EE 4CL4 – Control System Design

Solutions to Homework Assignment #10

1. **The output $y(t)$ of a continuous-time system having a unit step input $u(t)$ is sampled every 1 second. The expression for the sampled sequence $\{y[k]\}$ is given by:**

$$y[k] = 0.5 - 0.5(0.6)^k \qquad \forall k \geq 0 .$$

   a. **Determine $Y_q(z)$.**

   b. **Determine the transfer function from $U_q(z)$ to $Y_q(z)$.**

   c. **From the above result, derive the difference equation linking $\{y[k]\}$ to $\{u[k]\}$. (25 pts)**

   a. The Z-transform of the step response can be found from the table of Z-transform pairs:

$$Y_q(z) = 0.5\mathcal{Z}\left[1-(0.6)^k\right] = 0.5\left[\frac{z}{z-1} - \frac{z}{z-0.6}\right] = \frac{0.2z}{(z-1)(z-0.6)} .$$

   b. The transfer function is given by $Y_q(z)/U_q(z)$. The expression for $Y_q(z)$ obtained in part a corresponds to a unit step input, i.e., $U_q(z) = z/(z-1)$, such that:

$$\frac{Y_q(z)}{U_q(z)} = \frac{0.2z}{(z-1)(z-0.6)}\frac{z-1}{z} = \frac{0.2}{z-0.6} .$$

   c. The difference equation can be obtained from the above transfer function:

$$Y_q(z)(z-0.6) = 0.2U_q(z)$$
$$\Rightarrow y[k+1] - 0.6y[k] = 0.2u[k], \quad \text{or} \qquad y[k] - 0.6y[k-1] = 0.2u[k-1].$$

2. **The transfer function of a sampled-data system (in delta form) is given by:**

$$G_\delta(\gamma) = \frac{\gamma+0.5}{(\gamma+0.1)(\gamma+0.8)} .$$

   a. **If $\Delta = 3.5$ s, is the system stable?**

   b. **Find the corresponding Z-transform function for $\Delta = 3.5$ s.**

   c. **Repeat parts a and b for $\Delta = 1.5$ s. (25 pts)**

   a. We first recall that the stability region in the γ plane is given by $|\gamma\Delta + 1| < 1$. We observe that the system has poles at $\gamma = -0.1$ and $\gamma = -0.8$. The first pole is within the stability region $|-0.1 \cdot 3.5 + 1| = 0.65 < 1$ but the second is not $|-0.8 \cdot 3.5 + 1| = 1.8 > 1$, and hence the system is *unstable*.

b. To obtain the Z-transform transfer function, we use the relationship $\gamma = (z-1)/\Delta = (z-1)/3.5$. This yields:

$$G_q(z) = 3.5\frac{z+0.75}{(z-0.65)(z+1.8)}.$$

Note that this expression confirms the instability of the system because it has one pole outside the unit circle.

c. With this new sampling rate, both poles lie within the region of stability: $|-0.1\cdot 1.5+1| = 0.85 < 1$ and $|-0.8\cdot 1.5+1| = 0.2 < 1$.

The Z-transform transfer function is obtained using the relationship $\gamma = (z-1)/\Delta = (z-1)/1.5$, yielding:

$$G_q(z) = 1.5\frac{z-0.25}{(z-0.85)(z+0.2)},$$

which, as expected, is stable.

3. **A continuous-time plant has a transfer function given by:**

$$G_o(s) = \frac{1}{(s+1)^2(s+2)}.$$

a. **Compute the location of the sampling zeros for $\Delta = 0.2$ s.**

b. **How do the sampling zeros evolve when we vary Δ over the range [0.02 s, 2 s]? (25 pts)**

