COMP ENG 4TL4:

Digital Signal Processing

Notes for Lecture #9

Friday, September 26, 2003

3.2 Properties of the DTFT

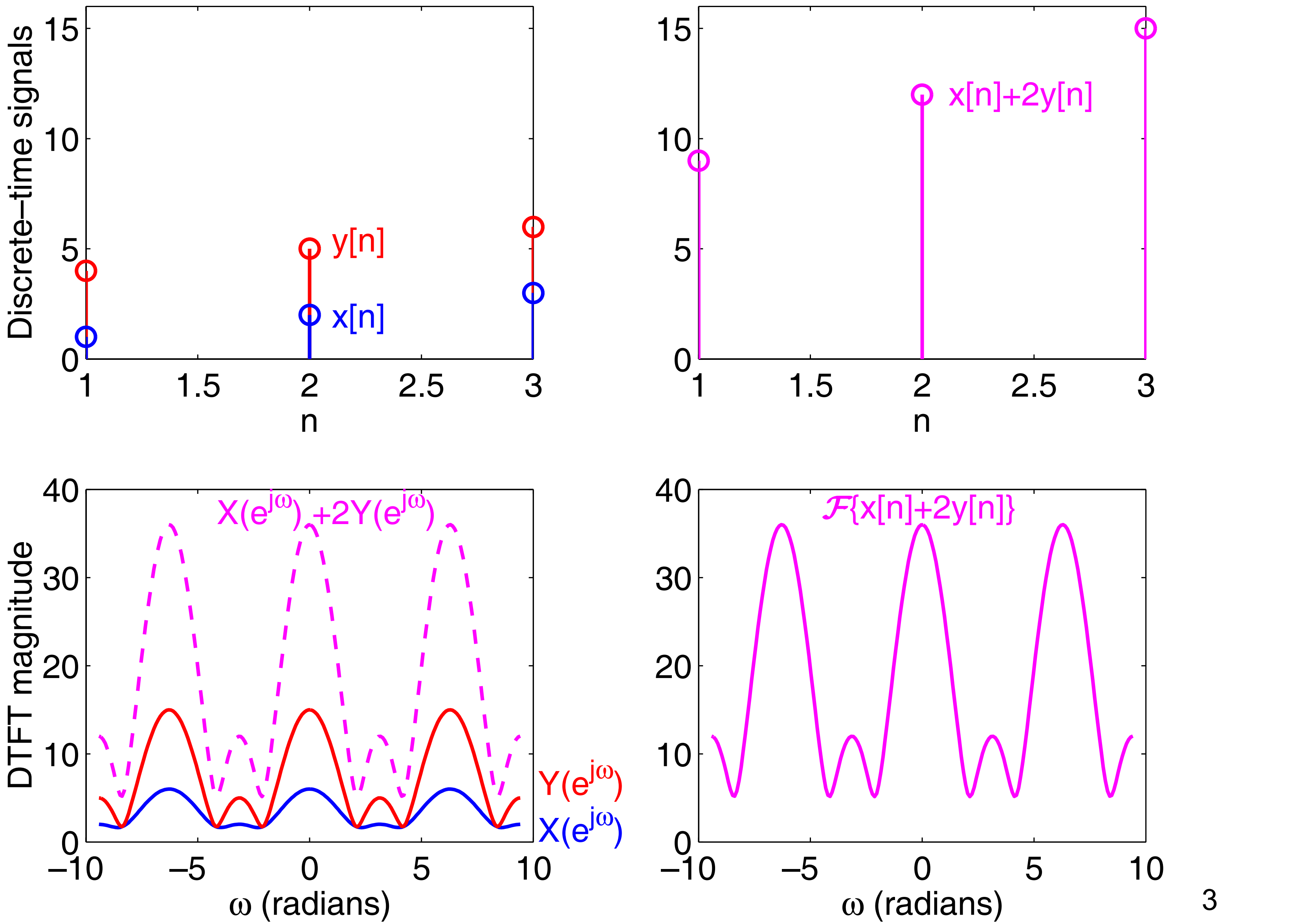
Linearity:

If $X(e^{j\omega}) = \mathcal{F}\{x[n]\}$ and $Y(e^{j\omega}) = \mathcal{F}\{y[n]\}$,
then $a\,X(e^{j\omega}) + b\,Y(e^{j\omega}) = \mathcal{F}\{a\,x[n] + b\,y[n]\}$.

