COMP ENG 4TL4:

# Digital Signal Processing

## Notes for Lecture #21

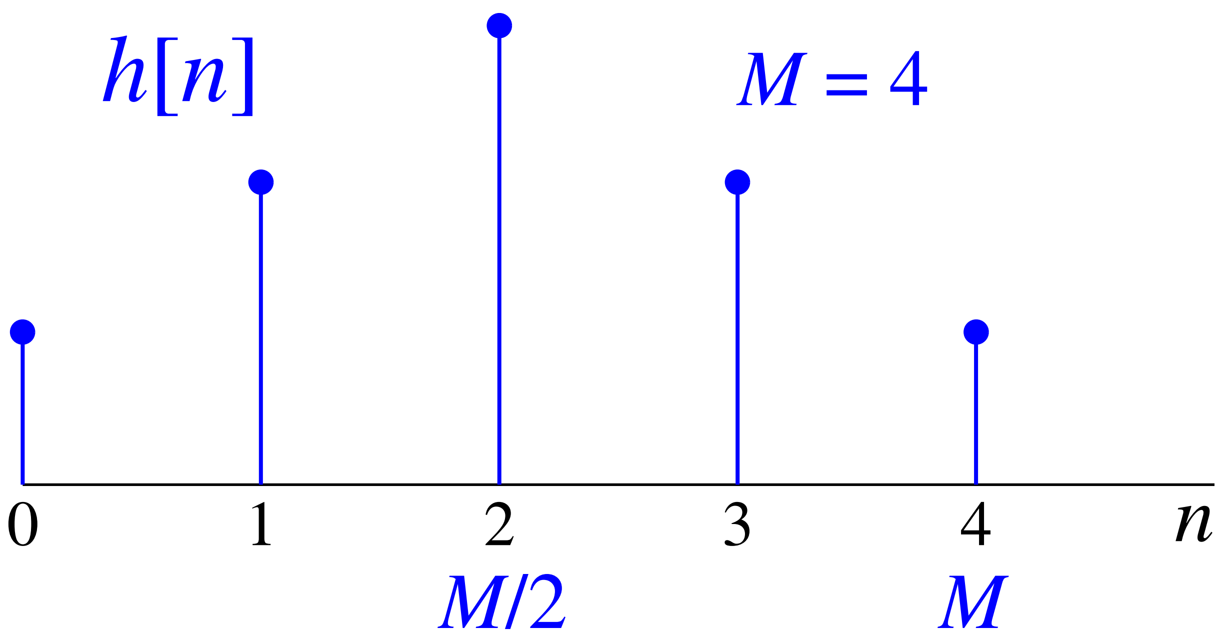
Friday, October 24, 2003

## Types of causal FIR (generalized) linear-phase filters:

Type I: Symmetric impulse response:

$$h[n] = h[M-n]\,, \qquad 0 \leq n \leq M,$$

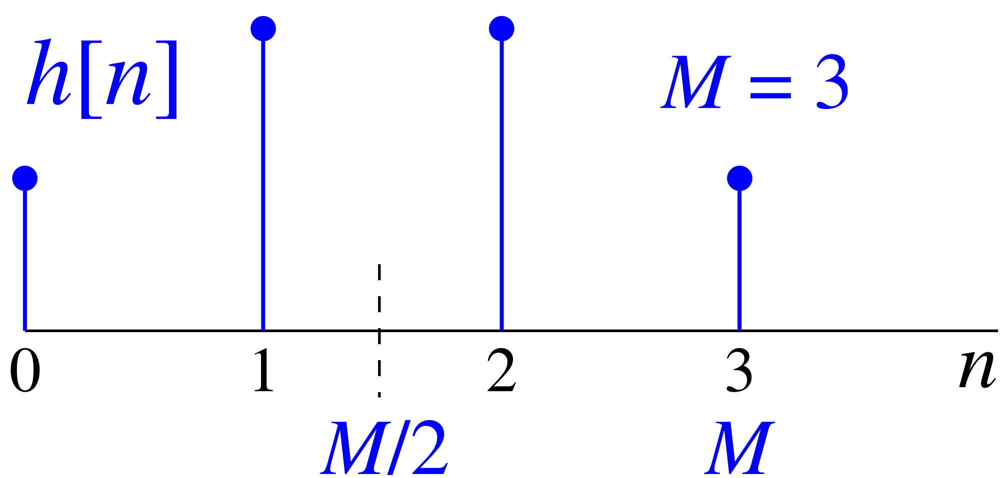
with order $M$ an even integer
($h[n]$ has length $M{+}1$, an odd integer) $\Rightarrow$
delay $\alpha = M/2$ is an integer, and $\beta = 0 \text{ or } \pi$.