The discrete-time transfer function, assuming a ZOH at the input to the continuous-time plant, is given by:

$$\begin{aligned}
G_{oq}(z) &= \frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left[\frac{1}{s(s+1)^2(s+2)}\right]\Bigg|_{t=k\Delta}\right\} \\
&= \frac{z-1}{z}\mathcal{Z}\left\{\left[\frac{1}{2} - t\,\mathrm{e}^{-t} - \frac{1}{2}\mathrm{e}^{-2t}\right]\Bigg|_{t=k\Delta}\right\} \\
&= \frac{z-1}{z}\mathcal{Z}\left[\frac{1}{2} - k\Delta\,\mathrm{e}^{-k\Delta} - \frac{1}{2}\mathrm{e}^{-2k\Delta}\right] \\
&= \frac{z-1}{z}\left[\frac{1}{2}\frac{z}{z-1} - \frac{\Delta\,\mathrm{e}^{-\Delta}z}{\left(z-\mathrm{e}^{-\Delta}\right)^2} - \frac{1}{2}\frac{z}{z-\mathrm{e}^{-2\Delta}}\right].
\end{aligned}$$

Letting $a = e^{-\Delta}$ and $b = e^{-2\Delta}$:

$$G_{oq}(z) = \frac{1}{2}\left[1 - \frac{2\Delta a(z-1)}{(z-a)^2} - \frac{(z-1)}{z-b}\right]$$
$$= \frac{1}{2}\left[\frac{(z-a)^2(z-b) - 2\Delta a(z-1)(z-b) - (z-1)(z-a)^2}{(z-a)^2(z-b)}\right]$$
$$= \frac{1}{2}\left[\frac{cz^2 + dz + e}{(z-a)^2(z-b)}\right],$$

where $c = -b-2\Delta a+1$, $d = 2ab+2\Delta a+2\Delta ab-2a$, and $e = -a^2b+a^2-2\Delta ab$.

a. For $\Delta = 0.2$ s, $a = 0.8187$ and $b = 0.6703$, giving $c = 2.1877\times10^{-3}$, $d = 7.1787\times10^{-3}$ and $e = 1.4664\times10^{-3}$. The numerator polynomial has roots at $z = -3.063$ and $z = -0.219$.

b. From the above results, we can build a "root locus" of the zeros of $G_{oq}(z)$ for Δ varying from 0.02 to 2 s. One root goes from -3.6582 to -0.6189 and the other from -0.2626 to -0.0296. This "root locus" is shown in Fig. 1.

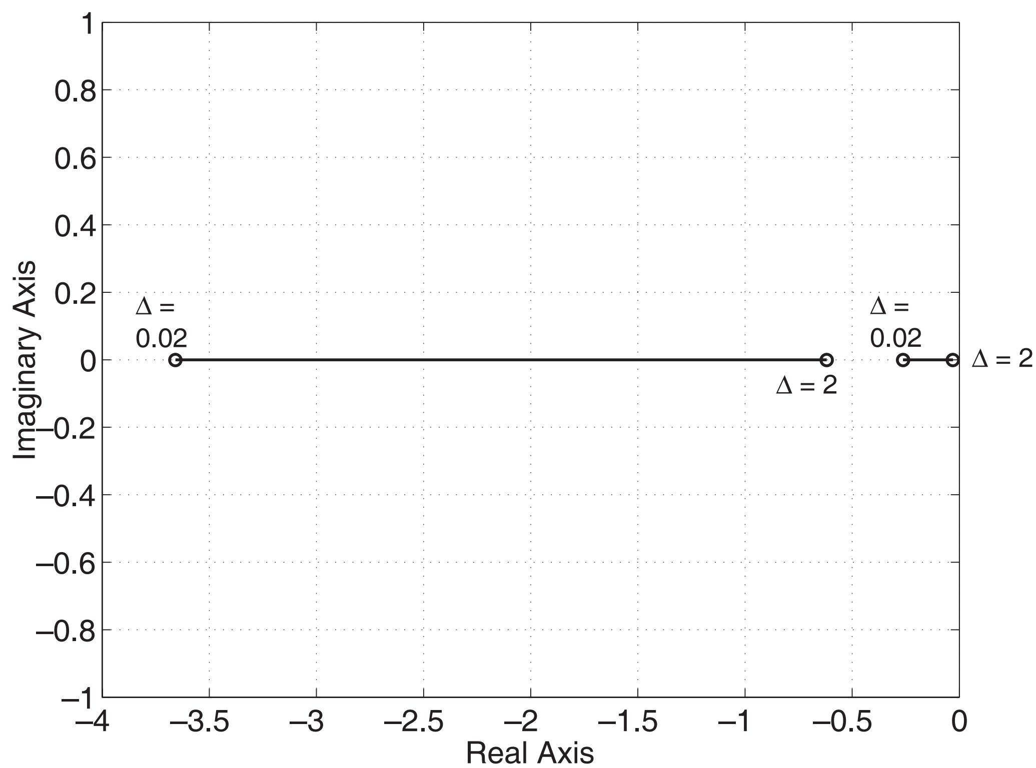

Figure 1 "Root locus" for different values of Δ.