Also, if $x[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\}$ and $y[n] = \mathcal{F}^{-1}\{Y(e^{j\omega})\}$,
then $a\,x[n] + b\,y[n] = \mathcal{F}^{-1}\{a\,X(e^{j\omega}) + b\,Y(e^{j\omega})\}$.

Proof: Elementary (direct substitution).

Example of DTFT linearity property:



Time shifting:

If $X\left(e^{j\omega}\right) = \mathcal{F}\{x[n]\}$ then $X\left(e^{j\omega}\right) e^{-j\omega m} = \mathcal{F}\{x[n-m]\}$.

Also, if $x[n] = \mathcal{F}^{-1}\left\{X\left(e^{j\omega}\right)\right\}$,

then $x[n-m] = \mathcal{F}^{-1}\left\{X\left(e^{j\omega}\right) e^{-j\omega m}\right\}$.

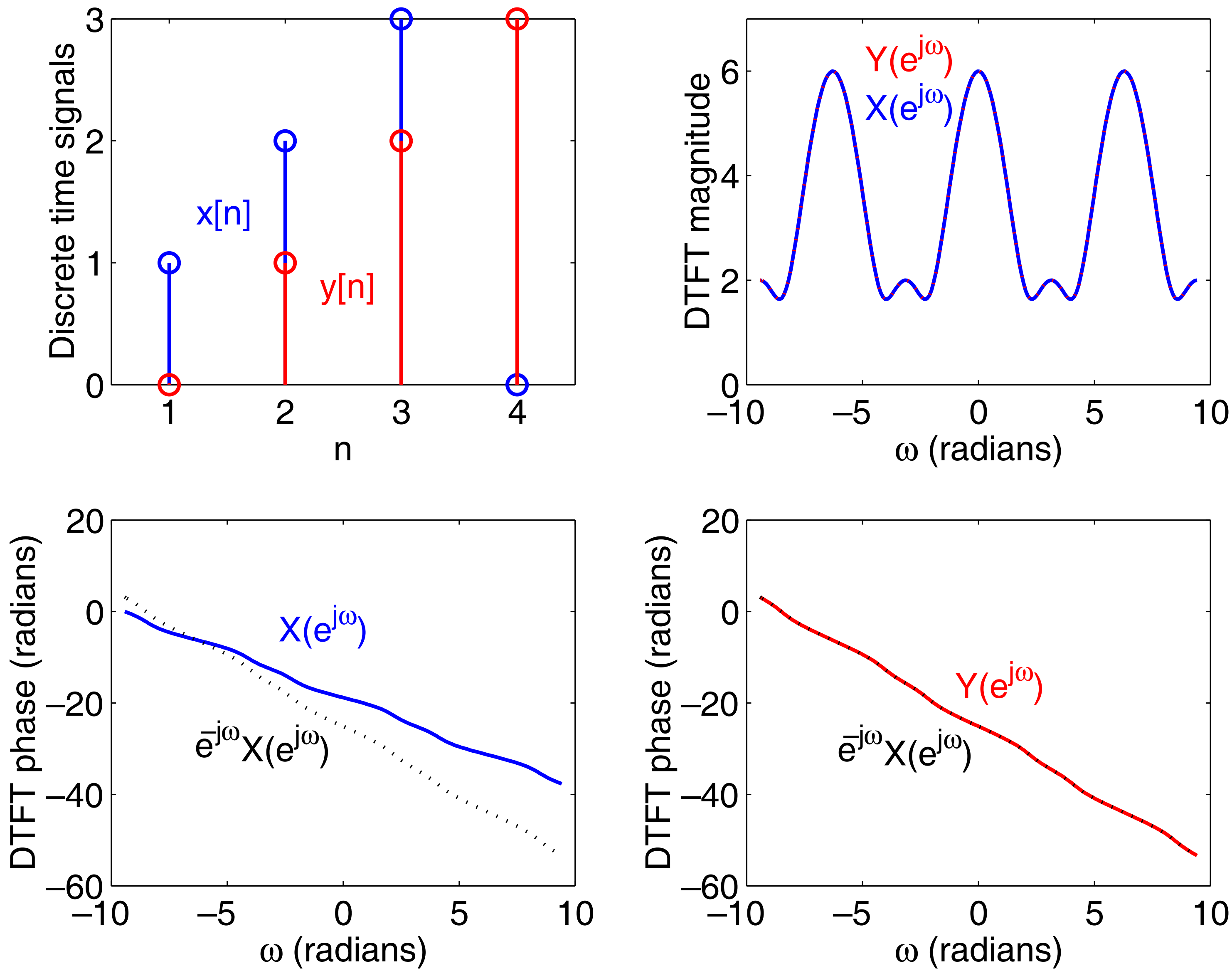
Proof:

$$\mathcal{F}\{x[n-m]\} = \sum_{n=-\infty}^{\infty} x[\underbrace{n-m}_{=k}]\, e^{-j\omega n} = \sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega(m+k)}$$

$$= e^{-j\omega m} \sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega k} = X\left(e^{j\omega}\right) e^{-j\omega m}$$

$$\mathcal{F}^{-1}\left\{X\left(e^{j\omega}\right) e^{-j\omega m}\right\} = \mathcal{F}^{-1}\{\mathcal{F}\{x[n-m]\}\} = x[n-m]$$

Example of DTFT time-shifting property: $m = 1$



Frequency shifting:

If $X\left(e^{j\omega}\right) = \mathcal{F}\{x[n]\}$ then $X\left(e^{j(\omega-\nu)}\right) = \mathcal{F}\left\{x[n]\,e^{j\nu n}\right\}$.

Also, if $x[n] = \mathcal{F}^{-1}\left\{X\left(e^{j\omega}\right)\right\}$,

then $x[n]\,e^{j\nu n} = \mathcal{F}^{-1}\left\{X\left(e^{j(\omega-\nu)}\right)\right\}$.

Proof:

$$\mathcal{F}\left\{x[n]\,e^{j\nu n}\right\} = \sum_{n=-\infty}^{\infty} x[n]\,e^{-j(\omega-\nu)n} = X\left(e^{j(\omega-\nu)}\right)$$

$$\mathcal{F}^{-1}\left\{X\left(e^{j(\omega-\nu)}\right)\right\} = \mathcal{F}^{-1}\left\{\mathcal{F}^{-1}\left\{x[n]\,e^{j\nu n}\right\}\right\} = x[n]\,e^{j\nu n}$$

Time reversal:

$$\text{If } X\left(e^{j\omega}\right) = \mathcal{F}\{x[n]\} \text{ then } X\left(e^{-j\omega}\right) = \mathcal{F}\{x[-n]\}.$$

$$\text{Also, if } x[n] = \mathcal{F}^{-1}\left\{X\left(e^{j\omega}\right)\right\},$$

