EE 4CL4 – Control System Design

Solutions to Homework Assignment #5

1. **Determine the open-loop transfer function of the system generating the root locus plot shown in Figure 1. (25 pts)**

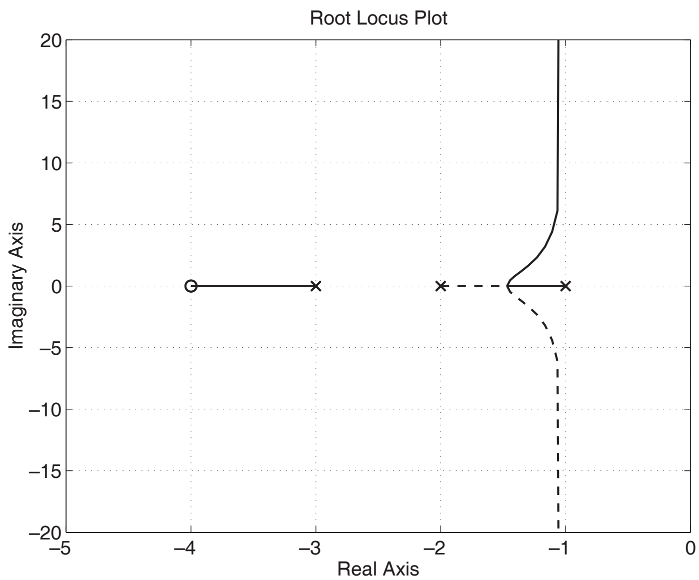

Figure 1

The open-loop poles of this system are at $s = -1$, -2 and -3. There is one finite open-loop zero at $s = -4$ and two implicit zeros at ∞. Thus the open-loop transfer function of this system is of the form:

$$H(s) = \lambda \frac{s+4}{(s+1)(s+2)(s+3)},$$

where λ is the gain-parameter varied to created the root-locus plot.

2. **Determine the PID controller parameters (for the *standard form*) for a plant with the nominal model:**

$$G_o(s) = \frac{-s+2}{(s+2)^2},$$

using the Ziegler-Nichols oscillation method. (25 pts)

The closed-closed characteristic polynomial for this nominal plant model in a one-d.o.f. unity-feedback loop with a proportional controller is:

$$1 + K_p G_o(s) = (s+2)^2 + K_p(-s+2) = 0.$$

At the point of critical stability, $K_p = K_c$ and $s = j\omega_c$, such that:

$$(s+2)^2 + K_p(-s+2) = (j\omega_c + 2)^2 + K_c(-j\omega_c + 2) = 0$$

$$\Rightarrow K_c = \frac{-(j\omega_c + 2)^2}{(-j\omega_c + 2)} = \frac{6\omega_c^{\,2} - 8 + j(\omega_c^{\,3} - 12\omega_c)}{\omega_c^{\,2} + 4}.$$

The critical gain $K_c \in \Re$, so the complex term in the equation above must equal zero, which gives:

$$\omega_c^{\,3} - 12\omega_c = 0 \Rightarrow \omega_c = 2\sqrt{3} \Rightarrow P_c = \frac{2\pi}{\omega_c} = \frac{\pi}{\sqrt{3}}.$$

Substituting the value for ω_c into the equation for K_c yields:

$$K_c = \frac{6\omega_c^{\,2} - 8}{\omega_c^{\,2} + 4} = \frac{6 \cdot 12 - 8}{12 + 4} = 4.$$

From Table 6.1 of Goodwin et al., the PID parameters are then:

$K_p = 0.6K_c = 2.4$,

$T_r = 0.5P_c = 0.9069$, and

$T_d = P_c/8 = 0.2267$.