4. **A continuous-time plant has a transfer function given by:**

$$G_o(s) = \frac{-s+1}{(s+2)(s+1)}.$$

a. **Is there any sampling frequency at which no zero appears in the Z-domain transfer function (assuming a ZOH at the plant input)?**

b. **Synthesize a minimal-prototype controller for Δ = 0.5 s.**

c. **Evaluate the control-loop performance to a unit step-output disturbance. (25 pts)**

a. The discrete-time transfer function, assuming a ZOH at the input to the continuous-time plant, is given by:

$$\begin{aligned}
G_{oq}(z) &= \frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left[\frac{-s+1}{s(s+1)(s+2)}\right]\Bigg|_{t=k\Delta}\right\} \\
&= \frac{z-1}{z}\mathcal{Z}\left\{\left[\frac{1}{2} - 2\,\mathrm{e}^{-t} + \frac{3}{2}\mathrm{e}^{-2t}\right]\Bigg|_{t=k\Delta}\right\} \\
&= \frac{z-1}{z}\mathcal{Z}\left[\frac{1}{2} - 2\,\mathrm{e}^{-k\Delta} + \frac{3}{2}\mathrm{e}^{-2k\Delta}\right] \\
&= \frac{z-1}{z}\left[\frac{1}{2}\frac{z}{z-1} - \frac{2z}{z-\mathrm{e}^{-\Delta}} + \frac{3}{2}\frac{z}{z-\mathrm{e}^{-2\Delta}}\right].
\end{aligned}$$

Letting $a = \mathrm{e}^{-\Delta}$:

$$\begin{aligned}
G_{oq}(z) &= \frac{1}{2}\left[1 - \frac{4(z-1)}{z-a} + \frac{3(z-1)}{z-a^2}\right] \\
&= \frac{1}{2}\left[\frac{(z-a)(z-a^2) - 4(z-1)(z-a^2) + 3(z-1)(z-a)}{(z-a)(z-a^2)}\right] \\
&= \frac{1}{2}\left[\frac{(3a^2-4a+1)z + a^3 - 4a^2 + 3a}{(z-a)(z-a^2)}\right].
\end{aligned}$$

Given this expression for $G_{oq}(z)$, there will be no sampling zeros when $3a^2 - 4a + 1 = 0 \Rightarrow a^2 - \frac{4}{3}a + \frac{1}{3} = 0 \Rightarrow (a-1)\left(a - \frac{1}{3}\right) = 0$, i.e., when a = 1 or 1/3. The first value for a corresponds to Δ = 0, which is not possible to implement, so only the second result is valid, i.e., $a = \mathrm{e}^{-\Delta} = 1/3 \Rightarrow \Delta = \ln(3)$. This produces the discrete-time (shift form) transfer function:

$$G_{oq}(z) = \frac{\left(\frac{1}{3}\right)^3 - 4\left(\frac{1}{3}\right)^2 + 3\,\frac{1}{3}}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{9}\right)} = \frac{0.2963}{(z-0.3333)(z-0.1111)}.$$

It is interesting to note that if we calculate the difference equation corresponding to this transfer function:

$$Y_q(z)\left(z^2 - 0.4444z + 0.03704\right) = 0.2963U_q(z)$$
$$y[k+2] - 0.4444y[k+1] + 0.03704y[k] = 0.2962u[k]$$
$$\Rightarrow y[k] - 0.4444y[k-1] + 0.03704y[k-2] = 0.2962u[k-2],$$

and assume zero initial conditions, then $y[k]$ must equal zero for both k = 0 and k = 1. Consequently, the step response of the *continuous-time* system $y(t)$ must be zero at times $t = 0$ and $t = \Delta = \ln(3)$. We note that $G_o(s)$ has a non-minimum-phase (NMP) zero, which must result in an *undershoot* in the step response. We deduce therefore that the second time that $y(t) = 0$ must be at the completion of the undershoot. That is, the case of no sampling zero in the discrete-time (shift form) transfer function corresponds to having a sampling interval equal to the time required to complete the undershoot produced by the non-minimum-phase zero in the continuous-time transfer function.