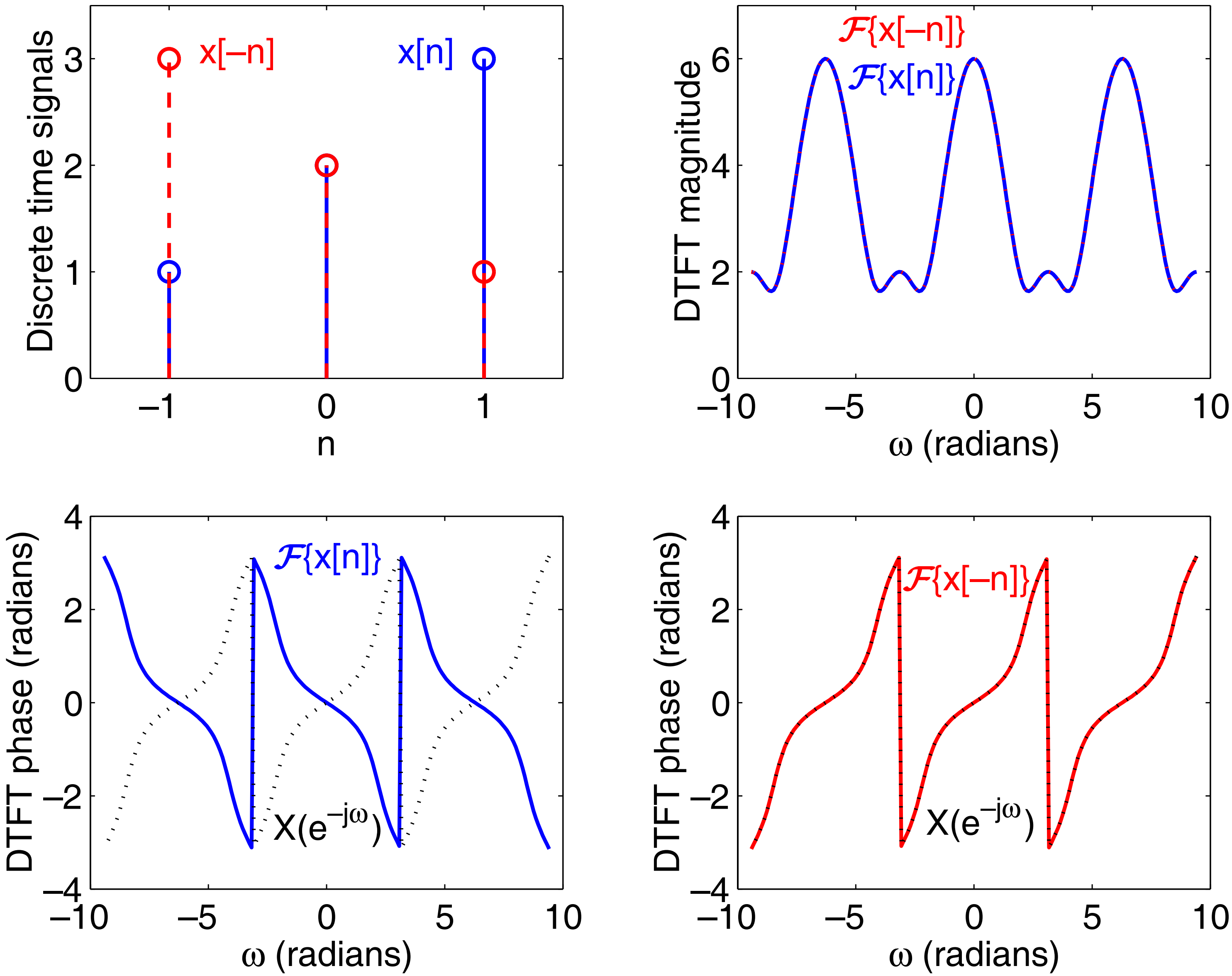
$$\text{then } x[-n] = \mathcal{F}^{-1}\left\{X\left(e^{-j\omega}\right)\right\}.$$

Proof:

$$\mathcal{F}\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[\underbrace{-n}_{=m}]e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x[m]\, e^{j\omega m} = X\left(e^{-j\omega}\right)$$

$$\mathcal{F}^{-1}\left\{X\left(e^{-j\omega}\right)\right\} = \mathcal{F}^{-1}\{\mathcal{F}\{x[-n]\}\} = x[-n]$$

Example of DTFT time-reversal property:



Differentiation in frequency:

If $X(e^{j\omega}) = \mathcal{F}\{x[n]\}$ then $j\dfrac{dX(e^{j\omega})}{d\omega} = \mathcal{F}\{nx[n]\}$.

Also, if $x[n] = \mathcal{F}^{-1}\left\{X(e^{j\omega})\right\}$,

then $nx[n] = \mathcal{F}^{-1}\left\{j\dfrac{X(e^{j\omega})}{d\omega}\right\}$.

Proof:

$$\mathcal{F}\{n\,x\,[n]\} = \sum_{n=-\infty}^{\infty} nx\,[n]\,e^{-j\omega n} = j\sum_{n=-\infty}^{\infty} x[n]\,\frac{d\left(e^{-j\omega n}\right)}{d\omega}$$

$$= j\frac{d}{d\omega}\left\{\sum_{n=-\infty}^{\infty} x[n]\;e^{-j\omega n}\right\} = j\,\frac{dX(e^{j\omega})}{d\omega}$$

$$\mathcal{F}^{-1}\left\{j\,\frac{dX(e^{j\omega})}{d\omega}\right\} = \mathcal{F}^{-1}\{\mathcal{F}\{nx[n]\}\} = nx[n]$$

Convolution theorem:

If $X\left(e^{j\omega}\right) = \mathcal{F}\{x[n]\}$, $H\left(e^{j\omega}\right) = \mathcal{F}\{h[n]\}$,

and $y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = x[n] * h[n]$,

then $Y\left(e^{j\omega}\right) = \mathcal{F}\{y[n]\} = X\left(e^{j\omega}\right) H\left(e^{j\omega}\right)$.

