EE 4CL4 – Control System Design

Solutions to Homework Assignment #9

1. **Compute the Z-transform for the discrete sequences that result from sampling the following signals a 1 Hz:**

   a. $e^{-0.1t}\cos(0.5t+\pi/4)$.

   b. $t^2\,e^{-0.25t}$. **(25 pts)**

   If the sampling frequency is 1 Hz, then the sampling period, Δ, is 1 s. To compute the Z-transform of a sampled signal $f(t)$ we first obtain the expression for $f(k\Delta) = f[k]$ and we then use the table of Z-transform pairs to obtain the Z-transform.

   a. If $f(t)=e^{-0.1t}\cos(0.5t+\pi/4)$, then:

$$\begin{aligned} f[k] &= a^k\cos(0.5k+\pi/4) \\ &= a^k\left[\cos(0.5k)\cos(\pi/4)-\sin(0.5k)\sin(\pi/4)\right] \\ &= \frac{a^k}{\sqrt{2}}\left[\cos(0.5k)-\sin(0.5k)\right], \end{aligned}$$

   where $a=e^{-0.1}$. Using the Z-transform pairs for sequences of the form $f[k]=a^k\cos(k\theta)$ and $f[k]=a^k\sin(k\theta)$ gives:

$$\begin{aligned} F(z) &= \frac{1}{\sqrt{2}}\left[\frac{z(z-a\cos(0.5))}{z^2-2az\cos(0.5)+a^2}-\frac{az\sin(0.5)}{z^2-2az\cos(0.5)+a^2}\right] \\ &= \frac{z(z-a\cos(0.5)-a\sin(0.5))}{\sqrt{2}\left(z^2-2az\cos(0.5)+a^2\right)} \\ &= \frac{z(z-1.228)}{\sqrt{2}\left(z^2-1.588z+0.819\right)}. \end{aligned}$$

   b. If $f(t)=t^2\,e^{-0.25t}$, then:

$$f[k]=k^2b^k,$$

   where $b=e^{-0.25}$. Using the Z-transform pair for a sequence of the form $g[k]=kb^k$ gives:

$$G(z)=\frac{bz}{(z-b)^2}.$$

   We now make use of the Z-transform property that $\mathcal{Z}[kg[k]]=-z\frac{\mathrm{d}\,G(z)}{\mathrm{d}\,z}$ to calculate:

$$F(z)=\mathcal{Z}[kg[k]]=-z\frac{\mathrm{d}\,G(z)}{\mathrm{d}\,z}=\frac{bz(z+b)}{(z-b)^3}.$$

2. **Consider the following recursive equation describing the relationship between the input $u[k]$ and the output $y[k]$ in a discrete-time (sampled-data) system:**

$$y[k]-0.5y[k-1]+0.06y[k-2]=0.6u[k-1]+0.3u[k-2].$$

a. **Determine the transfer function.**

b. **From the above result, compute the response of the system to a unit Kronecker delta. (25 pts)**

a. Taking the Z-transform of both sides of the difference equation with zero initial conditions gives:

$$Y_q(z)-0.5z^{-1}Y_q(z)+0.06z^{-2}Y_q(z)=0.6z^{-1}U_q(z)+0.3z^{-2}U_q(z).$$

The transfer function is normally expressed in positive powers of z only, so we multiply both sides by z^2 to yield:

$$H_q(z)=\frac{Y_q(z)}{U_q(z)}=\frac{0.6z+0.3}{z^2-0.5z+0.06}.$$

b. The response of the system to a unit Kronecker delta is the inverse Z-transform of the transfer function:

$$H_q(z)=\frac{0.6z+0.3}{z^2-0.5z+0.06}=\frac{0.6z+0.3}{(z-0.3)(z-0.2)}=\frac{1}{z}\left[\frac{4.8z}{(z-0.3)}-\frac{4.2z}{(z-0.2)}\right].$$

