EE 4CL4 – Control System Design

Solutions to Homework Assignment #8

1. **Consider a feedback control loop where:**

   $$G_o(s)C(s) = \frac{9}{s(s+4)}.$$

   a. **Verify Lemma 9.1 on page 242 of Goodwin et al. for this open-loop transfer function. Note that if the integral cannot be solved analytically, then you may calculate a numerical approximation using the MATLAB function `quadl()`. If you use this function, please include in your report the exact MATLAB command(s) that you used.**

   b. **Repeat but with** $G_o(s)C(s) = \frac{17s+100}{s(s+4)}$**.** **(25 pts)**

a. For $G_o(s)C(s) = \frac{9}{s(s+4)}$, the nominal sensitivity is:

$$S_o(s) = \frac{s(s+4)}{s^2+4s+9}.$$

Then, to verify Lemma 9.1 we compute:

$$\int_0^\infty \ln\left|S_o(\mathrm{j}\omega)\right| \mathrm{d}\,\omega = \int_0^\infty \ln\left|\frac{\mathrm{j}\omega(\mathrm{j}\omega+4)}{-\omega^2+4\,\mathrm{j}\omega+9}\right| \mathrm{d}\,\omega.$$

This can be approximated numerically using the MATLAB command:

```
quadl('log(abs(i*w.*(i*w+4)./(-w.^2+4*i*w+9)))',0,1e5)
```

giving the answer –9.0773e–005 ≈ 0.

From Lemma 9.1, for an open-loop transfer function without a delay term this integral should equal $-\kappa\frac{\pi}{2}$, where $\kappa \stackrel{\Delta}{=} \lim_{s\to\infty} sH_{ol}(s)$. For this $G_o(s)C(s)$, $\kappa = 0$ and hence the numerical approximation of the integral confirms Lemma 9.1 for this case.

b. For $G_o(s)C(s) = \frac{17s+100}{s(s+4)}$, the nominal sensitivity is:

$$S_o(s) = \frac{s(s+4)}{s^2+21s+100}.$$

Then, to verify Lemma 9.1 we compute:

$$\int_0^\infty \ln\left|S_o(\mathrm{j}\omega)\right| \mathrm{d}\,\omega = \int_0^\infty \ln\left|\frac{\omega(\mathrm{j}\omega+4)}{-\omega^2+21\,\mathrm{j}\omega+100}\right| \mathrm{d}\,\omega.$$

This integral can be approximated numerically using the MATLAB command:

```
quadl('log(abs(i*w.*(i*w+4)./(-w.^2+21*i*w+100)))',0,1e5)
```

giving the answer $-26.7024 \approx -8.5\pi$.

For this $G_o(s)C(s)$, $\kappa = 17$ and consequently this integral should equal $-\kappa\frac{\pi}{2} = -8.5\pi$. Once again the numerical approximation of the integral confirms Lemma 9.1 for this case.

2. **The nominal model for a plant is given by:**

$$G_o(s) = \frac{5(s-1)}{(s+1)(s-5)}.$$

   a. **Use the pole placement method to synthesize a controller such that the closed-loop poles are at (–2; –2; –2).**

   b. **Verify Lemma 9.2 on page 244 of Goodwin et al. for this plant and controller placed in a unity-feedback loop. Again, if you use the MATLAB function `quadl()`, please include in your report the exact MATLAB command(s) that you used.**

   c. **Verify Lemma 9.5 on page 249 of Goodwin et al. for this plant and controller placed in a unity-feedback loop. Again, if you use the MATLAB function `quadl()`, please include in your report the exact MATLAB command(s) that you used. (50 pts)**

a. The pole-assignment matrix equation for this system is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 5 & 0 \\ -5 & -4 & -5 & 5 \\ 0 & -5 & 0 & -5 \end{bmatrix}\begin{bmatrix} l_1 \\ l_0 \\ p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 12 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} l_1 \\ l_0 \\ p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} 1 \\ -4.375 \\ 2.875 \\ 2.775 \end{bmatrix},$$

giving the controller $C(s) = \dfrac{2.875s + 2.775}{s - 4.375}$.

b. For this controller and nominal plant model, the nominal sensitivity is:

$$S_o(s) = \frac{(s-5)(s+1)(s-4.375)}{(s+2)^3}.$$

Then, to verify Lemma 9.2 we compute:

$$\int_0^\infty \ln|S_o(\mathrm{j}\omega)|\,\mathrm{d}\,\omega = \int_0^\infty \ln\left|\frac{-\mathrm{j}\omega^3 + 8.375\omega^2 + 12.5\,\mathrm{j}\omega + 21.875}{-\mathrm{j}\omega^3 - 6\omega^2 + 12\,\mathrm{j}\omega + 8}\right|\mathrm{d}\,\omega.$$

This can be approximated numerically using the MATLAB command:

```
quadl('log(abs((-i*w.^3+8.375*w.^2+12.5*i*w+21.875)./(-
i*w.^3-6*w.^2+12*i*w+8)))',0,1e5)
```

giving the answer 6.8721.