Convolution of sequences in the *time domain* is equivalent to *multiplication* of the corresponding Fourier transforms in the *frequency domain*.

Proof of convolution theorem:

$$\begin{aligned}
Y\left(e^{j\omega}\right) &= \mathcal{F}\{y[n]\} = \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x[k]\, h[\underbrace{n-k}_{=m}] \right\} e^{-j\omega n} \\
&= \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]\, h[m]\, e^{-j\omega(m+k)} \\
&= \left\{ \sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega k} \right\} \left\{ \sum_{m=-\infty}^{\infty} h[m]\, e^{-j\omega m} \right\} \\
&= X\left(e^{j\omega}\right) H\left(e^{j\omega}\right)
\end{aligned}$$

Windowing theorem:

If $X\left(e^{j\omega}\right) = \mathcal{F}\{x[n]\}$, $W\left(e^{j\omega}\right) = \mathcal{F}\{w[n]\}$,

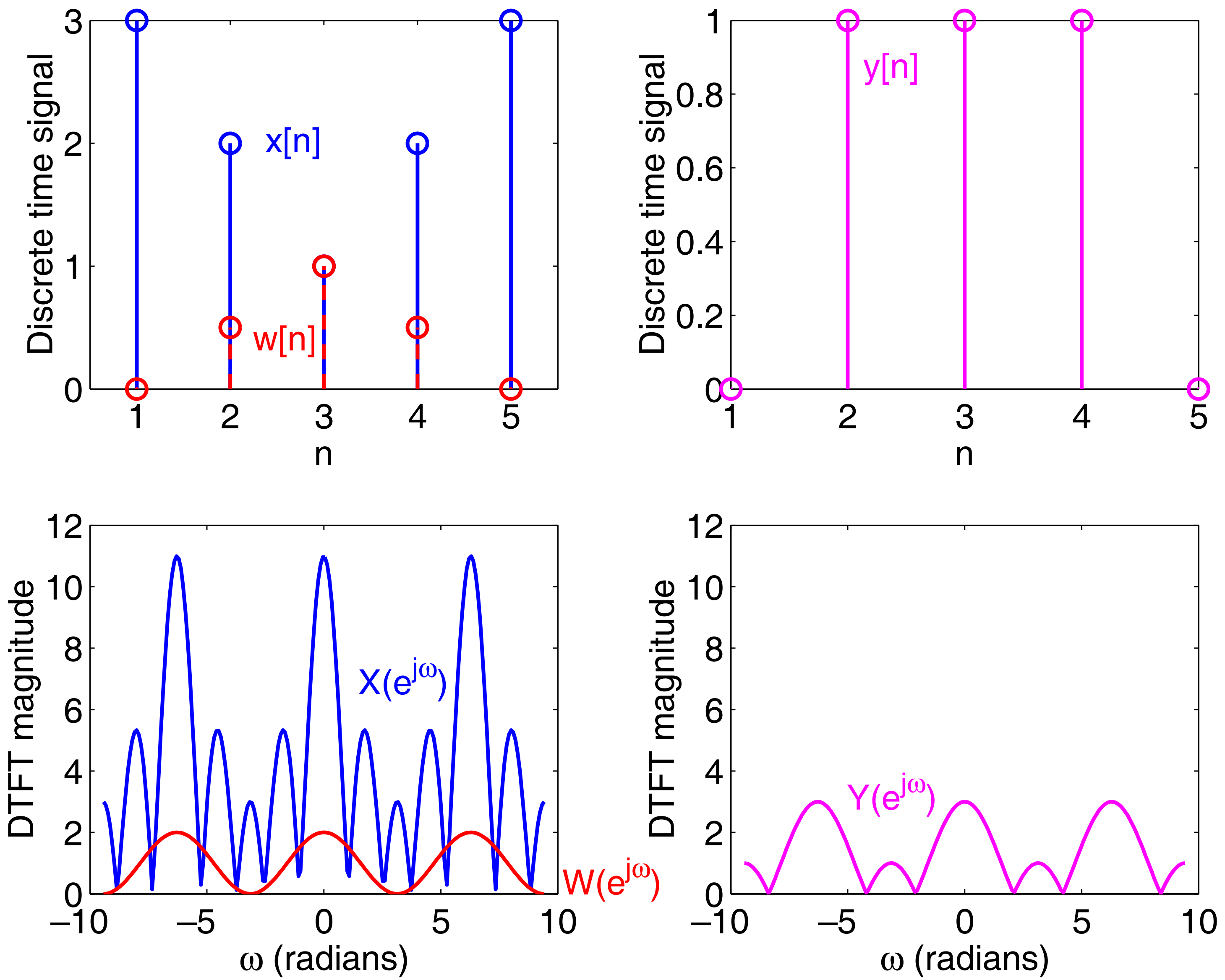
and $y[n] = x[n]\, w[n]$,

then $Y\left(e^{j\omega}\right) = \mathcal{F}\{y[n]\} = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X\left(e^{j\nu}\right) W\left(e^{j(\omega-\nu)}\right)\, d\nu.$

Multiplication of sequences in the *time domain* is equivalent to periodic *convolution* of the corresponding Fourier transforms in the *frequency domain*.

Proof: by means of direct substitution, similarly to the proof of the convolution theorem.

Example of DTFT windowing property:



Generalized Parseval theorem:

If $X\left(e^{j\omega}\right) = \mathcal{F}\{x[n]\}$, $Y\left(e^{j\omega}\right) = \mathcal{F}\{y[n]\}$,

then $\displaystyle\sum_{n=-\infty}^{\infty} x[n]\, y^* [n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\omega}\right) Y^*\left(e^{j\omega}\right)\, d\omega.$

