

COMP ENG 4TL4:

Digital Signal Processing

Notes for Lecture #29

Wednesday, November 19, 2003

Correlation-based methods of spectral estimation:

In the periodogram methods of spectral estimation, a direct estimation of the PSD is made from the Fourier transform of a windowed sequence.

An alternative approach is based on the property that the PSD is the Fourier transform of the autocorrelation function. Thus, we can first estimate the *autocorrelation function* and then take the Fourier transform to obtain an estimate of the PSD.

Let us consider a finite record $v[n]$ of the real-valued random signal $x[n]$:

$$v[n] = \begin{cases} x[n], & 0 \leq n \leq Q - 1, \\ 0, & \text{otherwise.} \end{cases}$$

An estimate of the autocorrelation function is:

$$\hat{\phi}_{xx}[m] = \frac{1}{Q} c_{vv}[m],$$

where $c_{vv}[m]$ is the autocorrelation of the finite sequence $v[n]$.

For a real-valued sequence, $c_{vv}[-m] = c_{vv}^*[m] = c_{vv}[m]$:

$$c_{vv}[m] = \sum_{n=0}^{Q-1} v[n] v[n+m]$$
$$= \begin{cases} \sum_{n=0}^{Q-|m|-1} x[n] x[n+|m|], & |m| \leq Q-1, \\ 0, & \text{otherwise.} \end{cases}$$

The mean of the estimate of the autocorrelation function is:

$$\begin{aligned} \mathbb{E} \left\{ \hat{\phi}_{xx}[m] \right\} &= \frac{1}{Q} \sum_{n=0}^{Q-|m|-1} \mathbb{E} \{ x[n] x[n + |m|] \} \\ &= \frac{1}{Q} \sum_{n=0}^{Q-|m|-1} \phi_{xx}[m] . \end{aligned}$$

Since $\phi_{xx}[m]$ is independent of n for a stationary random process:

$$\mathbb{E} \left\{ \hat{\phi}_{xx}[m] \right\} = \begin{cases} \left(\frac{Q-|m|}{Q} \right) \phi_{xx}[m] , & |m| \leq Q - 1, \\ 0, & \text{otherwise.} \end{cases}$$

We see that our estimator is biased because:

$$E \left\{ \hat{\phi}_{xx}[m] \right\} \neq \phi_{xx}[m] .$$

An unbiased estimator for the autocorrelation function would be:

$$\check{\phi}_{xx}[m] = \left(\frac{1}{Q - |m|} \right) c_{vv}[m] .$$

Why do we use the *biased* estimator? Because it:

1. has *lower variance*, and
2. is *asymptotically unbiased*, i.e., for large Q :

$$E \left\{ \hat{\phi}_{xx}[m] \right\} \simeq \phi_{xx}[m] .$$

Example #1:

Let $v[n]$ contain four points $\{x[0], x[1], x[2], x[3]\}$.

Unbiased estimate of the autocorrelation function:

$$\check{\phi}_{xx}[0] = \frac{1}{4}\{x[0]x[0] + x[1]x[1] + x[2]x[2] + x[3]x[3]\}$$

$$\check{\phi}_{xx}[1] = \frac{1}{3}\{x[0]x[1] + x[1]x[2] + x[2]x[3]\}$$

$$\check{\phi}_{xx}[2] = \frac{1}{2}\{x[0]x[2] + x[1]x[3]\}$$

$$\check{\phi}_{xx}[3] = x[0]x[3]$$

Notice that the estimates are averages, with the estimates becoming *less reliable* with increasing m .

Example #1 (cont.):

Biased estimate of the autocorrelation function:

$$\hat{\phi}_{xx}[0] = \frac{1}{4} \{x[0]x[0] + x[1]x[1] + x[2]x[2] + x[3]x[3]\}$$

$$\hat{\phi}_{xx}[1] = \frac{1}{4} \{x[0]x[1] + x[1]x[2] + x[2]x[3]\}$$

$$\hat{\phi}_{xx}[2] = \frac{1}{4} \{x[0]x[2] + x[1]x[3]\}$$

$$\hat{\phi}_{xx}[3] = \frac{1}{4} x[0]x[3]$$

Notice that the averages are given lower weightings as they become *less reliable* with increasing m .

In the case of the periodogram, all the autocorrelation lag values are implicitly involved in the spectral estimation, and the increased variance for increasing $|m|$ affects the estimated spectrum at all frequencies.

However, by explicitly calculating an estimate of the autocorrelation function, we are able to choose which lag values to include in estimating the PSD.

