COMP ENG 4TL4:

Digital Signal Processing

Notes for Lecture #7

Tuesday, September 23, 2003

<u>Property:</u> Suppose that for a given input $x[n]$ we have found *one particular* output sequence $y_{\mathrm{P}}[n]$ so that a LCCD equation is satisfied. Then, the same equation with the same input is satisfied by any output of the form:

$$y[n] = y_{\mathsf{P}}[n] + y_{\mathsf{H}}[n],$$

where $y_{\mathrm{H}}[n]$ is any solution to the LCCD equation with zero input, i.e., $x[n] = 0, \forall n \in [-\infty,\infty]$.

<u>Remark:</u> $y_{\mathrm{P}}[n]$ and $y_{\mathrm{H}}[n]$ are referred to as the <u>particular</u> and <u>homogeneous</u> solutions, respectively.

Proof: Taking the sum of:

$$\sum_{k=0}^{N} a[k]\, y_{\mathrm{P}}[n-k] = \sum_{k=0}^{M} b[k]\, x[n-k]$$

$$\sum_{k=0}^{N} a[k]\, y_{\mathrm{H}}[n-k] = 0 \quad \left(\Leftarrow \begin{array}{l} \text{homogeneous} \\ \text{equation} \end{array} \right)$$

we obtain:

$$\sum_{k=0}^{N} a[k]\, y[n-k] = \sum_{k=0}^{M} b[k]\, x[n-k],$$

with $y[n] = y_{\mathrm{P}}[n] + y_{\mathrm{H}}[n]$.

Property: An LCCD equation does not provide a unique specification of the output for a given input.

Corollary: Auxiliary information or conditions are required to specify uniquely the output for a given input.

Example: Let auxiliary information be in the form of N sequential output values (e.g, $\{y[-N],\ldots,y[-2],y[-1]\}$). Then:

- later values can be obtained by rearranging the LCCD equation as a *recursive* relation running forward in n, and
- prior values can be obtained by rearranging the LCCD equation as a *recursive* relation running backward in n.

LCCD equations as recursive procedures:

$$y[n] = \sum_{k=0}^{M} \frac{b[k]}{a[0]} x[n-k] - \sum_{k=1}^{N} \frac{a[k]}{a[0]} y[n-k]$$ forwards

$$y[n-N] = \sum_{k=0}^{M} \frac{b[k]}{a[N]} x[n-k] - \sum_{k=0}^{N-1} \frac{a[k]}{a[N]} y[n-k]$$ backwards

Example #1: First-order AR system $y[n] = ay[n-1] + x[n]$ with the given input $x[n] = b\delta[n-1]$ and the auxiliary condition $y[0] = y_0 \neq 0$.

Forwards recursion:

$$\begin{aligned}
y[1] &= ay_0 + b\,, \\
y[2] &= ay[1] + 0 \\
&= a(ay_0 + b) = a^2 y_0 + ab\,, \\
y[3] &= a(a^2 y_0 + ab) = a^3 y_0 + a^2 b\,, \\
\cdots & \quad \cdots \quad \cdots\cdots\cdots\cdots\cdots\cdots\,, \\
y[n] &= a^n y_0 + a^{n-1} b.
\end{aligned}$$

Note that:

$$y[n-1] = a^{-1}(y[n] - x[n]) \qquad \Rightarrow$$

Backwards recursion:

$$\begin{aligned} y[-1] &= a^{-1}(y_0 - 0) \\ &= a^{-1}y_0\,, \\ y[-2] &= a^{-2}y_0\,, \\ y[-3] &= a^{-3}y_0\,, \\ \cdots\cdots &\cdots \cdots\cdots\,, \\ y[-n] &= a^{-n}y_0. \end{aligned}$$

Question: Is this system LTI and causal?

Property: A linear system requires that the output be zero for all time when the input is zero for all time.

Proof: Represent zero input as a product of a zero constant $c = 0$ and an (arbitrary) non-zero signal $x[n]$:

$$c\,x[n] = 0 \cdot x[n] = 0.$$

Hence, for a linear system:

$$\begin{aligned} y[n] &= T\{c\,x[n]\} \\ &= c\,T\{x[n]\} \\ &= 0 \cdot T\{x[n]\} = 0. \end{aligned}$$

Example #1 (cont.): From the backwards recursion it follows that:

$$y\,[-n] = a^{-n} y_0 \;\neq 0 \quad \text{for} \quad a \neq 0\,,\; y_0 \neq 0,$$

whereas $x[-n] = 0, n \geq 0 \Rightarrow$

According to the property proven on the previous slide, the system is nonlinear!

The system was implemented in both positive and *negative* directions beginning with $n = 0$

$\Rightarrow$ the system is noncausal!

The forwards-backwards recursion can be rewritten for arbitrary n as:

$$y[n] = a^n y_0 + a^{n-1} b u[n-1].$$

Hence, a shift of the input by n_0 samples, $x_1[n] = x[n-n_0]$ $= b\delta[n-n_0-1]$, gives:

$$\begin{aligned} y_1[n] &= a^n y_0 + a^{n-n_0-1} b u[n-n_0-1] \\ &\neq y[n-n_0] \qquad \Rightarrow \end{aligned}$$

the system is not time-invariant!

Example #2: First-order AR system $y[n] = ay[n-1] + x[n]$ with the given input $x[n] = b\delta[n-1]$ and the auxiliary condition $y[0] = 0$.

Forwards-backwards recursion:

$$
\begin{aligned}
\cdots\cdots & \;\cdots\; \cdots\,, \\
y\,[-1] &= 0\,, \\
y\,[0] &= 0\,, \\
y\,[1] &= a \cdot 0 + b \\
&= b\,, \\
y\,[2] &= ab\,, \\
\cdots\cdots & \;\cdots\; \cdots\cdots\,, \\
y\,[n] &= a^{n-1}b.
\end{aligned}
$$

This recursion can be rewritten as:

$$y[n] = a^{n-1} b\, u[n-1]\ , \qquad \forall\ n.$$

It is easy to prove that this system is a causal LTI system.

$\Rightarrow$

For any system described by an LCCD equation:

- Linearity, time-invariance, and causality depend on auxiliary conditions!
- If an additional condition is that the system is initially at rest, then the system will be linear, time invariant, and causal.