The inverse Z-transform of the expressions inside the square brackets can be found in the table of Z-transform pairs, and then the "Backwards Shift" property from the Z-transform properties table can be used to find the effect of the factor $1/z$ outside the square bracket:

$$\begin{aligned}
h[k] &= \mathcal{Z}^{-1}\left[H_q(z)\right] \\
&= -\delta_K[k]\left\{4.8(0.3)^{-1}-4.2(0.2)^{-1}\right\}+4.8(0.3)^{k-1}-4.2(0.2)^{k-1} \\
&= 5\delta_K[k]+4.8(0.3)^{-1}(0.3)^k-4.2(0.2)^{-1}(0.2)^k \\
&= 5\delta_K[k]+16(0.3)^k-21(0.2)^k.
\end{aligned}$$

Alternatively, the "Unit Step" property from the Z-transform properties table can be used instead to find the effect of the factor $1/z$ outside the square bracket:

$$\begin{aligned}
h[k] &= \mathcal{Z}^{-1}\left[H_q(z)\right] \\
&= \mu[k-1]\left\{4.8(0.3)^{k-1}-4.2(0.2)^{k-1}\right\} \\
&= \mu[k-1]\left\{4.8(0.3)^{-1}(0.3)^k-4.2(0.2)^{-1}(0.2)^k\right\} \\
&= \mu[k-1]\left\{16(0.3)^k-21(0.2)^k\right\},
\end{aligned}$$

where $\mu[k]$ is the unit step function.

3. **Determine the step response of the discrete-time system with the transfer function:**

$$H(z) = \frac{1}{(z-0.6)^2}.$$ **(25 pts)**

The step response $y[k]$ of a given system with transfer function $H_q(z)$ can be computed from

$$Y_q(z) = H_q(z)\frac{z}{z-1}.$$

A partial fraction expansion is then used and known inverse Z-transforms for simple fractions are applied.

$$Y_q(z) = H_q(z)\frac{z}{z-1} = \frac{z}{(z-1)(z-0.6)^2} = z\left[\frac{25}{4}\frac{1}{z-1} - \frac{25}{4}\frac{1}{z-0.6} - \frac{5}{2}\frac{1}{(z-0.6)^2}\right]$$

$$= \frac{25}{4}\frac{z}{z-1} - \frac{25}{4}\frac{z}{z-0.6} - \frac{25}{6}\frac{0.6z}{(z-0.6)^2}$$

$$\Rightarrow y[k] = \mathcal{Z}^{-1}\left[\frac{25}{4}\frac{z}{z-1} - \frac{25}{4}\frac{z}{z-0.6} - \frac{25}{6}\frac{0.6z}{(z-0.6)^2}\right] = \frac{25}{4} - \frac{25}{4}(0.6)^k - \frac{25}{6}k(0.6)^k.$$

4. **Assume that, in Figure 12.6 on page 339 of Goodwin et al., $G_o(s)$ is given by:**

$$G_o(s) = \frac{2}{(s+1)(s+2)}.$$

   a. **Compute the Delta-Transform of the transfer function from $u[k]$ to $y[k]$, $H_{o\delta}(\gamma)$, as a function of the sampling interval Δ.**

   b. **Verify that, if we make $\Delta \to 0$, then:**

$$\lim_{\Delta \to 0} H_{o\delta}(\gamma)\Big|_{\gamma=s} = G_o(s).$$ **(25 pts)**

   a. One method for computing the Delta-transform transfer function makes use of the property:

$$Y_{o\delta}(\gamma) = \Delta Y_{oq}(z)\Big|_{z=\Delta\gamma+1}. \qquad (1)$$

   As in Fig. 12.6 (in the book) we assume a zero order hold. Then, the Z-transform for the system's response to the Kronecker delta is:

$$Y_{oq}(z) = \frac{z-1}{z}\mathcal{Z}\left[\mathcal{L}^{-1}\left[\frac{2}{s(s+1)(s+2)}\right]\Big|_{t=k\Delta}\right]$$

$$= \frac{z-1}{z}\mathcal{Z}\left[1 - 2\,\mathrm{e}^{-k\Delta} + \mathrm{e}^{-2k\Delta}\right]$$

$$= \frac{z-1}{z}\left[\frac{z}{z-1} - 2\frac{z}{z-e^{-\Delta}} + \frac{z}{z-e^{-2\Delta}}\right].$$