Type II: Symmetric impulse response:

$$h[n] = h[M-n]\,, \qquad 0 \le n \le M,$$

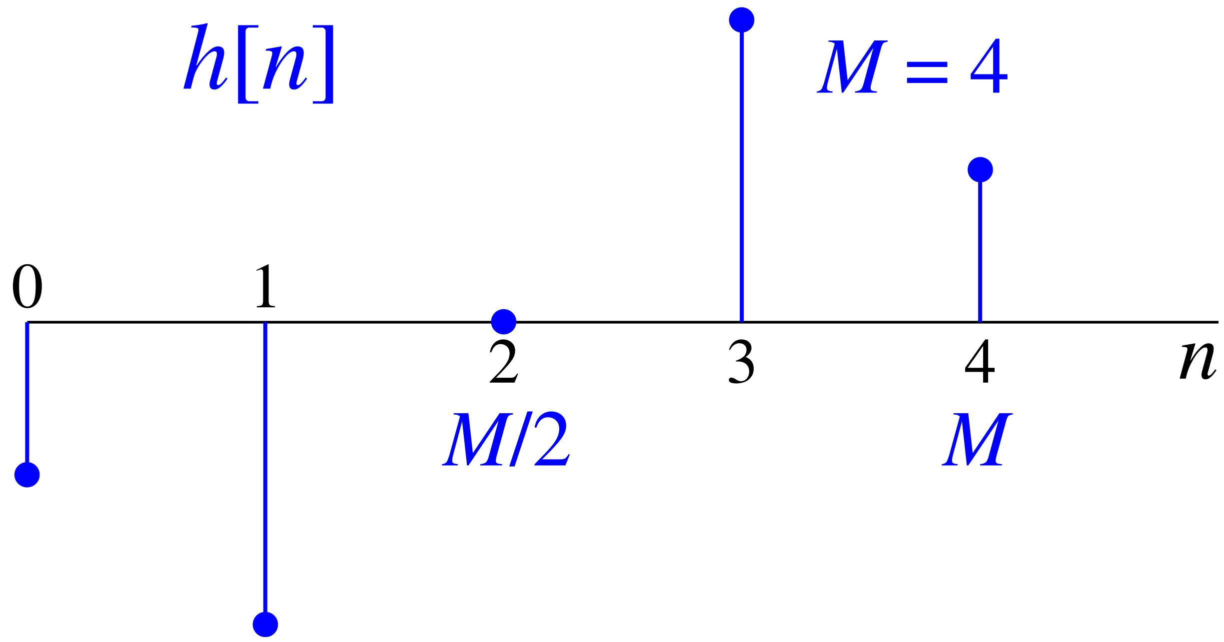
with order $M$ an odd integer
($h[n]$ has length $M+1$, an even integer) $\Rightarrow$
delay $\alpha = M/2$ is an integer $+\ 1/2$, and $\beta = 0$ or $\pi$.



Type III: Antisymmetric impulse response:

