular frequency component of an input signal*:

$$\Theta(\omega) = -\frac{\angle H(e^{j\omega})}{\omega},$$

measured in *samples*.

Result: To satisfy the distortionless response phase condition, the *phase delay* must be frequency-independent, i.e., uniform for each frequency $\Rightarrow \Theta(\omega) = \alpha = \mathrm{const}$.

That is, all frequency components will be delayed by α samples.

A filter having this property is called a linear-phase filter because its phase varies linearly with the frequency ω.

Hence in general, a *linear-phase frequency response* is given by:

$$H\left(e^{j\omega}\right) = \left|H\left(e^{j\omega}\right)\right| e^{-j\omega\alpha}.$$

<u>Example:</u> Ideal lowpass filter with linear phase:

$$H\left(e^{j\omega}\right) = \begin{cases} e^{-j\omega\alpha}, & |\omega| < \omega_C, \\ 0, & \omega_C < |\omega| < \pi. \end{cases}$$

The corresponding *impulse response* is:

$$h[n] = \frac{\sin\{\omega_C(n-\alpha)\}}{\pi(n-\alpha)}.$$

Taking an <u>integer</u> delay $\alpha = m$, we have:

$$h[2m-n] \;=\; \frac{\sin\{\omega_C(2m-n-m)\}}{\pi(2m-n-m)}$$

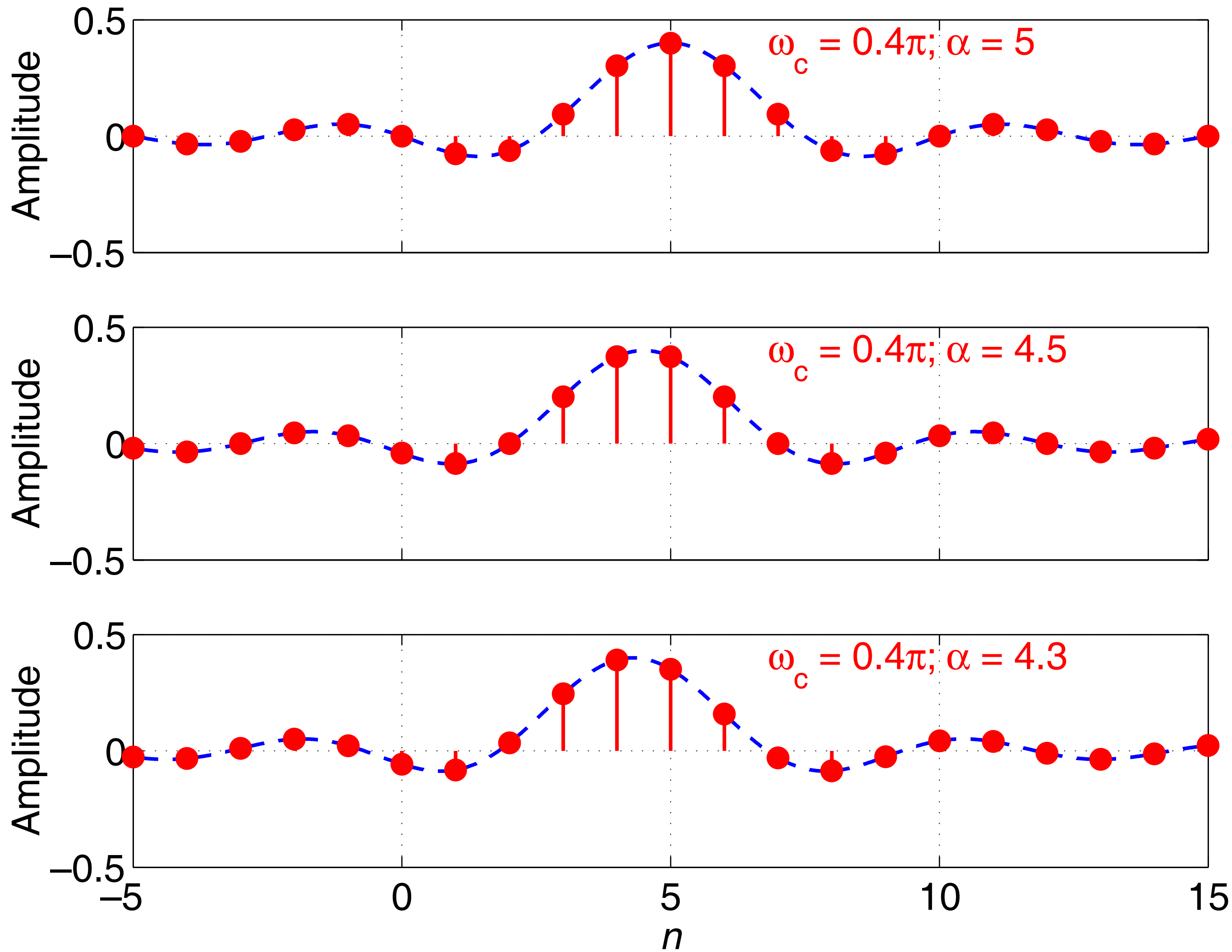
$$= \frac{\sin\{\omega_C(m-n)\}}{\pi(m-n)} \;=\; h[n]\,.$$

