Examples of z-Transforms:

(a) $x_1(n) = \{1, 2, 5, 7, 0, 1\}$

$X_1(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$

ROC: entire z-plane except $z=0$

(b) $x_2(n) = \delta(n-K), \quad K>0$

$X_2(z) = z^{-K}$

ROC: entire z-plane except $z=0$

(c) $x(n) = \alpha^n u(n) + b^n u(-n-1)$

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} (\alpha z^{-1})^n + \sum_{l=1}^{\infty} (b^{-1} z)^l$$

$$X(z) = \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-bz^{-1}}$$

ROC: $|\alpha| < |z| < |b|$

(d) using the time reversal property

$x(n) = u(-n)$ $\qquad$ $\cancel{x(n)} \leftrightarrow X(z)$

It is known that $u(n) \overset{z}{\longleftrightarrow} \frac{1}{1-z^{-1}}$

using the time reversal prop.: $x(-n) \overset{z}{\longleftrightarrow} X(z^{-1})$

$$u(-n) \overset{z}{\longleftrightarrow} \frac{1}{1-z}$$

(e) $x(n) = na^n u(n)$

we know that $a^n u(n) \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}$

and the differentiation prop.:

$x_1(n) = nx(n) \overset{z}{\longleftrightarrow} z\, dX(z)/dz = X_1(z)$

so

$$x(n) = na^n u(n) \overset{z}{\longleftrightarrow} -z\frac{dX_1(z)}{dz} = \frac{az^{-1}}{(1-az^{-1})^2}$$

2.1) (a) - The homogeneous solution $y_h[n]$ solves the difference equation when $x[n]=0$.

for $y_h[n] = Ac^n$

$$c^2 - \frac{1}{4}c + \frac{1}{8} = 0 \qquad \text{for} \qquad c = \frac{1}{2} \text{ and } -\frac{1}{4}$$

$$y_h[n] = A_1\left(\frac{1}{2}\right)^n + A_2\left(-\frac{1}{4}\right)^n$$

(b) By taking the Z-transform of both sides :

$$Y(z)\left(1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}\right) = 3X(z)$$

the impulse response, $H(z) = Y(z)/X(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$$

$$H(z) = \frac{3}{(1+\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{1}{1+\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{2}z^{-1}}$$

i- The causal impulse response corresponds to assuming that the region of convergence (ROC) extends outside the outermost pole.

$$h_c[n] = \left((-1/4)^n + 2(1/2)^n\right)u[n]$$ by taking the inverse Z transform.

ii- The anti-causal impulse response corresponds to assuming that the ROC is inside the innermost pole.

$$h_{ac}[n] = -\left((-1/4)^n + 2(1/2)^n\right)u[-n-1]$$

(c) $h_c[n]$ is summable while $h_{ac}[n]$ grows without bounds.

(d) $Y(z) = X(z)H(z)$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} \cdot \frac{1}{(1+\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})} =$$

$$Y(z) = \frac{1/3}{1+1/4\,z^{-1}} + \frac{2}{1-1/2\,z^{-1}} + \frac{2/3}{1-1/2\,z^{-1}}$$

$$y[n] = \frac{1}{3}\left(\frac{1}{4}\right)^n u[n] + 4(n+1)\left(\frac{1}{2}\right)^{n+1}u[n+1] + \frac{2}{3}\left(\frac{1}{2}\right)^n u[n]$$

2.17) (a) $r[n] = \begin{cases} 1 & \text{for } 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$

taking the fourier transform

$$R(e^{jw}) = \sum_{n=0}^{M} e^{-jwn} = \frac{1 - e^{jw(M+1)}}{1 - e^{-jw}}$$

$$= e^{-j\frac{M}{2}w} \left(\frac{e^{j\frac{M+1}{2}w} - e^{j\frac{M+1}{2}w}}{e^{jw} - e^{-jw}} \right)$$

$$= e^{-j\frac{M}{2}w} \left(\frac{\sin\left(\frac{M+1}{2}w\right)}{\sin(w/2)} \right)$$

(b) $w[n] = \frac{1}{2} + \frac{1}{2}\cos\frac{2\pi n}{M}$ for $0 \leq n \leq M$

0 otherwise

$$w[n] = \frac{r[n]}{2}\left[1 + \cos\left(\frac{2\pi n}{M}\right)\right] \quad \text{for } 0 \leq n \leq M$$

$$w(e^{jw}) = R(e^{jw}) * \sum_{n=-\infty}^{\infty} \frac{1}{2}\left(1 + \cos\left(\frac{2\pi n}{M}\right)\right) e^{-jwn}$$

$$= R(e^{jw}) * \sum_{n=-\infty}^{\infty} \frac{1}{2}\left(1 + \frac{1}{2}e^{j\frac{2\pi n}{M}} + \frac{1}{2}e^{-j\frac{2\pi n}{M}}\right) e^{-jw}$$

$$= R(e^{jw}) * \left[\frac{1}{2}\delta(w) + \frac{1}{4}\delta(w + 2\pi/M) + \frac{1}{4}\delta(w - 2\pi/M)\right]$$

2.1[illegible]) A system is causal iff. $h[n] = 0$ for $n < 0$

(a)

$$h[n] = \left(\tfrac{1}{2}\right)^n u[n]$$

Since $u[n] = 0$ for all $n < 0$
$h[n] = 0$ for all $n < 0$
So the system is causal

(b) $h[n] = (1/2)^n u[n-1]$

Since $u[n-1] = 0$ for all $n < 1$
$h[n] = 0$ for all $n < 0$
So the system is causal.

(d) $h[n] = u[n+2] - u[n-2]$

$u[n+2] = 1$ for $n > -2$ therefore
$h[n] \neq 0$ for $n < 0$ so
the system is non-causal.

2.19) (a) As $n \to \infty$ $h[n] \to \infty$ $h[n] = 4^n u[n]$

So the system is not stable.

(c) $h[n] = 3^n u[-n-1]$

$$\sum_n |h[n]| = \sum_{n=-\infty}^{-1} 3^n = \sum_{n=1}^{\infty} (1/3)^n = 1/2$$ so as $n \to \infty$ $h[n] \neq \infty$

the system is stable

(f) $h[n] = 2u[n+5] - u[n] - u[n-5]$

$$h[n] = \begin{cases} 2 & -5 \leq n \leq -1 \\ 1 & 0 \leq n \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

$\sum |h[n]| = 15$ So the system is stable

2.[illegible]) $y[n] + (1/a)\,y[n-1] = x[n-1]$

(a) $y[n] = x[n-1] - (1/a)\,y[n-1)$

$x[n] = \delta[n]$ to find the impulse response

$$y[n] = \delta[n-1] - \left(\tfrac{1}{a}\right) y[n-1]$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$y[0] = 0 - 0 = 0$

$y[1] = 1 - \frac{1}{a}\cdot 0 = 1$

$y[2] = 0 - \frac{1}{a}\cdot 1 = -1/a$

$y[3] = 0 - 1/a \cdot -1/a = (1/a)^2$

$y[4] = 0 - \frac{1}{a}\cdot (1/a)^2 = -(1/a)^3$

$\vdots$

$y[n] = (-1/a)^{n-1}\, u[n-1]$

$$h[n] = y[n] = (-1/a)^{n-1}\, u[n-1]$$

(b) If $h[n]$ is summable the system is stable.

$h[n]$ is summable if $|1/a| < 1$ or if $|a| > 1$
(or when $a \neq 1$).