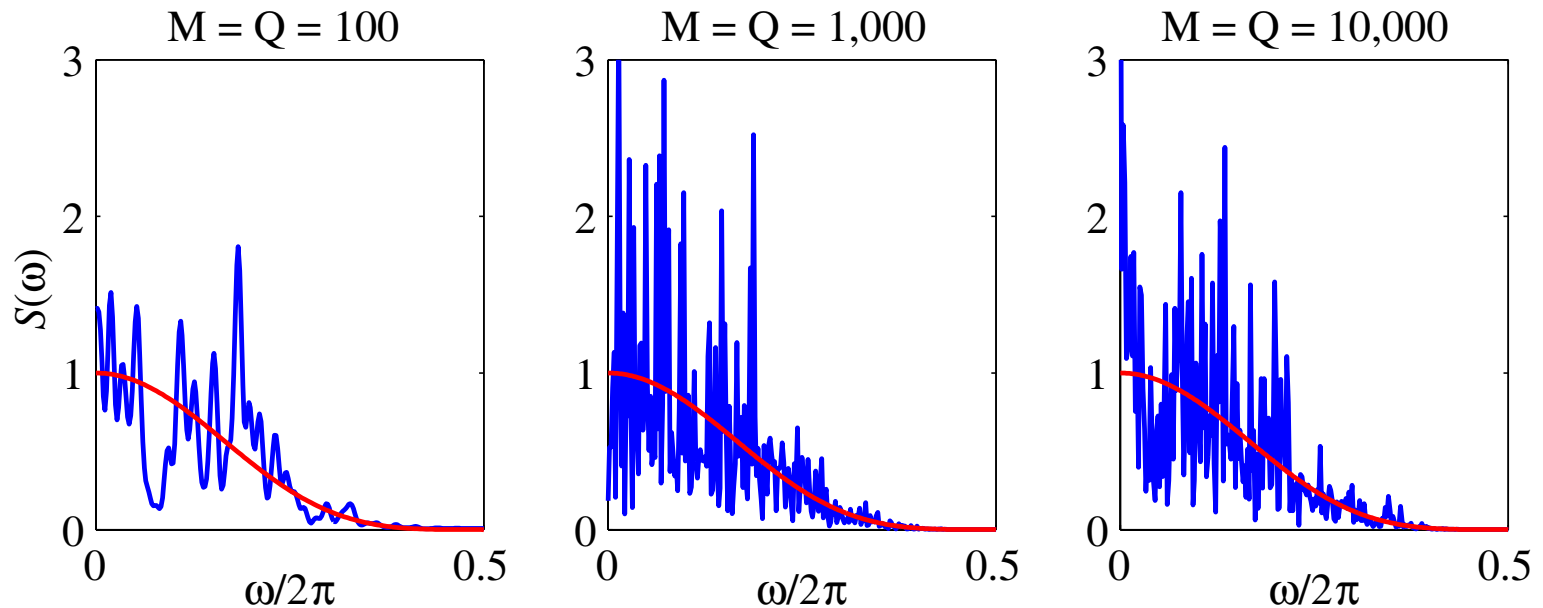
Let us define the correlation-based PSD estimator:

$$S(\omega) = \sum_{m=-(M-1)}^{M-1} \hat{\phi}_{xx}[m] w_c[m] e^{-j\omega m},$$

where $w_c[m]$ is a symmetric window of length $(2M-1)$.