Hence, the response is symmetric about $n = m$.

<u>Property:</u> Filters with <u>symmetric</u> impulse responses have *linear phase*.

However, filters with *nonsymmetric* impulse responses <u>may also have linear phase</u>!

Linear-phase filters with varying delay α:



Property: A filter with an impulse responses that is symmetric around the time origin $n = 0$ is a special case of linear-phase filters, i.e., $\alpha = m = 0$, referred to as a zero-phase filter.

Example #1: Ideal *zero-phase* lowpass filter:

$$\tilde{H}\left(e^{j\omega}\right) = \left|H\left(e^{j\omega}\right)\right| e^{j\omega 0} = \left|H\left(e^{j\omega}\right)\right|$$

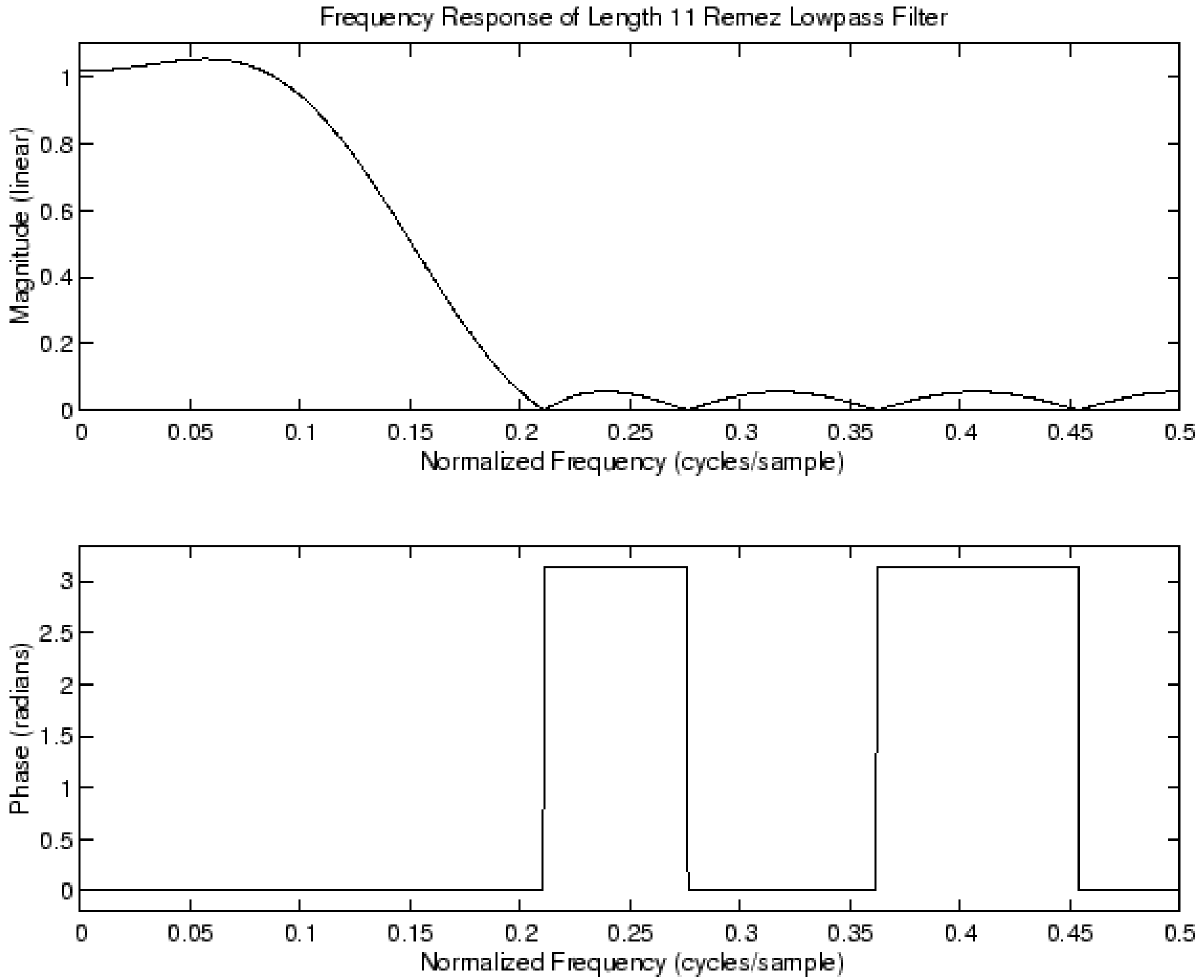
$$\Rightarrow \tilde{h}[n] = \frac{\sin(\omega_C n)}{\pi n}.$$