b. When $\Delta = 0.5$, $a = e^{-\Delta} = 0.6065$ and the discrete-time transfer function becomes:

$$G_{oq}(z) = \frac{-0.1612z + 0.2856}{z^2 - 0.9744z + 0.2231}.$$

Note that $G_{oq}(z)$ has a NMP zero at $z = 1.771$, so only the $A_{oq}(z)$ part can be cancelled by a minimal prototype controller. Therefore we let the controller denominator be $L_q(z) = (z-1)\bar{L}_q(z)$, the numerator be $P_q(z) = K_o A_{oq}(z) = K_o\left(z^2 - 0.9744z + 0.2231\right)$, and the closed-loop characteristic polynomial be $A_{cl}(z) = z^2 A_{oq}(z) = z^2\left(z^2 - 0.9744z + 0.2231\right)$, producing the Diophantine equation:

$$A_{oq}(z)L_q(z) + B_{oq}(z)P_q(z) = A_{cl}(z)$$
$$\Rightarrow \left(z^2 - 0.9744z + 0.2231\right)(z-1)(l_1 z + l_0) + (-0.1612z + 0.2856)K_o\left(z^2 - 0.9744z + 0.2231\right)$$
$$= z^2\left(z^2 - 0.9744z + 0.2231\right)$$
$$\Rightarrow (z-1)(l_1 z + l_0) + (-0.1612z + 0.2856)K_o = z^2.$$

The solution to this equation is $l_1 = 1$, $l_0 = 2.2958$ and $K_0 = 8.0386$, corresponding to the controller:

$$C_q(z) = \frac{K_o(z)A_{oq}(z)}{(z-1)\bar{L}_q(z)} = \frac{8.0386(z-0.6065)(z-0.3679)}{(z-1)(z+2.2958)} = \frac{8.039z^2 - 7.833z + 1.794}{z^2 + 1.296z - 2.296}.$$

c. The system response to a unit step-output disturbance is obtained via:

$$Y_q(z) = S_{oq}(z) D_{oq}(z).$$

From the table of Z-transform pairs $D_{oq}(z) = z/(z-1)$, and from the plant model and controller obtained in part b:

$$\begin{aligned}
S_{oq}(z) &= \frac{A_{oq}(z)L_q(z)}{A_{oq}(z)L_q(z) + B_{oq}(z)P_q(z)} \\
&= \frac{(z-1)(z+2.2958)(z-0.6065)(z-0.3679)}{(z-1)(z+2.2958)(z-0.6065)(z-0.3679) + 8.0386(z-0.6065)(z-0.3679)(-0.1612z+0.2856)} \\
&= \frac{(z-1)(z+2.2958)}{(z-1)(z+2.2958) + 8.0386(-0.1612z+0.2856)} \\
&= \frac{(z-1)(z+2.2958)}{z^2}.
\end{aligned}$$

We can now evaluate:

$$\begin{aligned}
Y_q(z) &= S_{oq}(z) D_{oq}(z) \\
&= \frac{(z-1)(z+2.2958)}{z^2} \frac{z}{z-1} \\
&= \frac{z+2.2958}{z} \\
&= 1 + \frac{2.2958}{z},
\end{aligned}$$

which, assuming zero initial conditions and zero reference signal, corresponds to the discrete-time sequence:

$$\begin{aligned}
y[k] &= \mathcal{Z}^{-1}\left[Y_q(z)\right] \\
&= \delta_K[k] + 2.2958\delta_K[k-1]\mu[k-1] \\
&= \delta_K[k] + 2.2958\delta_K[k-1].
\end{aligned}$$

Consequently, $y[0] = 1$, $y[1] = 2.2958$, and $y[k] = 0$ for $k \geq 2$, i.e., it takes just two time samples for the control-loop to compensate for the output disturbance.