$$h[n] = -h[M-n]\,, \qquad 0 \le n \le M,$$

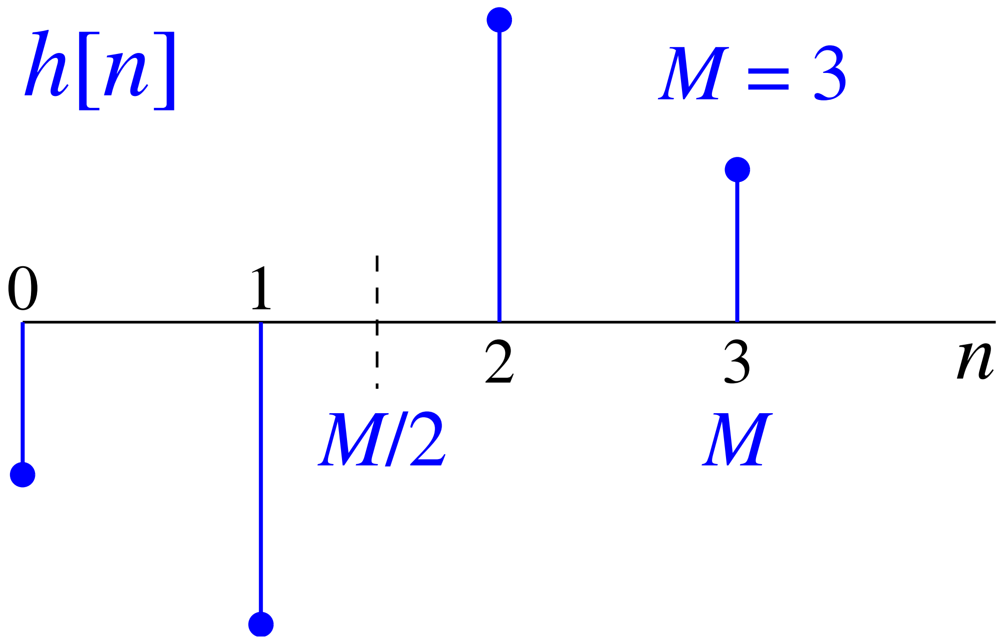
with order $M$ an even integer
($h[n]$ has length $M+1$, an odd integer) $\Rightarrow$
delay $\alpha = M/2$ is an integer, and $\beta = \pi/2$ or $3\pi/2$.



Type IV: Antisymmetric impulse response:

$$h[n] = -h[M-n]\,, \qquad 0 \le n \le M,$$

with order $M$ an odd integer
($h[n]$ has length $M+1$, an even integer) $\Rightarrow$
delay $\alpha = M/2$ is an integer $+\ 1/2$, and $\beta = \pi/2$ or $3\pi/2$.



## Location of zeros for FIR linear-phase filters:

$$H(z) = \sum_{n=0}^{M} b[n]\, z^{-n}$$

Types I & II: (Symmetric impulse responses)

$$H(z) = \sum_{n=0}^{M} h[M-n]\, z^{-n} = \sum_{k=M}^{0} h[k]\, z^{k} z^{-M}$$

$$= z^{-M} H\left(z^{-1}\right).$$

Evaluating $H(z)$ at $z = -1$ gives:

$$H(-1) = (-1)^{M}\, H(-1)\,.$$

If $M$ is even, then we have the simple identity:

$$H(-1) = H(-1)\,,$$

but if $M$ is odd, then we have:

$$H(-1) = -H(-1)\,,$$

so $H(-1)$ must be zero.

That is, for odd $M$ a type I or II FIR filter must have a zero at $z = -1$. Remembering that the DTFT is the $z$-transform evaluated on the unit circle, the magnitude response at $z = -1$ $(\rightarrow \omega = \pi)$ is zero.

Consequently, it is impossible to approximate a highpass filter using a symmetric FIR filter with odd order $M$—an even order $M$ must be used in this case.

## Types III & IV: (Antisymmetric impulse responses)

$$H(z) = -z^{-M} H\left(z^{-1}\right)$$

Evaluating $H(z)$ at $z = 1$ gives:

$$H(1) = -H(1)\,.$$

Consequently, $H(z)$ must have a zero at $z = 1$ $(\rightarrow \omega = 0)$ for both even and odd $M \Rightarrow$

A type III or IV antisymmetric filter cannot be lowpass, irrespective of the order $M$.

Evaluating $H(z)$ at $z = -1$ gives:

$$H(-1) = (-1)^{M+1}\, H(-1)\,.$$

If $M$ is odd $\Rightarrow$ $M{+}1$ is even, so again we have the simple identity:

$$H(-1) = H(-1)\,,$$

but if $M$ is even $\Rightarrow$ $M{+}1$ is odd and:

$$H(-1) = -H(-1)\,,$$

so $H(-1)$ must be zero.

Consequently, a type III or IV antisymmetric FIR filter with even order $M$ cannot be highpass.

These constraints on the zeros are important in designing FIR linear-phase systems since they impose limitations on the types of frequency responses that can be achieved!