Example #2: Linear interpolator: $(L = 2)$

$$h[n] = \frac{1}{2}\delta[n+1] + \delta[n] + \frac{1}{2}\delta[n-1]$$

$$\Rightarrow H\left(e^{j\omega}\right) = \frac{1}{2}e^{j\omega} + 1 + \frac{1}{2}e^{-j\omega} = 1 + \cos(\omega).$$

Note: In practice, zero-phase filters are only perfectly zero-phase in their passband. In the stopband, it is possible to have phase reversals of π radians.



Smith, J.O. *Introduction to Digital Filters,* http://www-ccrma.stanford.edu/~jos/filters/ (© 2003).

Definition: A system is referred to as a generalized linear-phase system if its frequency response can be expressed in the form:

$$H\left(e^{j\omega}\right) = A\left(e^{j\omega}\right)\ e^{j(\beta-\omega\alpha)}.$$

with a real function $A(e^{j\omega})$ and $\beta = \mathrm{const}$, $\alpha = \mathrm{const}$.

Note that the *phase delay*:

$$\Theta(\omega) = -\frac{\angle H(e^{j\omega})}{\omega} = \alpha - \frac{\beta}{\omega} \neq \text{const},$$

and consequently, generalized linear-phase systems produce phase distortions.

Example: Consider a discrete-time implementation of the ideal continuous-time differentiator:

$$y_c(t) = \frac{d}{dt} x_c(t) \quad \rightarrow \quad H(j\Omega) = j\Omega.$$

Restrict the input to be bandlimited:

$$\tilde{H}(j\Omega) = \begin{cases} j\Omega\,, & |\Omega| < \pi/T, \\ 0\,, & |\Omega| \geq \pi/T. \end{cases}$$

The corresponding discrete-time system has the frequency response:

$$H\left(e^{j\omega}\right) = \frac{j\omega}{T}\,, \qquad |\omega| < \pi.$$

Hence, $H(e^{j\omega})$ has the form of a generalized linear-phase filter with $\alpha = 0$, $\beta = \pi/2$, and $A(e^{j\omega}) = \omega/T$.

<u>Definition:</u> The <u>group delay</u> $\tau(\omega)$ of a system is:

$$\tau(\omega) = -\frac{d}{d\omega}\left\{\angle H(e^{j\omega})\right\},$$

measures in *samples*.

The *group delay* of a *generalized linear-phase system* is:

$$\tau(\omega) = -\frac{d}{d\omega}\{\beta - \omega\alpha\} = \alpha = \text{const},$$

as is the group delay of a truly linear-phase system.

Generalized linear-phase systems are useful in that they can produce near-distortionless transmission of the envelope of bandlimited signals.

<u>Property:</u> Filters with antisymmetry, i.e.:

$$h[2m-n] = -h[n],$$

have *generalized linear phase*.

Causal filters and (generalized) linear-phase:

- Causal FIR filters can be linear-phase or generalized linear-phase. In filter design, the symmetry or antisymmetry property is typically utilized to ensure linear-phase or generalized linear-phase, respectively.
- Causal FIR filters cannot be zero-phase, except for the trivial case of $h[n] = c\delta[n]$, where c is a constant.
- Causal IIR filters can be generalized linear-phase only for systems with nonrational system functions, so they are practically never used in filter design.