From Eq. (1) above, the Delta-transform for the system's response to the Kronecker delta is then:

$$
\begin{aligned}
Y_{o\delta}(\gamma) &= \Delta Y_{oq}(z)\Big|_{z=\Delta\gamma+1} \\
&= \Delta \frac{\Delta\gamma}{\Delta\gamma+1}\left[\frac{\Delta\gamma+1}{\Delta\gamma} - 2\frac{\Delta\gamma+1}{\Delta\gamma+1-e^{-\Delta}} + \frac{\Delta\gamma+1}{\Delta\gamma+1-e^{-2\Delta}}\right] \\
&= \Delta\left[1 - 2\frac{\Delta\gamma}{\Delta\gamma+1-e^{-\Delta}} + \frac{\Delta\gamma}{\Delta\gamma+1-e^{-2\Delta}}\right]
\end{aligned}
$$

We now note that the Delta-transform transfer function is the system's response to a scaled version of the Kronecker delta, $\frac{1}{\Delta}\delta_K[k]$, rather than the unscaled Kronecker delta. Therefore, the transfer function $H_{o\delta}(\gamma)$ can be obtained from:

$$
\begin{aligned}
H_{o\delta}(\gamma) &= \frac{1}{\Delta} Y_{o\delta}(\gamma) \\
&= 1 - 2\frac{\Delta\gamma}{\Delta\gamma+1-e^{-\Delta}} + \frac{\Delta\gamma}{\Delta\gamma+1-e^{-2\Delta}} \\
&= 1 - 2\frac{\gamma}{\gamma+a} + \frac{\gamma}{\gamma+b} \\
&= \frac{(2a-b)\gamma + ab}{(\gamma+a)(\gamma+b)},
\end{aligned}
$$

where $a = \left(1-e^{-\Delta}\right)/\Delta$ and $b = \left(1-e^{-2\Delta}\right)/\Delta$.

b. First we calculate the values of a and b when $\Delta \to 0$, making use of the Taylor series expansion of the exponential terms:

$$
a = \frac{1-e^{-\Delta}}{\Delta} = \frac{1-\left(1-\Delta+\frac{1}{2}\Delta^2-\frac{1}{6}\Delta^3+\ldots\right)}{\Delta} = \frac{\Delta-\frac{1}{2}\Delta^2+\frac{1}{6}\Delta^3-\ldots}{\Delta} = 1-\tfrac{1}{2}\Delta+\tfrac{1}{6}\Delta^2-\ldots
$$

$$
\Rightarrow \lim_{\Delta\to 0} a = 1
$$

$$
b = \frac{1-e^{-2\Delta}}{\Delta} = \frac{1-\left(1-2\Delta+\frac{1}{2}(2\Delta)^2-\frac{1}{6}(2\Delta)^3+\ldots\right)}{\Delta} = \frac{2\Delta-2\Delta^2+\frac{4}{3}\Delta^3-\ldots}{\Delta} = 2-2\Delta+\tfrac{4}{3}\Delta^2-\ldots
$$

$$
\Rightarrow \lim_{\Delta\to 0} b = 2
$$

Thus:

$$
\begin{aligned}
\lim_{\Delta\to 0} H_{o\delta}(\gamma)\Big|_{\gamma=s} &= \left[\frac{(2\cdot 1-2)\gamma + 1\cdot 2}{(\gamma+1)(\gamma+2)}\right]_{\gamma=s} \\
&= \frac{2}{(s+1)(s+2)} \\
&= G_o(s).
\end{aligned}
$$