Proof: similar to the proof of the Parseval theorem.

Summary of the main properties of the DTFT

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
$a\,x[n] + b\,y[n]$	$a\,X(e^{j\omega}) + b\,Y(e^{j\omega})$
$x^*[n]$	$X^*(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$x[n-m]$	$e^{-j\omega m} X(e^{j\omega})$
$e^{j\nu n} x[n]$	$X(e^{j(\omega-\nu)})$
$x[-n]$	$X(e^{-j\omega})$
$n\,x[n]$	$j\,\frac{dX(e^{j\omega})}{d\omega}$
$x[n] * h[n]$	$X(e^{j\omega})\,H(e^{j\omega})$
$x[n]\,w[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\nu})\,W(e^{j(\omega-\nu)})\,d\nu$
$\sum_{n=-\infty}^{\infty} \|x[n]\|^2$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left\|X(e^{j\omega})\right\|^2 d\omega$
$\sum_{n=-\infty}^{\infty} x[n]\,y^*[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\,Y^*(e^{j\omega})\,d\omega$

3.3 Frequency-Domain Representation of Discrete-Time Signals and Systems

Recall the impulse response $h[n]$ of an LTI system:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

Consider an input sequence: $x[n] = e^{j\omega n}, \; -\infty < n < \infty$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, e^{j\omega(n-k)} = e^{j\omega n} \underbrace{\left\{ \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k} \right\}}_{=H(e^{j\omega})}$$

$$= e^{j\omega n} H\left(e^{j\omega}\right)$$

The complex function:

$$H\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k}$$

is called the frequency response or the transfer function of the system.