# 6.5 All-Pass Systems

Consider a stable system function of the form:

$$H_{\mathrm{ap}}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}.$$

Note that this system has a pole at $z = a$ and a zero $z = 1/a^*$.

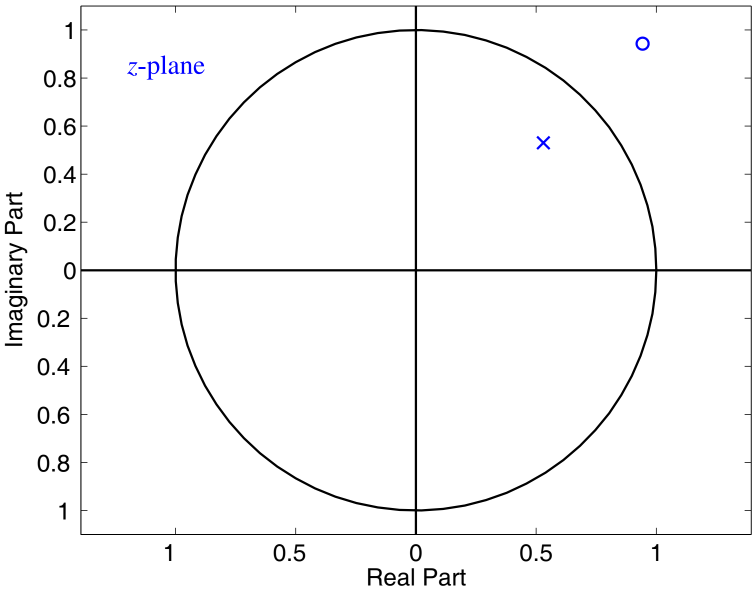
The frequency response of this system is:

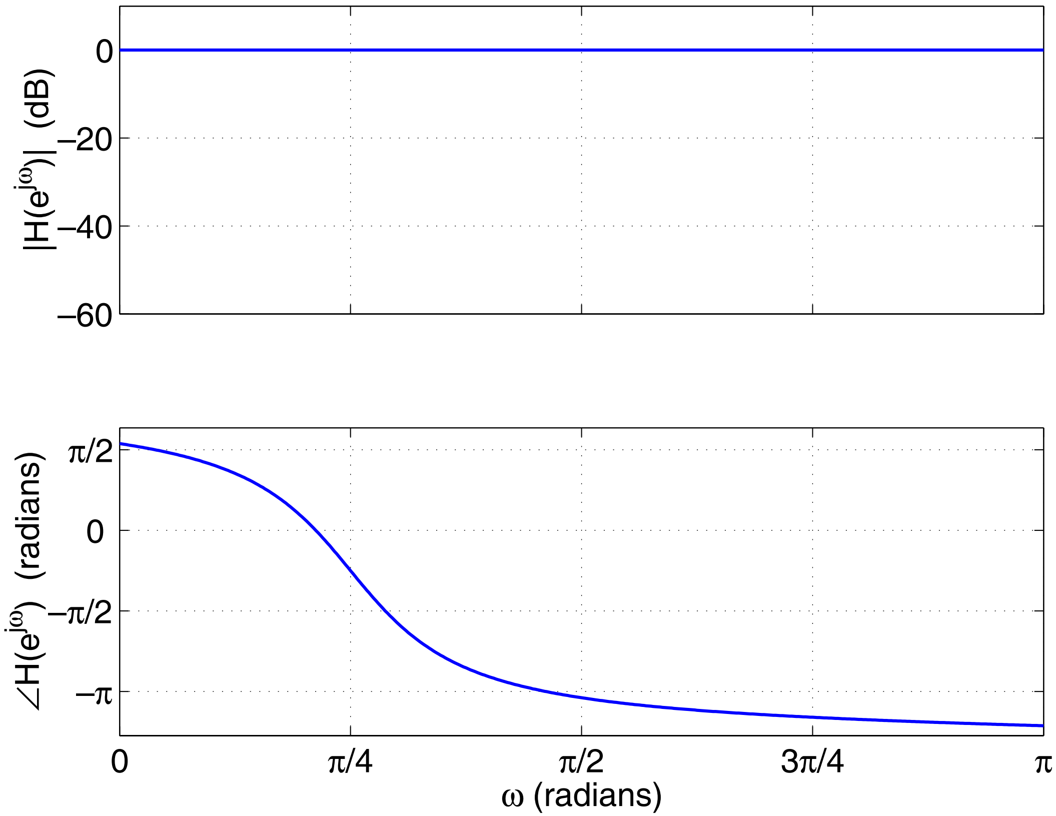
$$H_{\mathrm{ap}}\left(e^{j\omega}\right) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} = e^{-j\omega}\frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}}.$$