3. **Use the pole placement method to synthesize a controller $C(s)$ for the nominal plant model:**

$$G_o(s) = \frac{1}{(s+2)^2},$$

that produces the nominal closed-loop characteristic polynomial $A_{cl}(s) = (s^2 + 4s + 9)(s + 8)$**, using MATLAB to solve the matrix equations. (25 pts)**

The eliminant matrix for a nominal plant model with a maximum degree of 2 is:

$$\mathbf{M_e} = \begin{bmatrix} a_2 & 0 & b_2 & 0 \\ a_1 & a_2 & b_1 & b_2 \\ a_0 & a_1 & b_0 & b_1 \\ 0 & a_0 & 0 & b_0 \end{bmatrix},$$

and the controller to be synthesized is of the form:

$$C(s) = \frac{p_1 s + p_0}{l_1 s + l_0}.$$

The pole-assignment matrix equation for this system is then:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 4 & 4 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_1 \\ l_0 \\ p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 12 \\ 41 \\ 72 \end{bmatrix} \Rightarrow \begin{bmatrix} l_1 \\ l_0 \\ p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 5 \\ 40 \end{bmatrix},$$

giving the controller $C(s) = \dfrac{5s + 40}{s + 8}$.

4. **Find suitable PID controller parameters (for the *standard form*) for a plant with the nominal model:**

$$G_o(s) = \frac{10}{(s+1)(s+10)}, \tag{1}$$

using the reaction curve method with:

a. **the Ziegler-Nichols parameters, and**

b. **the Cohen-Coon parameters.** **(25 pts)**

The process reaction curve for this plant can be obtained by calculating the unit step response $y(t)$ of the plant in open loop:

$$Y(s) = G_o(s)U(s) = \frac{10}{s(s+1)(s+10)}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{10}{s(s+1)(s+10)}\right] = 1 - \frac{10}{9}\mathrm{e}^{-t} + \frac{1}{9}\mathrm{e}^{-10t}.$$

The slope of $y(t)$ is then:

$$\dot{y}(t) = \frac{10}{9}\mathrm{e}^{-t} - \frac{10}{9}\mathrm{e}^{-10t},$$

and the maximal slope can be found at the time t when the derivate of the slope is zero:

$$\ddot{y}(t) = -\frac{10}{9}\mathrm{e}^{-t} + \frac{100}{9}\mathrm{e}^{-10t} = 0$$

$$\begin{aligned}
\Rightarrow -\mathrm{e}^{-t} + 10\,\mathrm{e}^{-10t} &= 0 \\
10\,\mathrm{e}^{-10t} &= \mathrm{e}^{-t} \\
10\,\mathrm{e}^{-10t}\mathrm{e}^{t} &= 1 \\
\mathrm{e}^{-9t} &= 1/10 \\
-9t &= \log_{\mathrm{e}}(1/10) \\
t &= -\frac{\log_{\mathrm{e}}(1/10)}{9} \approx 0.2558
\end{aligned}$$

The maximal slope is then:

$$\dot{y}(2.558) = \frac{10}{9}\mathrm{e}^{-0.2558} - \frac{10}{9}\mathrm{e}^{-2.558} = 0.7743,$$

at the point $t = 0.2558$, $y(0.2558) = 0.1483$, giving the maximum slope tangent:

$$m.s.t. = 0.7743(t - 0.2558) + 0.1483.$$

The $m.s.t.$ is equal to $y_0 = 0$ at time $t_1 = 0.0643$ and is equal to $y_\infty = 1$ at $t_2 = 1.3558$. The unit step ($u_0 = 0$; $u_\infty = 1$) was applied at time $t_0 = 0$, giving the parameter model:

$$K_0 = \frac{y_\infty - y_0}{u_\infty - u_0} = 1; \qquad \tau_0 = t_1 - t_0 = 0.0643; \qquad \nu_0 = t_2 - t_1 = 1.2915.$$

a. From Table 6.2 of Goodwin et al., the *Ziegler-Nichols* PID parameters are then:

$$K_p = \frac{1.2\nu_0}{K_0\tau_0} = 24.1026,$$

$$T_r = 2\tau_0 = 0.1286, \text{ and}$$

$$T_d = 0.5\tau_0 = 0.0321.$$

b. From Table 6.3 of Goodwin et al., the *Cohen-Coon* PID parameters are:

$$K_p = \frac{\nu_0}{K_0\tau_0}\left[\frac{4}{3} + \frac{\tau_0}{4\nu_0}\right] = 27.0307,$$

$$T_r = \frac{\tau_0\left[32\nu_0 + 6\tau_0\right]}{13\nu_0 + 8\tau_0} = 0.1550, \text{ and}$$

$$T_d = \frac{4\tau_0\nu_0}{11\nu_0 + 2\tau_0} = 0.0232.$$