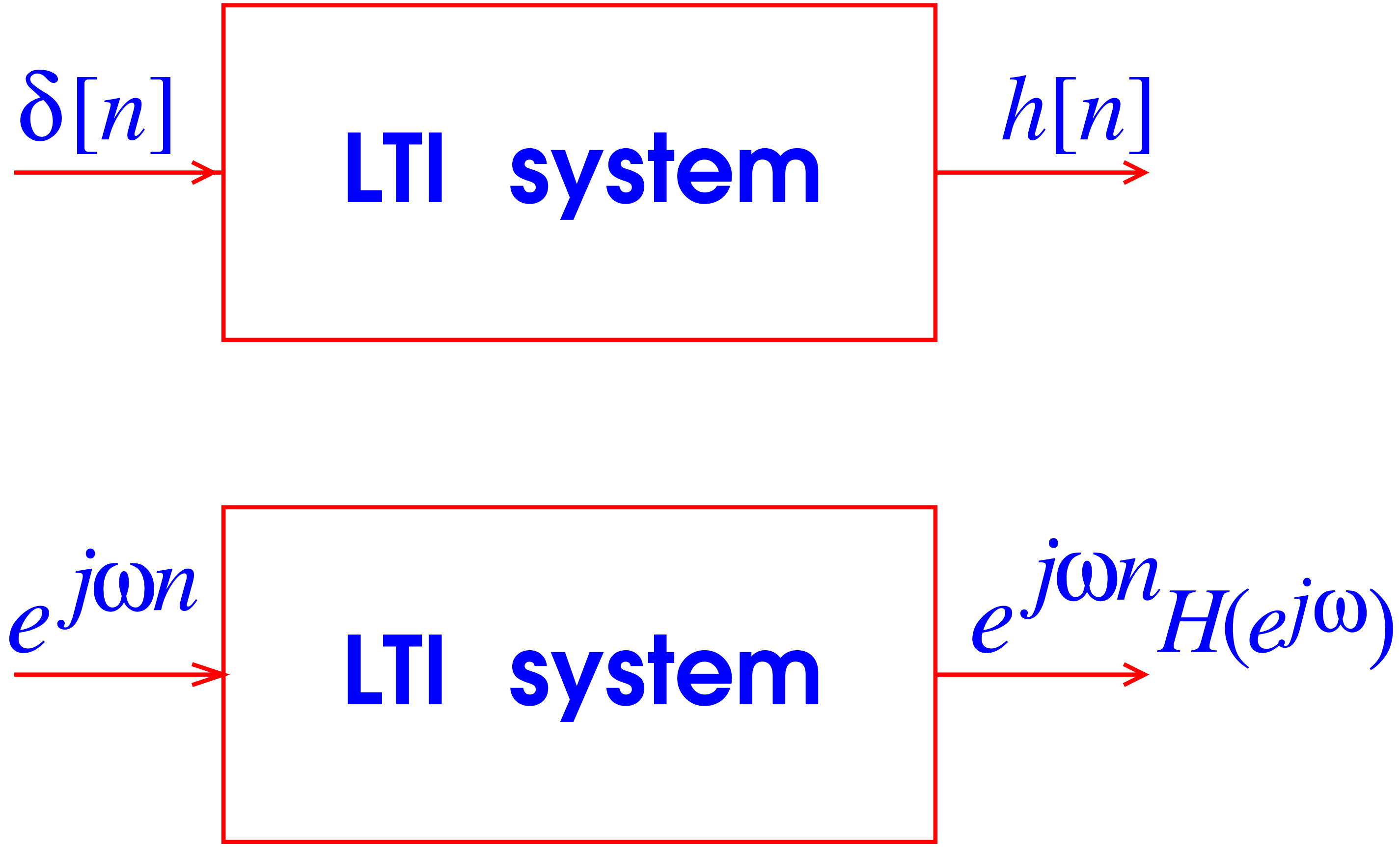
Remarks:

– The impulse response and transfer function represent a DTFT pair $\Rightarrow H(e^{j\omega})$ is a periodic function.

– The transfer function shows how different input frequency components are changed (e.g., attenuated) at the system output

– This function will be very useful for the consideration of signal *filtering* $\Rightarrow$ if $y[n] = h[n]*x[n]$, then $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$

Interpretation of impulse and frequency responses



Example: the delay system:

$$y[n] = x[n - n_d] \quad \text{with fixed integer } n_d$$

$$x[n] = e^{j\omega n} \;\Rightarrow\; y[n] = e^{j\omega(n-n_d)} \;\Rightarrow\; H\left(e^{j\omega}\right) = e^{-j\omega n_d}$$

Since $|H(e^{j\omega})| = 1$, this system is *frequency nonselective*. Such systems are often referred to as allpass systems. This was illustrated on slide #5.

(Examples of *frequency selective* systems will be given in Lab #2 and later in the course when the filtering operation is considered.)