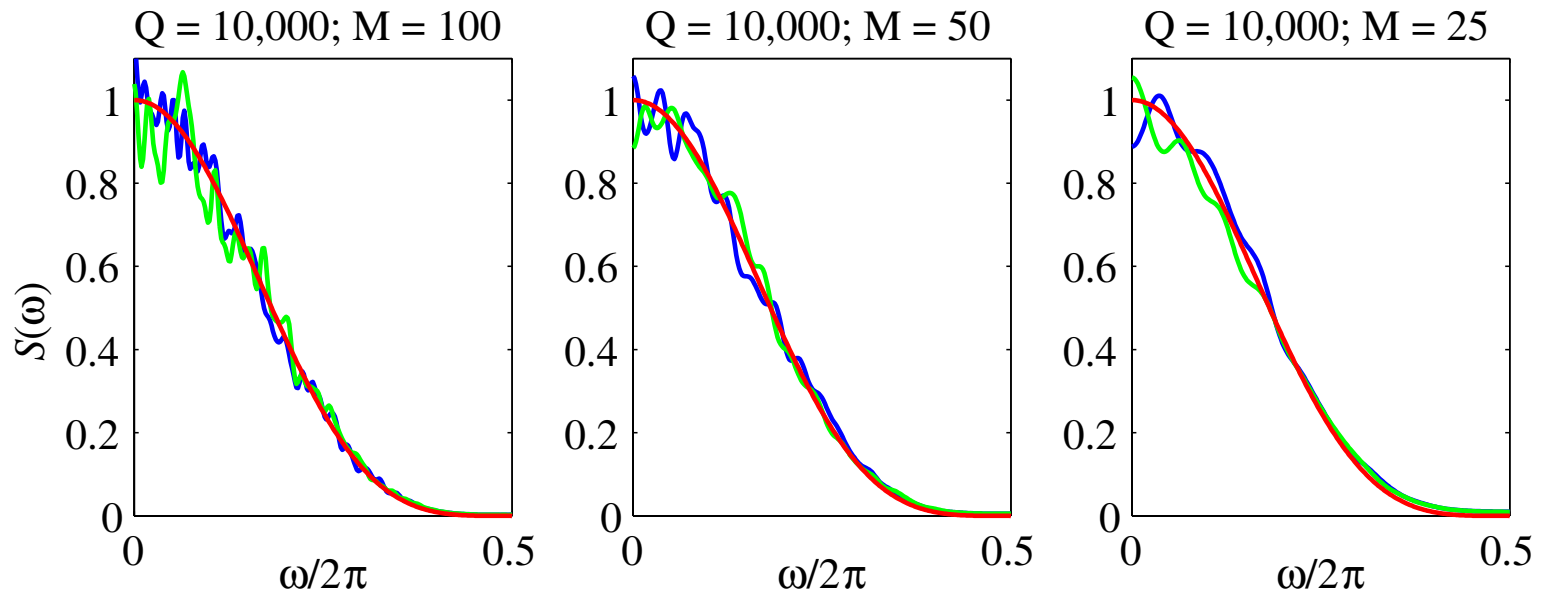
By limiting the length of the correlation window so that $M \ll Q$, we include only autocorrelation estimates for which the variance is low.

Example #2: Let $x[n]$, a zero-mean, unit-variance white Gaussian noise, be filtered by a lowpass filter with the magnitude-squared frequency response indicated by the red line in the plots below to give the lowpass Gaussian noise signal $y[n]$. Estimates of the PSD of $y[n]$ are shown below for $Q = 100, 1000$ and 10000 , where a *triangular* correlation window $w_c[m]$ with $M = Q$ was used.



Example #2 (cont.): Like the standard periodogram, the autocorrelation-based estimate of the PSD with $M = Q$ does not improve with increasing Q .

If $M \ll Q$, our estimator has a much lower variance, similar to the average periodogram methods, as shown in the plots below.



Note that if M is too small, however, then the spectral resolution of the PSD estimate will become too poor.

Choosing a correlation window:

In Ex. #2, an odd-length *triangular* correlation window $w_c[m]$ centered at $m = 0$ was used.

Question: Why this choice of a window?

Answer: The PSD has only *positive real values*, so our estimator $S(\omega)$ should only have *positive real values*.

Only an odd-length *triangular* window centered at $m = 0$ satisfies this requirement!

1. An odd-length symmetrical window centered at $m = 0$ always has a *real-valued* Fourier transform (i.e., it is a zero-phase system).
The autocorrelation estimate is also an odd-length symmetrical sequence centered at $m = 0$, so it too has a *real-valued* Fourier transform \Rightarrow
The product of the window and the autocorrelation sequence will have a real-valued Fourier transform, our estimator $S(\omega)$.
2. The triangular window, unlike the other common window types, always has a *positive-valued* Fourier transform.
The autocorrelation estimate also has a *positive-valued* Fourier transform \Rightarrow
The product of the window and the autocorrelation sequence will have a positive-valued Fourier transform, our estimator $S(\omega)$.

Computing the correlation estimate using the DFT:

The autocorrelation estimate can be computed using the DFT via the following method:

1. Form an N -point sequence by padding $x[n]$ with $(M-1)$ zeros.
2. Compute the N -point DFT $X[k]$.
3. Compute $|X[k]|^2 = X[k]X^*[k]$.
4. Compute the inverse DFT of $|X[k]|^2$ to obtain $\tilde{c}_{vv}[m]$.
5. Divide the resulting sequence by Q to obtain the autocorrelation estimate:

$$\hat{\phi}_{xx}[m] = \frac{1}{Q}\tilde{c}_{vv}[m] \quad \text{for } m = 0, 1, \dots, M - 1.$$

Zero-padding to $N \geq Q + M - 1$ in step 1 avoids time-aliasing in the autocorrelation calculation performed in the Fourier domain (step 3).