For this system, the open-loop transfer function is:

$$G_o(s)C(s)=\frac{14.375(s-1)(s+0.9652)}{(s-5)(s+1)(s-4.375)},$$

giving $\kappa=14.375$, $\sum_{i=1}^{2}R\{p_i\}=9.375$ and n_r = 1. Consequently the integral should equal $-\kappa\frac{\pi}{2}+\pi\sum_{i=1}^{2}R\{p_i\}=6.8691$, and thus the numerical approximation of the integral confirms Lemma 9.2 for this case.

c. To verify Lemma 9.5, we note that the open-loop transfer function has one non-minimum-phase zero at s = 1, and therefore $c_1=\gamma_1+\mathrm{j}\delta_1=1+\mathrm{j}0$. We then compute:

$$\int_0^{\infty}\ln\left|S_o(\mathrm{j}\omega)\right|\frac{2\gamma_k}{\gamma_k^2+(\delta_k-\omega)^2}\,\mathrm{d}\,\omega=\int_0^{\infty}\ln\left|\frac{-\mathrm{j}\omega^3+8.375\omega^2+12.5\,\mathrm{j}\omega+21.875}{-\mathrm{j}\omega^3-6\omega^2+12\,\mathrm{j}\omega+8}\right|\frac{2}{1+\omega^2}\,\mathrm{d}\,\omega\,.$$

This can be approximated numerically using the MATLAB command:

```
quadl('log(abs((-i*w.^3+8.375*w.^2+12.5*i*w+21.875)./(-
i*w.^3-6*w.^2+12*i*w+8))).*2./(1+w.^2)',0,1e5)
```

giving the answer 2.7354.

The open-loop transfer function has two open-loop poles, so the Blaschke product:

$$B_p(s)=\frac{(s-5)}{(s+5)}\frac{(s-4.375)}{(s+4.375)}\Rightarrow\ln\left|B_p(c_1)\right|=\ln\left|\frac{(1-5)}{(1+5)}\frac{(1-4.375)}{(1+4.375)}\right|=-0.8708.$$

Consequently the integral should equal $-\pi\ln\left|B_p(c_1)\right|=2.7358$, and thus the numerical approximation of the integral confirms Lemma 9.5 for this case.

3. **A plant model is given by:**

$$G(s) = \frac{e^{-s}}{s+1}.$$

Approximating the delay by $e^{-s} \approx \dfrac{s^2 - 6s + 12}{s^2 + 6s + 12}$**, use Lemma 9.5 and the result from Example 9.1 (on pages 249 and 251, respectively, of Goodwin et al.), with $\epsilon = 0.1$ and $\omega_l = 3$ to derive a lower bound for the nominal sensitivity peak.** **(25 pts)**

Using the approximation given for the delay, the nominal plant model is:

$$G_o(s) = \frac{s^2 - 6s + 12}{(s^2 + 6s + 12)(s+1)}.$$

We note that $G_o(s)$ has two NMP phase zeros, located at $s = 3.00 \pm j1.73$. Then, to apply Eq. (9.4.16) on page 251 of Goodwin et al., we need to compute $\Omega(c_i, \omega_l)$ for both zeros using Eq. (9.4.8) on page 250 of Goodwin et al. This yields:

$$\Omega(c_1, \omega_l) = 2\arctan\left(\frac{3 - 1.73}{3}\right) + 2\arctan\left(\frac{1.73}{3}\right) = 1.847$$

$$\Omega(c_2, \omega_l) = 2\arctan\left(\frac{3 + 1.73}{3}\right) - 2\arctan\left(\frac{1.73}{3}\right) = 0.965.$$

Thus, the worst case is for the zero located at $s = c_1$, since this maximizes the right hand side in Eq. (9.4.16) on page 251 of Goodwin et al., leading to:

$$\ln S_{\max} > \frac{1}{\pi - \Omega(c_1, \omega_l)}\left[\pi \ln\left|B_p(c_1)\right| + \left|\ln(\epsilon)\Omega(c_1, \omega_l)\right|\right] \approx 3.285 \Rightarrow S_{\max} > 26.709.$$