The term $e^{-j\omega}$ has unity magnitude, and the numerator and denominator factors are complex conjugates of each other and thus have the same magnitude $\Rightarrow |H_{\mathrm{ap}}(e^{j\omega})| = 1$.

Such systems with a *constant magnitude frequency response* are called all-pass systems.

# First-order all-pass filter with $a = 0.75e^{j\pi/4}$:



General form of all-pass filters with real-valued impulse responses:

$$H_{\mathrm{ap}}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d[k]}{1 - d[k] z^{-1}} \prod_{k=1}^{M_c} \frac{\left(z^{-1} - e^*[k]\right)\left(z^{-1} - e[k]\right)}{\left(1 - e[k] z^{-1}\right)\left(1 - e^*[k] z^{-1}\right)},$$

where $A$ is a positive constant, the $d[k]$s are the real poles, and the $e[k]$s are the complex poles (which much come in complex conjugate pairs for a real-valued impulse response).

For causal and stable all-pass systems, $|d[k]| < 1$ and $|e[k]| < 1$ for all $k$. That is, all the poles must fall within the unit circle, and all the zeros are the reciprocal complex conjugates of the poles and thus fall outside the unit circle.

- All-pole filters have a number of applications, including compensation for phase (or group delay) distortion.
- They are also useful in the theory of minimum-phase systems.

# 6.6 Minimum-Phase Systems

A *causal* and *stable* system $H(z)$ is considered to be minimum-phase if its inverse $1/H(z)$ is also causal and stable.

For this to be true all of the *system zeros* as well as the poles must lie within the unit circle.

Any rational system function can be expressed as:

$$H(z) = H_{\mathsf{min}}(z)\, H_{\mathsf{ap}}(z)\,,$$

where $H_{\min}(z)$ is a minimum-phase system and $H_{\mathrm{ap}}(z)$ is an all-pass system. This is referred to as *minimum-phase and all-pass decomposition*.

$H_{\mathrm{min}}(z)$ contains the poles and zeros of $H(z)$ that lie inside the unit circle, plus zeros that are conjugate reciprocals of the zeros of $H(z)$ that lie outside the unit circle.

$H_{\mathrm{ap}}(z)$ comprises all the zeros of $H(z)$ that lie outside the unit circle, plus poles to cancel out the reflected conjugate reciprocal zeros in $H_{\mathrm{min}}(z)$.

Consequently, a nonminimum-phase system can be created out of a minimum-phase system by reflecting one or more zeros inside the unit circle to their conjugate reciprocal locations outside the unit circle.

Conversely, a minimum-phase system can be created out of a nonminimum-phase system by reflecting all the zeros lying outside the unit circle to their conjugate reciprocal locations inside.

Both of these operations can be performed without changing the system's *magnitude frequency response*!

## Note:

– Minimum-phase systems are useful for compensating for frequency-response magnitude distortions. If we follow a filter with the transfer function $H(z)$ by a filter with the transfer function $1/H_{\min}(z)$, then the overall transfer function is:

$$H(z)\,\frac{1}{H_{\mathsf{min}}(z)} = \frac{H_{\mathsf{min}}(z)}{H_{\mathsf{min}}(z)} H_{\mathsf{ap}}(z) = H_{\mathsf{ap}}(z)\,,$$

leaving only the phase distortion given by $H_{\mathrm{ap}}(z)$.

– The name "minimum-phase" arises from the property that $\angle H(z) = \angle H_{\min}(z) + \angle H_{\mathrm{ap}}(z)$. All-pass filters always have positive *phase delay*, so the phase delay of $H(z)$ is always greater than that of $H_{\min}(z) \Rightarrow H_{\min}(z)$ has the minimum possible phase delay for a system with the frequency response $|H_{\min}(e^{j\omega})|$.